THE PERIODICITY OF THE FREE FERMIONS: A CONJECTURE

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Let *Fer* denote the free chiral Majorana fermion in one dimension. It is a generally covariant net of $\mathbb{Z}/2$-graded von Neumann algebras on the category of spin intervals. Given a spin interval I with real spinor bundle S, $S^{\otimes 2} \simeq T^*I$, the value *Fer* on I is the complexification of a completion of

$$\mathit{Cliff}\big(\Gamma(I,S)\big) := \bigoplus_{i\geq 0} \Gamma(I,S)^{\otimes i} \,/\, s\otimes s - \langle s,s\rangle,$$

where $\Gamma(I,S)$ is the space of sections of S, and the inner product is given by $\langle s,s\rangle = \int_I s^{\otimes 2}$, with $s^{\otimes 2} \in \Gamma(I,S^{\otimes 2}) \simeq \Gamma(I,T^*I) = \Omega^1(I)$.

Our periodicity conjecture says that $\mathit{Fer}^{\otimes n} \equiv \mathit{Fer}^{\otimes(n+576)}$, where the symbol $\equiv$ denotes an equivalence relation that I shall not define here. This conjecture is inspired by the speculations having to do with the generalized cohomology theory *TMF*, whose period is also 576. Here is a statement that is equivalent to the above one:

Conjecture. *Let H be a separable $\mathbb{Z}/2$-graded Hilbert space. There exists a net $\mathcal{B}$ on $[0,1]$ of $\mathbb{Z}/2$-graded von Neumann algebras on H subject to the following conditions:*

- *The net $\mathcal{B}$ is local in the $\mathbb{Z}/2$-graded sense:*
  If $I,J\subset[0,1]$ are intervals with disjoint interiors, then $\mathcal{B}(I)$ and $\mathcal{B}(J)$ graded-commute. More explicitly, the odd elements of $\mathcal{B}(I)$ and $\mathcal{B}(J)$ anti-commute, while the even elements $\mathcal{B}(I)$ and $\mathcal{B}(J)$ commute with both even and odd elements of the other algebra.
- *The net $\mathcal{B}$ is additive:*
  If $I,J,K\subset[0,1]$ are closed intervals satisfying $I\cup J=K$, then $\mathcal{B}(I)\vee\mathcal{B}(J)=\mathcal{B}(K)$. Here, the symbol $\vee$ denotes the von Neumann algebra generated by the two subalgebras. Note that $\mathcal{B}$ doesn't have to be regular at the end points: the algebra $\bigvee\mathcal{B}([\varepsilon,1-\varepsilon])$ doesn't have to be isomorphic to $\mathcal{B}([0,1])$, and actually it can't be.
- *The restriction of $\mathcal{B}$ to $(0,1)$ is isomorphic to $\mathit{Fer}^{\otimes 576}$:*
  There exists an isomorphism of $\mathbb{Z}/2$-graded Hilbert space $H\simeq(\text{Fock Space})^{\otimes 576}$ such that for every closed interval $I\subset(0,1)$ it yields an identification between $\mathcal{B}(I)$ and $\mathit{Fer}^{\otimes 576}(I)$. Here, Fock Space refers to the vacuum Hilbert space of the conformal net *Fer* on $\mathbb{R}$.
- *The net $\mathcal{B}$ satisfies the split property:*
  If $I,J\subset[0,1]$ are closed intervals that do not intersect, then the algebra $\mathcal{B}(I)\vee\mathcal{B}(J)$ is isomorphic to the spacial tensor product $\mathcal{B}(I)\,\bar{\otimes}\,\mathcal{B}(J)$.
- *The net $\mathcal{B}$ is irreducible:*
  $\mathcal{B}([0,1])=B(H)$ is the algebra of all bounded operators on H.
- *The net $\mathcal{B}$ satisfies Haag duality for disjoint unions of intervals:*
  if $0=x_1<x_2<\ldots<x_n=1$ are points delimiting intervals $I_i:=[x_i,x_{i+1}]$, then we have

  $$\mathcal{B}(I_1)\vee\mathcal{B}(I_3)\vee\ldots=\Big(\mathcal{B}(I_2)\vee\mathcal{B}(I_4)\vee\ldots\Big)',$$

  where the prime denotes the graded commutant inside $B(H)$.