Algebraic approach to quantum field theory on a class of noncommutative curved spacetimes

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Motivation

- QFT on curved spacetimes is important for physics [cf. T.P. Hack]

$\longrightarrow$ cosmology (CMB fluctuations) and black holes (Hawking radiation)

- precise formulation via algebraic approach [Wald, many people here, ...]
- But why should we make all of this noncommutative?
  - NC geometry from quantum gravity!?!?

  $\longrightarrow$ include some quantum gravity effects in QFTCS

  - NC geometry is natural generalization of classical geometry

  $\longrightarrow$ generalize standard methods of QFTCS as far as possible

  - NC in cosmology and black hole physics is of physical interest

  $\longrightarrow$ provide formal background for phenomenology

  - $\exists$ NC gravity solutions [Schupp, Solodukhin; T. Ohl, AS; Aschieri, Castellani]

  $\longrightarrow$ test their physical implications by using QFTCS

Reminder: Noncommutative spaces

What is a NC space and vector bundle?

- we know [Gelfan'd, Naimark]:

  topological spaces $\mathbb{X}$ $\Leftrightarrow$ commutative C^*-algebras $C(\mathbb{X})$

- example of a NC space (quantum mechanics):

  NC phase space $=$ NC algebra $[\hat{x},\hat{p}] = i\hbar\,\hat{1}$ $\xrightarrow{\hbar\to 0}$ $(x,p)\in\mathbb{R}^2$

$\longrightarrow$ **NC space** $:=$ NC (C^*-) algebra $\mathcal{A}$

- How to generalize vector bundles to the NC setting?
- we know [Serre, Swan]:

  vector bundles $E\to\mathbb{X}$ $\Leftrightarrow$ modules over $C(\mathbb{X})$ (= sections)

$\longrightarrow$ **NC vector bundle** $:=$ module over NC (C^*-) algebra $\mathcal{A}$

How to quantize classical spaces and vector bundles?

$\rightarrow$ we use (formal) deformation quantization ($\star$-products):

- smooth manifold $\mathcal{M}$ $\Rightarrow$ algebra of smooth functions $\mathcal{A} = (C^\infty(\mathcal{M}), \cdot)$
- **Basic idea:**
  replace $\cdot$-product by associative NC $\star$-product $\Rightarrow$ $\mathcal{A}_\star = (C^\infty(\mathcal{M})[[\lambda]], \star)$
- Example (Moyal-product): Let $\mathcal{M} = \mathbb{R}^d$. Define

$$h \star k = h\, e^{\frac{i\lambda}{2} \overleftarrow{\partial_\mu} \Theta^{\mu\nu} \overrightarrow{\partial_\nu}}\, k \qquad \rightarrow \qquad [x^\mu \stackrel{\star}{,} x^\nu] = i\lambda\Theta^{\mu\nu}$$

- smooth vector bundle $E \rightarrow \mathcal{M}$ $\Rightarrow$ module of smooth sections ${}_{\mathcal{A}}\Gamma^\infty(E, \mathcal{M})$
- **Basic idea:**
  replace left action $h \cdot v$ by $h \bullet v$, such that $(h \star k) \bullet v = h \bullet (k \bullet v)$
- Example: Let $\mathcal{M} = \mathbb{R}^d$ and $E = T\mathcal{M}$. Define for $v = v^\mu(x)\partial_\mu$

$$h \bullet v = \left(h \star v^\mu\right) \partial_\mu$$

NC spaces and vector bundles from Drinfel'd twists

- Drinfel'd twists arise in Hopf algebra theory (quantum symmetries)
- our motivation: $\exists$ canonical construction of $h \star k$ and $h \bullet v$ from a twist!

e.g. $\star$-product $\quad h \star k = h\, e^{\frac{i\lambda}{2}\overleftarrow{\partial_\mu}\Theta^{\mu\nu}\overrightarrow{\partial_\nu}}\, k \quad \leftrightsquigarrow \quad$ twist $\quad \mathcal{F}^{-1} = e^{\frac{i\lambda}{2}\Theta^{\mu\nu}\partial_\mu \otimes_{\mathbb{C}} \partial_\nu}$

- our class of twists: $\mathcal{F}^{-1} = \bar{f}^\alpha \otimes_{\mathbb{C}} \bar{f}_\alpha \in U\mathrm{Vec}[[\lambda]] \otimes_{\mathbb{C}} U\mathrm{Vec}[[\lambda]]$
  - normalization: $(\epsilon \otimes_{\mathbb{C}} \mathrm{id})\mathcal{F} = (\mathrm{id} \otimes_{\mathbb{C}} \epsilon)\mathcal{F} = 1$
  - cocycle condition: $\mathcal{F}_{12}\,(\Delta \otimes_{\mathbb{C}} \mathrm{id})(\mathcal{F}) = \mathcal{F}_{23}\,(\mathrm{id} \otimes_{\mathbb{C}} \Delta)(\mathcal{F})$
  - reality: $\mathcal{F}^{*\otimes *} = (S \otimes_{\mathbb{C}} S)(\mathcal{F}_{21}) \quad \big(\Rightarrow (h \star k)^* = k^* \star h^*\big)$
  - technical assumption: $S(\bar{f}^\alpha) \cdot \bar{f}_\alpha = 1 \quad \big(\Rightarrow \int \omega \wedge_\star \omega' = \int \omega \wedge \omega'\big)$

NB: most studied NC gravity solutions are of this type

- twist deformation quantization [Wess group]:
  - algebra of functions $(C^\infty(\mathcal{M})[[\lambda]], \star)$, where $h \star k := \bar{f}^\alpha(h) \cdot \bar{f}_\alpha(k)$
  - exterior algebra $(\Omega^\bullet[[\lambda]], \wedge_\star, \mathrm{d})$, where $\omega \wedge_\star \omega' := \bar{f}^\alpha(\omega) \wedge \bar{f}_\alpha(\omega')$
  - pairing $\langle v, \omega \rangle_\star := \langle \bar{f}^\alpha(v), \bar{f}_\alpha(\omega)\rangle$ among vectorfields and 1forms $\quad \ldots$

Scalar field theory on curved NC spacetimes

- action:

$$S_\Phi = \int L_\Phi = -\frac{1}{2}\int \big(\langle\langle d\Phi, g_\star^{-1}\rangle_\star, d\Phi\rangle_\star + M^2\,\Phi\star\Phi\big)\star \mathrm{vol}_\star$$

- use local basis: $\langle \partial_\mu, \widetilde{dx}^\nu\rangle_\star = \delta_\mu^\nu$

$$\rightarrow \quad g_\star^{-1} = \partial_\mu^* \otimes_\star g^{\mu\nu}\star\partial_\nu \quad , \qquad d\Phi = dx^\mu\,\partial_\mu\Phi =: \widetilde{dx}^\mu \star \partial_{\star\mu}\Phi$$

$$L_\Phi = -\frac{1}{2}\big((\partial_{\star\mu}\Phi)^*\star g^{\mu\nu}\star\partial_{\star\nu}\Phi + M^2\,\Phi\star\Phi\big)\star\mathrm{vol}_\star$$

- equation of motion:

$$0 = \frac{1}{2}\Big(\square_\star[\Phi]\star\mathrm{vol}_\star + \mathrm{vol}_\star\star\big(\square_\star[\Phi^*]\big)^* - M^2\,\Phi\star\mathrm{vol}_\star - M^2\,\mathrm{vol}_\star\star\Phi\Big)$$

$$=: P_\star[\Phi]\star\mathrm{vol}_\star$$

NB: $P_\star$ is formally self adjoint w.r.t. SP $(\varphi,\psi)_\star = \int \varphi^*\star\psi\star\mathrm{vol}_\star$, i.e.

$$\big(\varphi, P_\star[\psi]\big)_\star = \big(P_\star[\varphi],\psi\big)_\star$$

- slice of de Sitter space: $ds^2 = -dt^2 + e^{2Ht}\,(dx^2 + dy^2 + dz^2)$
- the following NC spacetimes solve NC Einstein equations [T. Ohl, AS]

1.) $\mathcal{F}^{-1} = \exp\left(\frac{i\lambda}{2}(\partial_t \otimes_{\mathbb{C}} \partial_\varphi - \partial_\varphi \otimes_{\mathbb{C}} \partial_t)\right) \;\Rightarrow\; \left[e^{i\varphi} \stackrel{\star}{,} t\right] = \lambda\, e^{i\varphi}$

$$-\left(\partial_t^2 + 3H\partial_t + M^2\right) \frac{1 + e^{i3\lambda H\partial_\varphi}}{2}\,\Phi + e^{-2Ht}\triangle\,\frac{e^{-i\lambda H\partial_\varphi} + e^{i4\lambda H\partial_\varphi}}{2}\,\Phi = 0$$

- depends only on $\lambda H \;\Rightarrow$ no deformation for $H \to 0$!

  very rough estimate: $\lambda H \approx t_{\mathrm{pl}} H_{\mathrm{today}} \approx 10^{-60}$

2.) $\mathcal{F}^{-1} = \exp\left(\frac{i\lambda}{2}(\partial_t \otimes_{\mathbb{C}} x^i\partial_i - x^i\partial_i \otimes_{\mathbb{C}} \partial_t)\right) \;\Rightarrow\; \left[t \stackrel{\star}{,} x^i\right] = i\lambda\, x^i$

$$-\left(\partial_t^2 + 3H\partial_t + M^2\right) \frac{1 + e^{-i3\lambda\mathcal{D}}}{2}\Phi + e^{-2Ht}\triangle\frac{e^{i\lambda\mathcal{D}} + e^{-i4\lambda\mathcal{D}}}{2}\Phi = 0$$

where $\mathcal{D} := \partial_t - Hx^i\partial_i$

NC scalar fields: Green's operators, solution space and canonical quantization

- let $P_\star = \sum_{n=0}^{\infty} \lambda^n P_{(n)}$ be a deformed Klein-Gordon operator (defined above)
- $(\mathcal{M}, g_\star, \star)\big|_{\lambda\to 0}$ time-oriented, connected, globally hyperbolic
- technical assumption: $P_{(n)} : C^\infty(\mathcal{M}) \to C_0^\infty(\mathcal{M})$ for all $n > 0$
  - fulfilled for all twists of compact support
  - or $g_\star$ asymptotically (outside compact region) symmetric under $\mathcal{F}$
- based on strong results for the commutative case we find:

there exist unique Green's operators $\Delta_{\star\pm} := \sum_{n=0}^{\infty} \lambda^n \Delta_{(n)\pm}$ satisfying

(i) $P_\star \circ \Delta_{\star\pm} = \mathrm{id}_{C_0^\infty(\mathcal{M})[[\lambda]]}$,

(ii) $\Delta_{\star\pm} \circ P_\star\big|_{C_0^\infty(\mathcal{M})[[\lambda]]} = \mathrm{id}_{C_0^\infty(\mathcal{M})[[\lambda]]}$,

(iii) $\mathrm{supp}\big(\Delta_{(n)\pm}(\varphi)\big) \subseteq J_\pm\big(\mathrm{supp}(\varphi)\big)$, for all $n \in \mathbb{N}^0$ and $\varphi \in C_0^\infty(\mathcal{M})$,

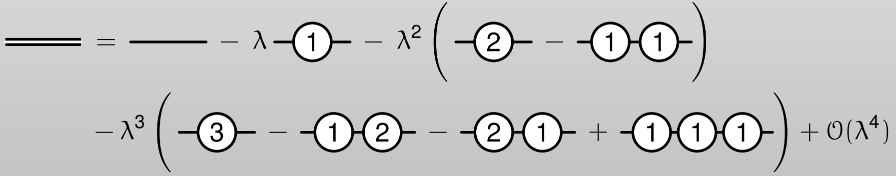
where $J_\pm$ is the causal future/past with respect to the metric $g_\star|_{\lambda\to 0}$.

NB: also possible for deformed normally hyperbolic operators $P_\star$

Explicit formula for $\Delta_{\star\pm}$ in terms of $\Delta_{\pm} := \Delta_{(0)\pm}$:

$$\begin{aligned}\Delta_{\star\pm} = \Delta_{\pm} &\\ &- \lambda\, \Delta_{\pm} \circ P_{(1)} \circ \Delta_{\pm}\\ &- \lambda^2 \left(\Delta_{\pm} \circ P_{(2)} \circ \Delta_{\pm} - \Delta_{\pm} \circ P_{(1)} \circ \Delta_{\pm} \circ P_{(1)} \circ \Delta_{\pm}\right)\\ &+ \mathcal{O}(\lambda^3) \quad \text{[higher orders follow the same structure]}\end{aligned}$$

Graphically:



→ perturbative approach to deformed Green's operators

- define fundamental solution $\Delta_\star := \Delta_{\star +} - \Delta_{\star -}$
- this sequence of maps is an exact complex

$$0 \longrightarrow C_0^\infty(\mathcal{M})[[\lambda]] \xrightarrow{P_\star} C_0^\infty(\mathcal{M})[[\lambda]] \xrightarrow{\Delta_\star} C_{sc}^\infty(\mathcal{M})[[\lambda]] \xrightarrow{P_\star} C_{sc}^\infty(\mathcal{M})[[\lambda]]$$

- since Φ is real $\rightarrow$ restriction to real solutions
- space of “physical sources”:

$$H := \left\{\varphi \in C_0^\infty(\mathcal{M})[[\lambda]] : \left(\Delta_{\star\pm}(\varphi)\right)^* = \Delta_{\star\pm}(\varphi)\right\}$$

NB: Let ψ be a real solution of the deformed wave equation, then there is a $\varphi \in H$, such that $\psi = \Delta_\star(\varphi)$.

$\Rightarrow$ $H/\mathrm{Ker}(\Delta_\star)$ is isomorphic to the space of real solutions of $P_\star$

Proposition (T. Ohl, AS)

$(V_\star, \omega_\star)$ with $V_\star := H/\mathrm{Ker}(\Delta_\star)$ and

$$\omega_\star([\varphi],[\psi]) := \left(\varphi, \Delta_\star(\psi)\right)_\star = \int \varphi^* \star \Delta_\star(\psi) \star \mathrm{vol}_\star$$

is a symplectic vector space.

What are possible $*$-algebras of field observables?

- Let $\mathcal{A}$ be a unital $*$-algebra over $\mathbb{C}[[\lambda]]$ [math/0408217 (Waldmann), . . .]

1.) $*$-algebra of Weyl-type $W_\star : V_\star \to \mathcal{A}$, such that

$$W_\star(0) = 1 \ ,$$
$$W_\star(-\varphi) = W_\star(\varphi)^* \ ,$$
$$W_\star(\varphi) \cdot W_\star(\psi) = e^{-i\omega_\star(\varphi,\psi)/2} \, W_\star(\varphi + \psi) \ .$$

- not a C^*-algebra, since everything is over $\mathbb{C}[[\lambda]]$
- can be made C^*, if we could find convergent deformation $(V_\star, \omega_\star)$

2.) $*$-algebra of field polynomials $\Phi_\star : V_\star \to \mathcal{A}$ (linear), such that

$$\Phi_\star(\varphi)^* = \Phi_\star(\varphi) \ ,$$
$$\big[\Phi_\star(\varphi), \Phi_\star(\psi)\big] = i\,\omega_\star(\varphi,\psi)\, 1 \ .$$

- here **not** twisted commutator $\big[a, b\big]_\star = \big[\bar{f}^\alpha(a), \bar{f}_\alpha(b)\big]$

- *Observation:* Let ω_1 and ω_2 be two symplectic structures on $\mathbb{R}^n$. Then $\exists\, T : \mathbb{R}^n \to \mathbb{R}^n$, such that $\omega_1(Tu, Tv) = \omega_2(u, v)$ for all u, v.

$\longrightarrow$ deformation theory of finite dim. symplectic vector spaces is "trivial"

- T also exists for $\left(V = \frac{C_0^\infty(\mathcal{M},\mathbb{R})[[\lambda]]}{\mathrm{Ker}(\Delta)}, \omega\right)$ and $\left(V_\star = \frac{H}{\mathrm{Ker}(\Delta_\star)}, \omega_\star\right)$ in QFT!

$\longrightarrow$ (formal) symplectomorphism between commutative and NC QFT

- Sketch of the construction of T:
  - proof that H is isomorphic to $C_0^\infty(\mathcal{M}, \mathbb{R})[[\lambda]]$ using Hodge and $\star$-Hodge
  - use this and "Ker"-isomorphism to map $\omega_\star$ to an $\hat{\omega}_\star$ on V
  - show that $\omega(Tu, Tv) = \hat{\omega}_\star(u, v)$, for all $u, v \in V$, can be solved for T
- Induction of representations:
  - let $\mathcal{A}$ be field polynomials of comm. QFT acting on Hilbert space $\mathcal{H}$
  - define $\Phi_\star(\varphi) := \Phi(T\varphi) \Rightarrow$ representation of algebra of NC QFT on $\mathcal{H}[[\lambda]]$
  - NC correlation functions from commutative correlation functions

$$\langle\Psi|\Phi_\star(\varphi_1)\cdots\Phi_\star(\varphi_n)|\Psi\rangle = \langle\Psi|\Phi(T\varphi_1)\cdots\Phi(T\varphi_n)|\Psi\rangle$$

- Locality properties of T? Study nets of algebras! [work in progress]

Power spectrum in NC cosmology (still naive!)

We had the deformed cosmological model (slice of de Sitter space):

$$P_\star[\Phi] = -\left(\partial_t^2 + 3H\partial_t\right)\frac{1 + e^{i3\lambda H\partial_\varphi}}{2}\Phi + e^{-2Ht}\triangle\frac{e^{-i\lambda H\partial_\varphi} + e^{i4\lambda H\partial_\varphi}}{2}\Phi = 0$$

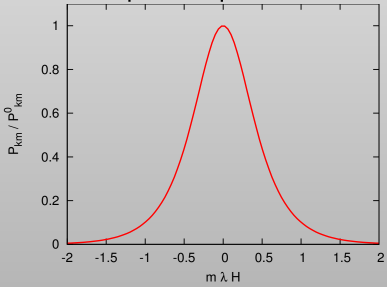
- performing the spherical wave expansion we can solve the EOM

$$\widetilde{\Phi}_{km}(\eta) = c_1\frac{e^{i\epsilon_{km}\eta}}{\sqrt{\epsilon_{km}}}\left(\frac{1}{\epsilon_{km}} - i\eta\right) + c_2\frac{e^{-i\epsilon_{km}\eta}}{\sqrt{\epsilon_{km}}}\left(\frac{1}{\epsilon_{km}} + i\eta\right)$$

where η is conformal time and $\epsilon_{km}^2 = k^2\frac{\cosh(5\lambda Hm/2)}{\cosh(3\lambda Hm/2)}$

- Fock space representation of NC field polynomials
- for Bunch-Davies vacuum ($c_1 = 0$) we find the power spectrum

$$\mathcal{P}_{km} := \lim_{\eta\to 0}\widetilde{\Phi}_{km}(\eta)\,\widetilde{\Phi}^*_{km}(\eta) = \frac{H^2}{\pi k^3}\sqrt{\frac{\cosh(3\lambda Hm/2)}{\cosh^3(5\lambda Hm/2)}}$$



$\rightarrow$ decreasing power for large $|m|$

- scalar field actions on curved NC spacetimes:
  - formally self adjoint EOM operators $\mathrm{P}_\star$
  - explicit models for NC cosmology (also black holes) [arXiv:1003.3190]
- NC Green's operators, solution space and quantization [arXiv:0912.2252]:
  - existence, uniqueness and construction of the deformed Green's operators
  - symplectic structure on the space of real solutions of $\mathrm{P}_\star$
  - quantization via $*$-algebras of field observables (no C^*-algebras, yet)
  - symplectomorphism between commutative and NC QFT
- Outlook and future work:
  - properties of states on $*$-algebras of NC field observables
    $\longrightarrow$ NC cosmo, NC Hawking radiation, ...
  - more detailed investigations in NC cosmology
  - Can one include convergent deformations? $\longrightarrow$ hopefully C^*-properties