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A remark on a theorem of Chan-Jackson-Trang

F. Schlutzenberg observed that the methods from [We] could be used to strengthen said theorem.

Let $M^\#$ denote the sharp for an inner model with a proper class of measurable cardinals.

Let $D \subset \omega_1$ be a club such that each $\alpha \in D$ is an $M^\#$-admissible. Chan-Jackson-Trang had basically shown that $L_{\omega_1}[D] \models \text{ZFC} + \text{GCH}$.

We here show that there is an iterate, M, of $M^\#$ together with a Prikry generic, C, over M such that $L_{\omega_1}[D] = M|\omega_1[C]$.

Let $(\kappa_i : i < \omega_1)$ be the monotone enumeration of D. Let M' denote the iterate of $M^\#$ obtained by hitting the top measure of $M^\#$ and its images ω_1 times, so that ω_1^V is the critical point of the top measure of M'.

Let $E \subset \omega_1$ be the set of critical points used in the iteration from $M^{\#}$ to M'. So E is club, and as every $\alpha \in D$ is $M^{\#}$-admissible, $E \cap \alpha$ is club in α for every $\alpha \in D$.

We may then iterate M' below ω_1^V and normally in such a fashion that if M denotes the iterate, then $\kappa_{\omega \cdot i + \omega}$ is the i^{th} measurable cardinal of M, $i < \omega_1$. If $\mathcal{I}$ denotes the (linear) iteration from M' to M, then for all $i < \omega_1$ and all $n < \omega$, $n > 0$,

$$\pi^{\mathcal{I}}_{\omega \cdot i + n, \omega \cdot i + \omega}(\kappa_{\omega \cdot i + n}) = \kappa_{\omega \cdot i + \omega}$$

and $\kappa_{\omega \cdot i + n} = \mathrm{crit}(\pi^{\mathcal{I}}_{\omega \cdot i + n, \omega \cdot i + \omega})$, so that

$$(\kappa_{\omega \cdot i + n} : 0 < n < \omega)$$

is Prikry-generic over M w.r.t. the (unique) measure of M on $\kappa_{\omega \cdot i + \omega}$. Moreover,

$C = (\kappa_{\omega \cdot i+n} : i < \omega_1 \wedge 0 < n < \omega)$ is Prikry generic over M, cf. [We].

C consists exactly of the successor points of D, so that $L_{\omega_1}[C] = L_{\omega_1}[D]$. C is class generic over $M|\omega_1$. We claim that

$$(*) \qquad L_{\omega_1}[D] = M|\omega_1[C] .$$

"$\subset$" is trivial, so let us verify that "$\supset$" holds true. By "$\subset$" and $M^\# \notin M|\omega_1[C]$, $M^\# \notin L_{\omega_1}[D]$, so that $K^{L_{\omega_1}[D]}$ exists and is "$M^\#$-small," i.e., $M^\# \not\trianglelefteq K^{L_{\omega_1}[D]}$. Moreover, $K^{M|\omega_1[C]} = K^{M|\omega_1}$ (tail ends of C don't add new bounded sets, K doesn't change by set forcing, and K is locally definable), so that by $K^{M|\omega_1} = M|\omega_1$, $M|\omega_1$ is the core model of $M|\omega_1[C]$.

By "$\subset$", $K^{L_{\omega_1}[D]} \leq^* M|\omega_1$ then in the mouse order.

Suppose that $K^{L_{\omega_1}[D]} <^* M|\omega_1$, so that a proper initial segment of $M|\omega_1$ iterates past $K^{L_{\omega_1}[D]}$. The witnessing iteration, $\mathcal{U}$, on the $M|\omega_1$-side must then use a single measure and its images ω_1 times, which implies that the collection of total measures in $K^{L_{\omega_1}[D]}$ is bounded below ω_1. Let $\bar{\mathcal{U}}$ be the iteration on $K^{L_{\omega_1}[D]}$ arising in this comparison.

By $K^{L_{\omega_1}[D]} <^* M|\omega_1 <^* M^*$ and the fact that D only consists of M^*-admissibles, every $\alpha \in D$ is a fixed point of $\pi^{\mathcal{U}}_{0\alpha}$, hence of $\pi^{\mathcal{U}}_{0\infty}$ for all sufficiently big $\alpha \in D$. Also, the set $\{(\pi^{\mathcal{I}}_{\beta\infty})^{-1}(\omega_1^V) : \omega_1^V \in \operatorname{ran}(\pi^{\mathcal{I}}_{\beta\infty}) \wedge (\pi^{\mathcal{I}}_{\beta\infty})^{-1}(\omega_1^V) = \operatorname{crit}(\pi^{\mathcal{I}}_{\beta\infty})\}$, which is the set of all critical points of the single measure and its images which is used on a tail end of I, covers a tail end of D.

We may then pick $i < \omega_1$ such that

$\pi^{\mathcal{U}}_{0\infty}(\kappa_{\omega\cdot i+\omega}) = \kappa_{\omega\cdot i+\omega}$ and $\kappa_{\omega\cdot i+\omega} =$

$\mathrm{crit}(\pi^{\mathcal{I}}_{\omega\cdot i+\omega,\,\omega_1^V}) = (\pi^{\mathcal{I}}_{\omega\cdot i+\omega,\,\omega_1^V})^{-1}(\omega_1^V)$.

$\kappa_{\omega\cdot i+\omega}$ is then inaccessible in $\mathcal{M}^{\mathcal{U}}_\infty$, hence in $K^{L_{\omega_1}[D]}$, but $(\kappa_{\omega\cdot i+n} : n<\omega) \in L_{\omega_1}[D]$ witnesses that $\kappa_{\omega\cdot i+\omega}$ is of cofinality ω in $L_{\omega_1}[D]$. By the Dodd-Jensen covering lemma, then, $\kappa_{\omega\cdot i+\omega}$ must be measurable in $K^{L_{\omega_1}[D]}$. But we could have chosen $\kappa_{\omega\cdot i+\omega}$ above the sup of the measurables of $K^{L_{\omega_1}[D]}$. Contradiction!

Hence $K^{L_{\omega_1}[D]} =^* M|\omega_1$ in the mouse order and $K^{L_{\omega_1}[D]}$ has unboundedly many measurable cardinals in ω_1^V, in fact by a theorem of Jensen's there is an elementary embedding

$\pi : M|\omega_1 \longrightarrow K^{L_{\omega_1}[D]}$ resulting from a normal iteration, call it $\mathcal{I}$, on $M|\omega_1$ ($K^{L_{\omega_1}[D]}$ was just shown to be a universal weasel in $M_1|\omega_1[C]$.)

Let us now prove that $M|\omega_1 = K^{L_{\omega_1}[D]}$, which will finish the proof of $(*)$, as then $M|\omega_1 \subset L_{\omega_1}[D]$.

It obviously suffices to verify that the $\kappa_{\omega\cdot i+\omega}$, $i<\omega_1^V$, are the measurable cardinals of $K^{L_{\omega_1}[D]}$.

Fix $i<\omega_1^V$ and suppose that that's true for all $\kappa_{\omega\cdot j+\omega}$, $j<i$. Then $\kappa_{\omega\cdot i+\omega}$ is the next measurable of $M|\omega_1$, so the next measurable of $K^{L_{\omega_1}[D]}$ must be $\geq \kappa_{\omega\cdot i+\omega}$. We have that $\mathcal{P}(\kappa_{\omega\cdot i+\omega}) \cap K^{L_{\omega_1}[D]} = \mathcal{P}(\kappa_{\omega\cdot i+\omega}) \cap M|\omega_1$ and $(\kappa_{\omega\cdot i+n} : 0<n<\omega) \in K^{L_{\omega_1}[D]}$ generates the (unique) measure in M on $\kappa_{\omega\cdot i+\omega}$. But then this measure is in $L_{\omega_1}[D]$ as well and in fact it will be on the $K^{L_{\omega_1}[D]}$ - sequence.

References.

[We] Philip Welch, paper on L[Card] ?

William Chan, talk at UC Berkeley, July 2019.