# The Destruction of the Axiom of Determinacy by Forcings on $\mathbb{R}$ when $\Theta$ is regular.

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(Joint work with Stephen Jackson)

__Question__ Can a nontrivial forcing which is a surjective image of $\mathbb{R}$ preserve the axiom of determinacy?

Ikegami and Trang studied various aspects of forcing over AD. They showed many forcing, such as Cohen forcing, can not preserve AD.

__Fact:__ (Ikegami and Trang) $ZF + AD^+ + V = L(P(\mathbb{R}))$. If $\mathbb{P}$ is a forcing and $1_{\mathbb{P}} \Vdash AD$, then $1_{\mathbb{P}} \Vdash \dot{\Theta} = \check{\Theta}$.

__Fact__ (Ikegami and Trang) Assume the consistency of $ZF + AD^+ + \Theta > \Theta_0$. Then it is consistent that ZF+AD holds and there is a forcing $\mathbb{P}$ such that $1_{\mathbb{P}} \Vdash AD$ and $1_{\mathbb{P}} \Vdash \check{\Theta} < \dot{\Theta}$.

The main theorems are

__Thm__ $(ZF + AD)$ If $\mathbb{P}$ is a nontrivial wellorderable forcing of cardinality less than $\Theta$ then $1_{\mathbb{P}} \Vdash \neg AD$.

__Thm__ $(ZF + AD + \Theta \text{ regular})$ If $\mathbb{P}$ is a nontrivial forcing which is a surjective image of $\mathbb{R}$, then $1_{\mathbb{P}} \Vdash \neg AD$.

Fix $\langle U^{(n)}(\dot{A},\dot{B}) : n \in \omega \rangle$ a good universal sequence for $\Sigma_1^1(A,B)$ which is defined uniformly in $\dot{A},\dot{B}$.

$\dot{A},\dot{B}$ represent subsets of $\mathbb{R}$. $U^{(n)}(\dot{A},\dot{B}) \subseteq \mathbb{R}^{n+1}$.

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<u>Uniform Coding Lemma</u> (Moschovakis) $ZF + AD$. $X \subseteq \mathbb{R}$. $\pi : X \to On$.

For each $a \in X$, let $Q_{<a} = \{ b \in X : \pi(b) < \pi(a) \}$ and $Q_a = \{ b \in X : \pi(b) = \pi(a) \}$.

Let $Z \subseteq X \times \mathbb{R}$. Then there is an $e \in \mathbb{R}$ so that for all $a \in X$

① $U_e^{(2)}(Q_{<a}, Q_a) \subseteq Z \cap (Q_a \times \mathbb{R})$

② $U_e^{(2)}(Q_{<a}, Q_a) \neq \emptyset \iff Z \cap (Q_a \times \mathbb{R}) \neq \emptyset$.

If $\pi : X \to \kappa$ is surjective, then for each $e \in \mathbb{R}$, let

$S_e^\pi = \{ \alpha \in \kappa : \exists a\ (\pi(a) = \alpha \wedge U_e^{(2)}(Q_{<a}, Q_a) \neq \emptyset \}$.

By the coding lemma, for all $A \subseteq \kappa$, there exists an $e \in \mathbb{R}$ so that $S_e^\pi = A$.

<u>Definition</u>: Let $A \subseteq \mathbb{R}$. Let $\delta_A$ be the least $\delta$ so that $L_\delta(A,\mathbb{R}) \prec_1 L(A,\mathbb{R})$.

Also ~~$L_\delta$~~ $\delta_A$ is least $\delta$ so that $L_\delta(A,\mathbb{R}) \prec_1^{\mathbb{R} \cup \{\mathbb{R}, A\}} L(A,\mathbb{R})$, meaning elementary of $\Sigma_1$ formula with parameters from $\mathbb{R} \cup \{\mathbb{R}, A\}$.

There is a $\Sigma_1(L(A,\mathbb{R}), \{\mathbb{R},A\})$ surjection $\varphi_A : U_A \to \delta_A$ where $U_A$ is universal for $\Sigma_1(L(A,\mathbb{R}), \mathbb{R} \cup \{\mathbb{R},A\})$.

There is a $\Sigma_1(L(A,\mathbb{R}), \{\mathbb{R},A\})$ formula $\Sigma$ so that for all $\alpha, e$

$L(A,\mathbb{R}) \models \quad \alpha \in S_e^{\varphi_A} \iff \Sigma(\alpha, e, A, \mathbb{R})$.

<u>Definition</u>: Let $\lambda \leq \kappa$. $\kappa \to (\kappa)_2^\lambda$ asserts that for all $\Phi : [\kappa]^\lambda \to 2$ there is a club $C \subseteq \kappa$ and $i \in 2$ so that for all $f \in [\kappa]^\lambda$ of the "correct type", $\Phi(f) = i$.

If $\kappa \to (\kappa)_2^\kappa$, then $\kappa$ is said to be a strong partition cardinal.

<u>Fact</u> (Martin) $ZF + AD$. $\omega_1 \to (\omega_1)_2^{\omega_1}$.

<u>Fact</u>: (Kechris, Kleinberg, Moschovakis, Woodin) $ZF + AD$. For all $A \subseteq \mathbb{R}$ $\delta_A \to (\delta_A)_2^{\delta_A}$. Hence the strong partition cardinals are cofinal in $\Theta$.

<u>Definition</u>: Let $\kappa$ be a regular cardinal. $\mathbb{P}$ is a forcing. $\mathbb{P}$ has the ground club property at $\kappa$ iff for all $p \in \mathbb{P}$ and $\mathbb{P}$-name $\dot{D}$ st $p \Vdash$ "$\dot{D} \subseteq \check{\kappa}$ is club", there is some club $C \subseteq \kappa$ so that $p \Vdash \check{C} \subseteq \dot{D}$.

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<u>Main Lemma 1</u>: (ZF) Let $\mathbb{P}$ be a forcing such that $\mathbb{P}$ has the ground club property at $\kappa$ and $1_{\mathbb{P}} \Vdash \check{\kappa} \longrightarrow (\check{\kappa})^{\check{\omega}}_{2}$, then $1_{\mathbb{P}} \Vdash \dot{\mathbb{R}} = \check{\mathbb{R}}$.

<u>Proof</u>: Let $G \subseteq \mathbb{P}$ be $\mathbb{P}$-generic. Note $([\kappa]^{\omega})^{V} \in V[G]$.

In $V[G]$, define

$$\Phi(f) = \begin{cases} 0 & \text{if } f \in ([\kappa]^{\omega})^{V} \\ 1 & \text{otherwise} \end{cases}$$

In $V[G]$, let $D \subseteq \kappa$ be club which is homogeneous for $\Phi$. By the ground club property, let $C \subseteq D$ with $C \in V$. Pick $f \in ([C]^{\omega})^{V}$ of the correct type. $\Phi(f) = 0$ so $C$ is homogeneous for $\Phi$ taking value $0$.

Let $c_i = C(\omega \cdot i + \omega)$ i.e. the $(\omega \cdot i + \omega)^{th}$ element of $C$. As $C \in V$, $\langle c_i : i \in \omega \rangle \in C$. Let $z \in \mathbb{R}^{V[G]}$. Let $f_z = \{c_i : i \in z\}$.

$f_z \in ([C]^{\omega})^{V[G]}$ of correct type. Hence $\Phi(f_z) = 0$. So $f_z \in V$.

$z = \{ i \in \omega : c_i \in f_z \}$. So $z \in \mathbb{R}^{V}$. $\mathbb{R}^{V[G]} = \mathbb{R}^{V}$. ◻

<u>Fact 2</u> (ZF) Let $\mathbb{P}$ be well orderable forcing of cardinality $\lambda$. Then $\mathbb{P}$ has ground club property at $\kappa$ for all regular $\kappa > \lambda$.

<u>Pf</u>: Let $1_{\mathbb{P}} \Vdash \dot{D} \subseteq \check{\kappa}$ club. Let $A_\alpha = \{ p \in \mathbb{P} : \exists \beta\ p \Vdash \dot{D}(\check{\alpha}) = \beta \}$.

Let $B_\alpha = \{ \beta \in \kappa : \exists p \in A_\alpha\ \ p \Vdash \dot{D}(\check{\alpha}) = \check{\beta} \}$.

$|B_\alpha| \leq |\mathbb{P}| = \lambda < \kappa$ and $\kappa$ regular so $\sup B_\alpha < \kappa$. Let $F(\alpha) = \sup B_\alpha$.

Let $C = \{ \alpha < \kappa : (\forall \eta < \alpha)(F(\eta) < \alpha) \}$. $C$ is club.

Let $G \subseteq \mathbb{P}$ be generic. Let $D = \dot{D}[G]$. Fix $\alpha \in C$. For each $\eta < \alpha$, $G \cap A_\eta \neq \emptyset$. If $q \in G \cap A_\eta$, then $q \Vdash \eta \leq \dot{D}(\check{\eta}) \leq F(\eta) < \alpha$.

Since $D$ is club, $\alpha \in D$. So $C \subseteq D$. ◻

**Fact 3** (ZF + AD) If $\mathbb{P}$ is a nontrivial well orderable forcing of size less than $\Theta$, then $1_{\mathbb{P}} \Vdash \neg AD$.

**Proof:** $|\mathbb{P}| = \delta < \Theta$. Assume $\mathbb{P} \subseteq \delta$. Let $G \subseteq \mathbb{P}$ be generic and $V[G] \models AD$.

Find $\kappa$ with $\delta < \kappa < \Theta^{V[G]}$ so that $V[G] \models \kappa \to (\kappa)^{\omega}_{2}$. $\kappa$ is regular.

Fact 2 implies $\mathbb{P}$ has ground club at $\kappa$. Main lemma 1 implies $\mathbb{R}^V = \mathbb{R}^{V[G]}$.

Since $\mathbb{P}$ is nontrivial, $G$ is new subset of $\delta$. In $V$, let $\pi : \mathbb{R} \to \delta$ be a surjection. As $V[G] \models AD$, the coding lemma states there is some $e \in \mathbb{R}$ such that $G = S^{\pi}_{e}$. If $\mathbb{R}^V = \mathbb{R}^{V[G]}$, then $G = S^{\pi}_{e} \in V$. Contradiction ▨

**Fact 4**: (ZF) Let $\mathbb{P}$ be forcing which is surjective image of $\mathbb{R}$. For each regular $\kappa \geq \Theta$, $\mathbb{P}$ has ground club property at $\kappa$.

**Pf:** Let $\pi : \mathbb{R} \to \mathbb{P}$ be a surjection. $1_{\mathbb{P}} \Vdash \dot{D} \subseteq \kappa$ club.

Let $A_\alpha = \{ p \in \mathbb{P} : \exists \beta \ p \Vdash \dot{D}(\check{\alpha}) = \beta \}$. $B_\alpha = \{ \beta < \kappa : \exists p \in A_\alpha \ p \Vdash \dot{D}(\check{\alpha}) = \check{\beta} \}$.

In $V$, define $\Phi : \mathbb{R} \to \kappa$ by

$$\Phi(r) = \begin{cases} \beta & \text{if } \pi(r) \in A_\alpha \wedge \pi(r) \Vdash \dot{D}(\check{\alpha}) = \check{\beta} \\ 0 & \text{otherwise.} \end{cases}$$

Let $r \preceq s \iff \Phi(r) \leq \Phi(s)$. $\preceq$ is prewellordering on $\mathbb{R}$. It has some length $\delta < \Theta \leq \kappa$. Since $\kappa$ is regular, this implies $\Phi$ is bounded below $\kappa$. So $\sup B_\alpha < \kappa$. Let $F(\alpha) = \sup B_\alpha$.

Now finish the proof as in Fact 2 ▨

**Fact 5**: (ZF + AD) Let $\mathbb{P}$ be nontrivial forcing which is a surjective image of $\mathbb{R}$ and $1_{\mathbb{P}} \Vdash AD$. Then $1_{\mathbb{P}} \Vdash \check{\mathbb{R}} \subsetneq \dot{\mathbb{R}}$.

**Pf:** Let $\pi : \mathbb{R} \to \mathbb{P}$ be a surjection. Suppose $p \Vdash \dot{\mathbb{R}} = \check{\mathbb{R}}$. Then for all $A \in \mathcal{P}(\mathbb{R})^V$, $p \Vdash A \leq_L \pi^{-1}[G]$, since $G$ is a new set.

In $V$, define $\Phi : \mathbb{R} \times \mathbb{R} \to \Theta$ by

$$\Phi(r,s) = \begin{cases} \alpha & \text{if } \pi(r) \Vdash \Xi_s^{-1}[\pi^{-1}[G]] \in V \text{ and is a p.w.o. of length } \alpha \\ 0 & \text{otherwise.} \end{cases}$$

($\Xi_s$ is the Lipschitz reduction coded by $r$.) $\Phi$ is a surjection of $\mathbb{R} \times \mathbb{R}$ onto $\Theta$. This is impossible. ▨

**Fact 6** (ZF + AD) Let $\mathbb{P}$ be a nontrivial forcing which is a surjective image of $\mathbb{R}$.
~~B~~ $1_{\mathbb{P}} \Vdash AD$ implies $1_{\mathbb{P}} \Vdash \dot{\Theta} = \check{\Theta}$.

**Proof**: Suppose not. Let $G \subseteq \mathbb{P}$ be $\mathbb{P}$-generic so that $V[G] \models AD$ and $\Theta^V < \Theta^{V[G]}$.
By AD, let $\kappa$ be such that $\Theta^V < \kappa < \Theta^{V[G]}$ ~~but~~ and $\kappa \to (\kappa)^{\omega}_2$. $\kappa$ has the ground club property by Fact 4. Main Lemma 1 states $\mathbb{R}^V = \mathbb{R}^{V[G]}$. This contradicts Fact 5. $\square$

**Fact 7** (ZF + AD) Let $A \subseteq \mathbb{R}$ ~~and~~ such that $V = L(A, \mathbb{R})$. Let $\mathbb{P}$ be a forcing on $\mathbb{R}$, i.e. $\mathbb{P} \subseteq \mathbb{R}$, with $\mathbb{P} \leq_L A$ and $1_{\mathbb{P}} \Vdash AD$.
Then $1_{\mathbb{P}} \Vdash \dot{V} = L(A \oplus \check{\mathbb{R}}, \dot{\mathbb{R}})$. (Here $A \oplus B$ for $A, B \subseteq \mathbb{R}$ refer to some simple coding of two sets of reals into one.)

**Proof**: Let $G \subseteq \mathbb{P}$ be generic. Denote $\mathbb{R} = \dot{\mathbb{R}}^V$ and $\mathbb{R}^* = \dot{\mathbb{R}}^{~~V[G]~~ L(A,\mathbb{R})[G]}$.
$\mathbb{P}, \mathbb{R} \in L(A \oplus \mathbb{R}, \mathbb{R}^*)$. $L(A,\mathbb{R})$ is definable inner model of $L(A \oplus \mathbb{R}, \mathbb{R}^*)$.
So $\Theta^{L(A,\mathbb{R})} \leq \Theta^{L(A \oplus \mathbb{R}, \mathbb{R}^*)}$. Since $\mathbb{P}, \mathbb{R} \in L(A \oplus \mathbb{R}, \mathbb{R}^*)$, $L(A,\mathbb{R})[G] \neq L(A \oplus \mathbb{R}, \mathbb{R}^*)$ implies that $G \notin L(A \oplus \mathbb{R}, \mathbb{R}^*)$. As $L(A,\mathbb{R})[G]$ and $L(A \oplus \mathbb{R}, \mathbb{R}^*)$ have the same reals all sets in $\mathcal{P}(\mathbb{R})^{L(A \oplus \mathbb{R}, \mathbb{R}^*)}$ is Lipschitz below $G$. In $L(A,\mathbb{R})[G]$ define $\Phi : \mathbb{R}^* \longrightarrow \Theta^{L(A \oplus \mathbb{R}, \mathbb{R}^*)}$ by

$$\Phi(r) = \begin{cases} \text{length}(\Xi_r^{-1}[G]) & \text{if } \Xi_r^{-1}[G] \in L(A \oplus \mathbb{R}, \mathbb{R}^*) \text{ is pwo} \\ 0 & \text{otherwise} \end{cases}$$

$\Phi$ is surjective so $\Theta^{L(A \oplus \mathbb{R}, \mathbb{R}^*)} < \Theta^{L(A,\mathbb{R})[G]}$.
Hence $\Theta^{L(A,\mathbb{R})} \leq \Theta^{L(A \oplus \mathbb{R}, \mathbb{R}^*)} < \Theta^{L(A,\mathbb{R})[G]}$. This contradicts Fact 6. $\square$

**Fact 8** (ZF + AD + $\Theta$ regular) $\mathbb{P}$ is a forcing which is surjective image of $\mathbb{R}$ so that $1_{\mathbb{P}} \Vdash AD$. Then $1_{\mathbb{P}} \Vdash \Theta$ regular.

**Proof**: Let $\pi : \mathbb{R} \to \mathbb{P}$ surjective. $G \subseteq \mathbb{P}$ generic. Fact 6 states that $\Theta^V = \Theta^{V[G]}$.
Suppose $\Theta$ is not regular in $V[G]$. There is some $\eta < \Theta$ and function $f : \eta \longrightarrow \Theta$ which is cofinal. Let $\dot{f}$ be a name for $f$. In $V$, let $g : \eta \times \mathbb{R} \to \Theta$ by

$$g(\alpha, r) = \begin{cases} \beta & \text{if } \pi(r) \Vdash \dot{f}(\alpha) = \beta \\ 0 & \text{otherwise} \end{cases}$$

Let $\phi : \mathbb{R} \to \eta$ be a surjection in $V$. Let $h : \mathbb{R} \times \mathbb{R} \to \Theta$ be defined by $h(x_0, x_1) = g(\phi(x_0), x_1)$. $h$ induces p.w.o on $\mathbb{R}$ so it has length $\delta$ less than $\Theta$. Then $h$ induces a cofinal map of $\delta$ into $\Theta$ in $V$. But $\Theta$ is regular in $V$. Contradiction. $\square$

Every forcing which is a surjective image of $\mathbb{R}$ is equivalent to a forcing which is on $\mathbb{R}$. In the following, we will assume the forcing is on $\mathbb{R}$.

So $\mathbb{P} \subseteq \mathbb{R}$. A $\mathbb{P}$-name for a real consisting of $(\check{n}, p)$ $n \in \omega$ and $p \in \mathbb{R}$ can be regarded as a set of reals. In the following, a name is a set reals implies that it is of the form above.

<u>Definition</u>: Let $\mathbb{P} \subseteq \mathbb{R}$ be a forcing. $\mathbb{P}$ has the name condition iff there is an $A \subseteq \mathbb{R}$ so that $\mathbb{P} \leq_L A$ and $1_{\mathbb{P}} \Vdash$ "for all $r \in \dot{\mathbb{R}}$, there is $\sigma \in \mathcal{P}(\mathbb{R})^{L(\check{A}, \mathbb{R})}$ such that $\sigma[\dot{G}] = r$ and $L(\check{A}, \mathbb{R}) \models \sigma \leq_L \check{A}$". (The use of Lipschitz reduction here is important.)

<u>Fact 9</u>: $(ZF + AD)$ $A \subseteq \mathbb{R}$. Let $C_A$ be the set of $\alpha < \Theta$ so that there is some p.w.o on $\mathbb{R}$ $\sqsubseteq$ of length $\alpha$ so that $\sqsubseteq \leq_L A$. Then $C_A$ is bounded below $\Theta$.

<u>Proof</u>: The argument is similar to many earlier arguments.

<u>Fact 10</u>: $(ZF + AD + \Theta$ regular$)$ Let $\mathbb{P} \subseteq \mathbb{R}$ be a forcing. $1_{\mathbb{P}} \Vdash AD$ implies that $\mathbb{P}$ has the name condition.

<u>Proof</u>: Let $p \in \mathbb{P}$. $G \subseteq \mathbb{P}$ with $p \in \mathbb{P}$ be a generic. Fact 6 and 8 imply $\Theta^V = \Theta^{V[G]}$ and $V[G] \models \Theta$ is regular. If $r \in \mathbb{R}^{V[G]}$, let $\tau$ be a name for it. Let $\sigma = \{(\check{n}, p) : p \Vdash \check{n} \in \tau\}$. $\sigma[G] = r$ and $\sigma \subseteq \mathbb{R}$ (as in the above remark.)

In $V[G]$, define $\Phi : \mathbb{R}^{V[G]} \longrightarrow \Theta$ by

$$\Phi(r) = \min\{\sup(C_\sigma)^V + 1 : \sigma \in \mathcal{P}(\mathbb{R})^V \wedge \sigma[G] = r\}$$

where $C_\sigma$ is the set from Fact 9. $\Phi$ induce p.w.o on $\mathbb{R}^{V[G]}$ so it has length $\gamma$ below $\Theta^{V[G]} = \Theta^V$.

Fix some p.w.o on $\mathbb{R}$, $\preceq^*$, of length $\gamma$ in $V$.

Let $r \in \mathbb{R}$. Let $\sigma \subseteq \mathbb{R}$ be so that $\sigma[G] = r$ and $\sup(C_\sigma)^V + 1 = \Phi(r)$.

Since $\gamma > \Phi(r) = \sup(C_\sigma)^V + 1$, $\neg(\preceq^* \leq_L \sigma)^V$. By Wadge's lemma, $(\sigma \leq_L \preceq^*)^V$. So it has been shown that for all $r \in \mathbb{R}^{V[G]}$, there is some $\mathbb{P}$-name $\sigma \subseteq \mathbb{R}$ so that $V \models \sigma \leq_L \preceq^*$. Some $q \in \mathbb{R}$, this statement about $\gamma$.

Define $\Psi : \mathbb{R} \longrightarrow \Theta$ be such that $\Psi(q)$ is least $\gamma$ such that $q$ forces the above statement about $\gamma$; (and $0$ ~~other~~ is no $\gamma$ exists).

$\Psi$ induces p.w.o on $\mathbb{R}$ and so has length less than $\Theta$. The regularity of $\Theta$ implies $\Psi$ is bounded by some $\eta$. Let $\preceq^*$ be p.w.o of length $\eta$.

$\preceq^*$ witnesses the name condition. $\square$

<u>Fact 11</u>: (ZF + AD) Let $\mathbb{P} \subseteq \mathbb{R}$ st $1_{\mathbb{P}} \Vdash AD$ and has the name condition as witnessed by some $A \subseteq \mathbb{R}$. Then in $L(A, \mathbb{R})$, $1_{\mathbb{P}} \Vdash AD$, $\delta_A$ has the ground club property, and $1_{\mathbb{P}} \Vdash \check{\delta}_A \longrightarrow (\check{\delta}_A)^{\omega}_2$.

<u>Proof</u>: <u>Notation</u>: whenever $G \subseteq \mathbb{P}$ is a generic, $\mathbb{R}$ denotes $\dot{\mathbb{R}}^V$ and $\mathbb{R}^*$ denotes $\dot{\mathbb{R}}^{V[G]}$.

Let $p \in \mathbb{P}$ and $G \ni p$ generic over $V$ with $p \in G$. By the name condition $\dot{\mathbb{R}}^{L(A,\mathbb{R})[G]} = \dot{\mathbb{R}}^{V[G]}$. Since $V[G] \models AD$, $L(A,\mathbb{R})[G] \models AD$. Pick $q \leq p$ $q \in G$ so that $L(A,\mathbb{R}) \models q \Vdash AD$. By density, $L(A,\mathbb{R}) \models 1_{\mathbb{P}} \Vdash AD$.

By Fact 7, $L(A,\mathbb{R})[G] = L(A \oplus \mathbb{R}, \mathbb{R}^*)$. Let $p \in \mathbb{P}$ and $p \in G \subseteq \mathbb{P}$ be a generic.

<u>Claim 1</u>: $\delta_A = \left(\dot{\delta}_{A \oplus \mathbb{R}}\right)^{L(A \oplus \mathbb{R}, \mathbb{R}^*)}$.

<u>Pf</u>: Let $r \in \mathbb{R}^*$. Let $\tau \in \mathbb{R}$ be a $\mathbb{P}$-name so that $\tau[G] = r$ and $\tau \leq_L A$.

Let $\varphi(\dot{v}, A \oplus \mathbb{R}, \dot{\mathbb{R}})$ be $\Sigma_1$ formula. Suppose $L(A \oplus \mathbb{R}, \mathbb{R}^*) \models \varphi(r, A \oplus \mathbb{R}, \mathbb{R}^*)$.

Since $L(A \oplus \mathbb{R}, \mathbb{R}^*) = L(A,\mathbb{R})[G]$, there is some $q_0 \leq p$ $q_0 \in G$ so that

$$L(A,\mathbb{R}) \models q_0 \Vdash_{\mathbb{P}} L(\check{A} \oplus \check{\mathbb{R}}, \dot{\mathbb{R}}) \models \varphi(\tau, \check{A} \oplus \check{\mathbb{R}}, \dot{\mathbb{R}}).$$

$$L(A,\mathbb{R}) \models \exists \alpha.\ L_\alpha(A,\mathbb{R}) \models q_0 \Vdash L(\check{A} \oplus \check{\mathbb{R}}, \dot{\mathbb{R}}) \models \varphi(\tau, \check{A} \oplus \check{\mathbb{R}}, \dot{\mathbb{R}})$$

This a $\Sigma_1(L(A,\mathbb{R}), \mathbb{R} \cup \{\mathbb{R}, A\})$ statement. (Note that $\tau$ in the above is an abreviation for some statement involving $\mathbb{R}$ and $A$ since $\tau \leq_L A$.)

There is $\alpha < \delta_A$ so that

$$L_\alpha(A,\mathbb{R}) \models q_0 \Vdash L(\check{A} \oplus \check{\mathbb{R}}, \dot{\mathbb{R}}) \models \varphi(\tau, \check{A} \oplus \check{\mathbb{R}}, \dot{\mathbb{R}}).$$

Since $q_0 \in G$

$$L_\alpha(A,\mathbb{R})[G] \models L(A \oplus \mathbb{R}, \dot{\mathbb{R}}^{L_\alpha(A,\mathbb{R})[G]}) \models \varphi(r, A \oplus \mathbb{R}, \dot{\mathbb{R}}^{L_\alpha(A,\mathbb{R})[G]})$$

By the name condition, $\dot{\mathbb{R}}^{L_\alpha(A,\mathbb{R})[G]} = \mathbb{R}^*$ so

$$L_\alpha(A \oplus \mathbb{R}, \mathbb{R}^*) \models \varphi(r, A \oplus \mathbb{R}, \mathbb{R}^*).$$

By upward absoluteness of $\Sigma_1$ formulas

$$L_{\delta_A}(A \oplus \mathbb{R}, \mathbb{R}^*) \models \varphi(r, A \oplus \mathbb{R}, \mathbb{R}^*).$$

So $\left(\dot{\delta}_{A \oplus \mathbb{R}}\right)^{L(A \oplus \mathbb{R}, \mathbb{R}^*)} \leq \delta_A$.

Now suppose $r \in \mathbb{R}$ and $L(A,\mathbb{R}) \models \varphi(r, A, \mathbb{R})$ where $\varphi$ is $\Sigma_1$.

Note $L(A \oplus \mathbb{R}, \mathbb{R}^*) \models \mathbb{R} \leq_L A \oplus \mathbb{R}$. So

$$L(A \oplus \mathbb{R}, \mathbb{R}^*) \models L(A,\mathbb{R}) \models \varphi(r, A, \mathbb{R}).$$

$$L(A \oplus \mathbb{R}, \mathbb{R}^*) \models \exists \alpha.\ L_\alpha(A,\mathbb{R}) \models \varphi(r, A, \mathbb{R}).$$

Again this is a $\Sigma_1(L(A\oplus\mathbb{R}, \mathbb{R}^*, \mathbb{R}^* \cup \{A, \mathbb{R}^*\}))$ statement. There is some $\alpha < (\dot{\delta}_{A\oplus\mathbb{R}})^{L(A\oplus\mathbb{R},\mathbb{R}^*)}$ so that  (8)

$$L_\alpha(A,\mathbb{R}) \models \varphi(r, A, \mathbb{R}).$$

By upward absoluteness

$$L_{(\delta_{A\oplus\mathbb{R}})^{L(A\oplus\mathbb{R},\mathbb{R}^*)}}(A,\mathbb{R}) \models \varphi(r,A,\mathbb{R}).$$

So $\delta_A \leq (\dot{\delta}_{A\oplus\mathbb{R}})^{L(A\oplus\mathbb{R},\mathbb{R}^*)}$. Claim 1 has been shown.

So there is some $q \leq p$ st $L(A,\mathbb{R}) \models q \Vdash \check{\delta}_A = \dot{\delta}_{\check{A}\oplus\check{\mathbb{R}}}$. Again by a density argument, $L(A,\mathbb{R}) \models 1_{\mathbb{P}} \Vdash \check{\delta}_A = \dot{\delta}_{\check{A}\oplus\check{\mathbb{R}}}$.

By the Kechris, Kleinberg, Moschovakis, Woodin result above,

if $1_{\mathbb{P}} \Vdash AD$, then $1_{\mathbb{P}} \Vdash \check{\delta}_A \longrightarrow (\check{\delta}_A)^{\delta_A}_2$.

<u>Claim 2</u> In $L(A,\mathbb{R})$, $\delta_A$ has ground club ~~below~~ property.

<u>Pf.</u> $p \in \mathbb{P}$. $G \subseteq \mathbb{P}$ $\mathbb{P}$-gen over $V$ with $p \in G$.

Let $D \in L(A,\mathbb{R})[G]$ be a club subset of $\delta_A$. Consider $D : \delta_A \to \delta_A$ as its own increasing enumeration and code the graph as a subset of $\delta_A \times \delta_A$.

Recall $\rho_A$ and $\Xi$ the formula $\Xi$ from earlier.

Since $L(A,\mathbb{R})[G] \models AD$, there is some $e \in \mathbb{R}^*$ so that $D = S_e^{\rho_A}$.

By the name condition, some $\dot{e} \in \mathbb{R}$, $\dot{e} \leq_L A$ in $L(A,\mathbb{R})$ with $\dot{e}[G] = e$.

Let $q_0 \leq p$ st $q_0 \Vdash$ "$S_{\dot{e}}^{\rho_A}$ is club". For each $\beta$

$$L(A,\mathbb{R}) \models \exists\alpha\; L_\alpha(A,\mathbb{R}) \models \forall k \leq_{\mathbb{P}} q_0\; \exists j \leq_{\mathbb{P}} k\; \exists \gamma\; j \Vdash \Xi(\langle\check{\beta},\gamma\rangle, \dot{e}, \check{A}, \dot{\mathbb{R}})$$

So there is $\alpha < \delta_A$ so that $\downarrow$

$$L_\alpha(A,\mathbb{R}) \models \sim\!\sim\!\sim$$

Let $\varepsilon_\beta$ be the least $\alpha$ with this property. Thus for all $H \subseteq \mathbb{P}$ generic with $q_0 \in H$, $S_{\dot{e}}^{\rho_A}[H](\beta) < \varepsilon_\beta$. In particular, $D(\beta) < \varepsilon_\beta$.

Let $F : \delta_A \to \delta_A$ be defined by $F(\beta) = \varepsilon_\beta$. Let $C = \{\alpha < \delta_A : (\forall \eta < \alpha)\; F(\eta) < \alpha\}$. As in Fact 2, $C$ is club and $L(A,\mathbb{R})[G] \models C \subseteq D$. This completes the proof. $\blacksquare$

Thm (ZF + AD + $\Theta$ is regular) $\mathbb{P}$ is nontrivial forcing which is a surjective image of $\mathbb{R}$. Then $1_{\mathbb{P}} \Vdash \neg AD$.

Proof: Suppose not. ~~$G \subseteq \mathbb{P}$ generic with $V[G] \models AD$.~~ Fact 10 states $\mathbb{P}$ has name condition as witnessed by some $A \subseteq \mathbb{R}$. ~~Apply~~ Now work in $L(A,\mathbb{R})$. Let $G \subseteq \mathbb{P}$ be $\mathbb{P}$-generic. Main Lemma 1 and Fact 11 imply

$$\mathbb{R}^{L(A,\mathbb{R})} = \mathbb{R}^{L(A,\mathbb{R})[G]}.$$

However Fact 5 states that

$$\mathbb{R}^{L(A,\mathbb{R})} \subsetneq \mathbb{R}^{L(A,\mathbb{R})[G]}.$$

Contradiction. $\boxed{///}$

~~If $L(\mathbb{R}) \models AD$, then $\Theta$ is regular in~~ $\Theta$ is regular in $L(\mathbb{R})$. If $ZF + AD^+ + \neg AD_{\mathbb{R}} + V = L(\mathcal{P}(\mathbb{R}))$ holds then Woodin has shown that there is some $J \subseteq On$ so that $V = L(J,\mathbb{R})$. $\Theta$ is regular in $L(J,\mathbb{R})$.

Corollary: If $L(\mathbb{R}) \models AD$, then for any nontrivial forcing $\mathbb{P} \in L_{\Theta}(\mathbb{R})$, $L(\mathbb{R}) \models 1_{\mathbb{P}} \Vdash \neg AD$.

$ZF + AD^+ + \neg AD_{\mathbb{R}} + V = L(\mathcal{P}(\mathbb{R})) \vdash$ "if $\mathbb{P}$ is nontrivial forcing which is surjective image of $\mathbb{R}$", then $1_{\mathbb{P}} \Vdash \neg AD$".