steve jackson : det. + the Suslin hypothesis

q (foreman) does $AD + V = L(\mathbb{R}) \Rightarrow SH$?

thm. (with w. chan)

$$ZF + AD^+ + V = L(P(\mathbb{R})) \Rightarrow \cancel{SH} \; SH.$$

a tree is a set $(T, <)$ s.t.

$\forall t \in T \; \{s < t\}$ is well-ordered

an ω_1-tree is a tree s.t. $\forall t \in T \; |t|_T < \omega_1$

+ f.a. $\alpha < \omega_1$ $\{t \in T : |t|_T = \alpha\}$ ctble.

an aronszajn tree is a ω_1-tree with no ω_1-branch.

a suslin tree is an aronszajn tree with no unctble. antichain.

a suslin line is a lin order $(X, <)$ which is c.c.c. but not separable.

[w.l.o.g. may assume $(X,<)$ is complete, dense in itself + no endpoints]

fact (ZFC) $\exists$ suslin tree $\leftrightarrow$ $\exists$ suslin line

<u>fact</u> (2F)

if there is a Suslin tree on a well-ordered set, then there is a Suslin line.

[use lex order on maximal branches]

<u>fact</u> (2F) if $(X, <)$ is a Suslin line with a well-ordered <u>separating family</u>, then there is a Suslin tree.

↑

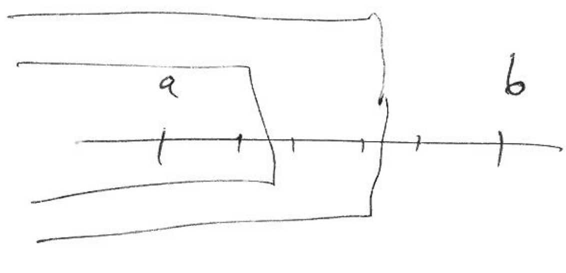
a family $\mathcal{A}$ of ~~sets~~ cuts (A, A') s.t. if $a < b$, then $\exists A \in \mathcal{A}$ $a \in A$, $b \notin A$.

[by ind. on $\alpha < \omega_1$, non empty intervals (a_α, b_α) : let $\{A_\alpha\}$ = a w.o. separating family

(a_0, b_0) arb.

let $E_\alpha = \bigcup_{\beta < \alpha} \{a_\beta, b_\beta\}$

as $(X, <)$ is not separable, $\exists (a, b) \neq \emptyset$ with $(a, b) \cap E_\alpha = \emptyset$



$\exists \eta_1, \eta_2$ s.t.

$A_{\eta_2} \setminus A_{\eta_1} \subset (a', b') \subset (a, b)$

$T = \{(a_\alpha, b_\alpha)\}$ $(a_\alpha, b_\alpha) \trianglelefteq (a_\beta, b_\beta)$ iff $(a_\alpha, b_\alpha) \subset (a_\beta, b_\beta)$.

th. (ZF + AD^+ + $V = L(\mathcal{P}(\mathbb{R}))$) there is no aronszajn tree.

pf. 1st, there is no aronszajn tree on a well-ordered set. supp. $(T, <)$ is such an aronszajn tree. since ω_1 is measurable, we easily get a branch thru T

(define $\{s_\alpha : \alpha < \omega_1\}$ by ind. so that for each α, $\forall^* \beta < \omega_1$ $s_\alpha < \beta$.)

for general case (T not well-ordered)

th. (same hyp.) (caicedo-ketchersid)

for any X, either X can be w.o. or $\mathbb{R}$ inject into ~~$\mathbb{R}$~~ X.

latter case: $\mathbb{R} = \bigcup_{i < \omega_1} A_i$, $\overline{\overline{A_i}} = \aleph_0$.

rmk: will use arguments from woodin's AD^+ thy. also from harrington-marker-shelah paper.

$(P, \leq)$ prelinear order: $\leq$ is conn., transitive. (i.e., lin order on its equivalence classes) main lemma. (ZF + AD^+)

let $\leq$ be a prelin. order on $\mathbb{R}$. one of the following holds.

(1) $\exists$ perfect set of disjoint closed intervals in $\leq$ $\left(\{ [a_x, b_x] : x \in 2^\omega \} \right)$

(2) there is ~~no~~ a well-ordered separating family.

pf.: fix an ∞-borel code $(\underset{\text{OR}}{S}, \underset{\text{fla}}{\varphi})$

$$z \in \leq \iff L[S, z] \models \varphi(S, z).$$

for all $M \models ZF$, $S \in M$, $\mathbb{O}^M_S$ = all OD for S in M subsets of $\mathbb{R}$. view: $\mathbb{O}^M_S \in \mathrm{HOD}^M_S$

τ = can. name for a real added.

- for every $x \in M$, the corr. filter $G_x \subset \mathbb{O}^M_S$ is gen. over ~~M~~ HOD^M_S, $\tau[G_x] = x$.

- Supp. $N \supset \mathrm{HOD}^M_S$, $N \models ZF$, $N \cap OR = M \cap OR$.
Supp. $p \in \mathbb{O}^M_S$ defined as

$$p = \{ z \in \mathbb{R}^M : L[\underset{\mathrm{OD}^M_S \cap \mathcal{P}(OR)}{K}, z] \models \psi(K, z) \}.$$

then $N \models p \Vdash_{\mathbb{O}^M_S} (L[K,\tau] \models \psi(K,\tau))$.

pf.: otherwise

$\exists q' \leq p \; N \models q' \Vdash_{\mathbb{O}^M_S} L[K,\tau] \not\models \psi(K,\tau)$.

by vopenka + forcing thm, $\exists q \leq q'$

$HOD^M_S \models q \Vdash_{\mathbb{O}^M_S} L[K,\tau] \models \psi(K,\tau)$

this gives a contradiction.

<u>notation.</u> $M \models ZF$, $S \in M$.

let ${}_n\mathbb{O}^M_S$ = "n dim" version of vopenka,

i.e., all OD^M_S subsets of $\mathbb{R}^n$.

<u>fact.</u> if M, S as above, if $N \supset HOD^M_S$,

then if $(g_0, -, g_{n-1})$ is gen. / N

for ${}_n\mathbb{O}^M_S$, then each g_i is gen. / N

for ${}_1\mathbb{O}^M_S$.

pf. of main lemma: recall $\leq$ is a prelinear

order on $\mathbb{R}$ with ∞-borel code S.

case 1. for all $x \in \mathbb{R}$,

f.a. $a, b \in L[S, x]$ with $a < b$
there is a set $A \in \mathbb{U}_S^{L[S,x]}$ with
$a \in A$, $b \notin A$.

$\mathbb{U}_S^{L[S,x]}$ = downward closure (by $<$)
of sets in $\mathbb{D}_S^M$.

let μ = martin measure on degrees $\mathcal{D}$.
for $f : \mathcal{D} \longrightarrow \omega_1$
let $A_{[f]} = \{ y \in \mathbb{R} : \forall^* x \in \mathcal{D} \;\; y \in f(x)^{\text{th}}$ set
in $\mathbb{U}_S^{L[S,x]} \}$.

the set $\{ A_{[f]} \}$ is a separating family for $\leq$.

th. (woodin) (from current hypo's)
the ultrapower by μ is well-founded.

So have a w.o. sep. family.

case 2. there is some $x \in \mathbb{R}$ s.t.
$\exists\, a, b \in L[S, x]$, $a < b$, but there is no
$A \in \mathbb{U}_S^{L[S,x]}$ with $a \in A$, $b \notin A$.

let's fix this x.

let $u = \{ [c_0, c_1] \in (\mathbb{R}^2)^{L[S,x]} :$

$c_0 < c_1 \wedge \neg \exists A \in \mathcal{U}_S^{L[S,x]}$

with $c_0 \in A,\ c_1 \notin A \}$

so $u \in {}_2\mathbb{O}_S^{L[S,x]}$.

we're not leaving $L[S,x]$ for now.

<u>claim.</u> let $M \models ZF$, $S \in M$, $M \supset HOD_S^{L[S,x]}$.

let $r \in M$. then

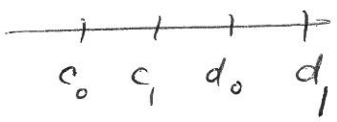
$$M \models u \Vdash_{{}_2\mathbb{O}_S^{L[S,x]}} \neg(\tau_{left} < \check{r} < \tau_{right}).$$

pf. of claim: supp. it fails.

so $\exists v \leq u$ $M \models v \Vdash \tau_{left} < \check{r} < \tau_{right}$.

• <u>first</u> supp. have $(c_0, c_1), (d_0, d_1) \in v$

with $c_1 < d_0$



consider ${}_4\mathbb{O}_S^{L[S,x]}$:

let $w = \{ (e_0, e_1, e_2, e_3) : (e_0, e_1) \in v,\ (e_2, e_3) \in v,$
$e_1 < e_2 \} \neq \emptyset$.

$w \in {}_4\mathcal{D}_S^{L[S,x]}$.

let (g_0, g_1, g_2, g_3) be ${}_4\mathcal{D}_S^{L[S,x]}$ gen. / M.

then by the Vopenka lea,

$g_0 < g_1$ and $g_2 < g_3$, and $g_1 < g_2$.

so $g_0 < g_1 < g_2 < g_4$.

but would have to have $r \in (g_0, g_1)$
$r \in (g_2, g_3)$ $\lightning$

• second assume f.a. $(c_0, c_1), (d_0, d_1) \in V$,

$\neg c_1 < d_0$.

so all left endpoints are $<$ all right endpoints.

let $A = \{x : \exists (e_0, e_1) \in V, \ x \leq e_0\}$

$\in \mathcal{U}_S^{L[S,x]}$.

take any $(e_0, e_1) \in V$.

by df $e_0 \in A$.

but $e_1 \notin A$ by case hypo.

contradiction!

$\dashv$ (claim)

from the claim, if (a,b), (c,d) are mut. gen. for $\mathbb{P}\ {}_2\mathbb{B}_S^{L[S,x]} \times {}_2\mathbb{B}^{L[S,x]}$,

then $[a,b] \cap [c,d] = \emptyset$.

in V, get a perfect set $\{(a_x, b_x) : x \in {}^\omega 2\}$ of mutual generic over $L[S,x]$ for ${}_2\mathbb{B}_S^{L[S,x]}$.

gives a perfect set of disjoint closed intervals.

<u>thm.</u> let $J \subset OR$, if $L(J, \mathbb{R}) \models AD$, then $L(J, \mathbb{R}) \models AD$.

Pf.: $L(J, \mathbb{R}) \models AD^+$.

let $(P, <)$ be a Suslin line.

$P \in OD_{J,z}^{L(J,\mathbb{R})}$, $z \in {}^\omega\omega$.

fix def. the $\varphi : OR \times \mathbb{R} \longrightarrow L(J, \mathbb{R})$ onto.

let $Q_\alpha = \{x \in \mathbb{R} : \varphi(\alpha, x) \in p\}$.

$$Q = \bigcup_{\alpha \in OR} Q_\alpha .$$

by a thm. of Woodin, there is an ∞-borel code S_α for Q_α in $HOD_{J,z}^{L[S,\mathbb{R}]}$.

if any of the Q_α has a perfect set of disjoint intervals, then so does P, a contradiction.

we have a uniformly def.ble sep. families S_α for Q_α.

$S = \bigcup_\alpha S_\alpha$ is a w.o. sep. family for P.

gives a Suslin tree, T. $\lightning$ $\dashv$