Strategic Ramsey cardinals

DAN SAATTRUP NIELSEN AND RALF SCHINDLER

ABSTRACT. We show that strategic ω-Ramsey cardinals are downwards absolute to L, are equiconsistent with virtually measurable cardinals, and are limits of ω-Ramseys in L. Further, we present Schindler's argument showing that strategic ω_1-Ramsey cardinals are measurable in the core model K below a Woodin cardinal, improving a result of Philip Welch.

1 Setting the scene

In this section we'll recall the relevant definitions we'll need.

DEFINITION 1.1. For a cardinal κ, a **weak κ-model** is a set $\mathcal{M}$ of size κ satisfying that $\kappa+1\subseteq\mathcal{M}$ and $(\mathcal{M},\in)\models\mathsf{ZFC}^-$. If furthermore $\mathcal{M}^{<\kappa}\subseteq\mathcal{M}$, $\mathcal{M}$ is a **κ-model**.[1] $\dashv$

Recall that μ is an **$\mathcal{M}$-measure** if $(\mathcal{M},\in,\mu)\models\ulcorner\mu$ is a κ-complete ultrafilter on $\kappa\urcorner$.

DEFINITION 1.2. Let $\mathcal{M}$ be a weak κ-model and μ an $\mathcal{M}$-measure. Then μ is

- **weakly amenable** if $x\cap\mu\in\mathcal{M}$ for every $x\in\mathcal{M}$ with $\mathcal{M}$-cardinality κ;
- **countably complete** if $\bigcap\vec{X}\neq\varnothing$ for every ω-sequence $\vec{X}\in{}^\omega\mu$;
- **$\mathcal{M}$-normal** if $(\mathcal{M},\in,\mu)\models\forall\vec{X}\in{}^\kappa\mu:\triangle\vec{X}\in\mu$;
- **normal** if $\triangle\vec{X}$ is stationary in κ for every κ-sequence $\vec{X}\in{}^\kappa\mu$;
- **good** if it has a well-founded ultrapower.

$\dashv$

Note that a normal $\mathcal{M}$-measure is $\mathcal{M}$-normal, countably complete and good.

[1] Note that our (weak) κ-models do not have to be transitive, in contrast to the models considered in Gitman (2011) and Gitman and Welch (2011). Not requiring the models to be transitive was introduced in Holy and Schlicht (2018).

Definition 1.3 (Holy, Schlicht, N.). Let $\kappa = \kappa^{<\kappa}$ be an uncountable cardinal, $\gamma \leqslant \kappa$ an ordinal and $\theta > \kappa$ a regular cardinal. Then define the following game $\tilde{G}^\theta_\gamma(\kappa)$ with $(\gamma{+}1)$-many rounds:

$$\begin{array}{llllllllll} \text{I} & \mathcal{M}_0 & & \mathcal{M}_1 & & \cdots & & \mathcal{M}_\gamma & \\ \text{II} & & \mu_0 & & \mu_1 & & \cdots & & \mu_\gamma \end{array}$$

Here $\mathcal{M}_\alpha \prec H_\theta$ is a weak κ-model for every $\alpha \leqslant \gamma$, μ_α is a normal $\mathcal{M}_\alpha$-measure for $\alpha < \gamma$, μ_γ is an $\mathcal{M}_\gamma$-normal good $\mathcal{M}_\gamma$-measure and the $\mathcal{M}_\alpha$'s and μ_α's are $\subseteq$-increasing. For limit ordinals $\alpha \leqslant \gamma$ we furthermore require that $\mathcal{M}_\alpha = \bigcup_{\xi<\alpha} \mathcal{M}_\xi$ and $\mu_\alpha = \bigcup_{\xi<\alpha} \mu_\xi$. Player II wins iff she could continue to play throughout all $(\gamma{+}1)$-many rounds. ⊣

Holy and Schlicht (2018) have shown that the game doesn't depend upon θ, so that we may simply call the game $\tilde{G}_\gamma(\kappa)$.

Definition 1.4 (Holy, Schlicht). Let κ be a cardinal and $\gamma \leqslant \kappa$ an ordinal. Then κ is **γ-Ramsey** if player I does not have a winning strategy in $\tilde{G}_\gamma(\kappa)$, and it's **strategic γ-Ramsey** if player II *does* have a winning strategy in $\tilde{G}_\gamma(\kappa)$. ⊣

2 Strategic ω-Ramseys

In this section we'll show that every virtually measurable cardinal is strategic ω-Ramsey and that every strategic ω-Ramsey is virtually measurable in L. This will also show that strategic ω-Ramseys are downwards absolute to L and that they're limits of ω-Ramseys in L. This section is joint work with Ralf Schindler.

Definition 2.1. A cardinal is **virtually measurable** if there exists a transitive M and a forcing poset $\mathbb{P}$ such that, in $V^{\mathbb{P}}$, there exists an elementary embedding $j : H_{\kappa^+} \to M$ with critical point κ. ⊣

We'll need the following well-known lemmata.

Lemma 2.2 (Ancient Kunen Lemma). *Let $M \models \mathsf{ZFC}^-$ and $j : M \to N$ an elementary embedding with critical point κ such that $\kappa + 1 \subseteq M \subseteq N$. Assume that $X \in M$ has M-cardinality κ. Then $j \restriction X \in N$.* ⊣

Lemma 2.3 (Absoluteness of embeddings on countable structures). *Let M be a countable first-order structure and $j : M \to N$ an elementary embedding. If W is a transitive (set or class) model of (some sufficiently large fragment of) ZFC such that M is countable in W and $N \in W$, then for any finite subset of M, W has some elementary embedding $j^* : M \to N$, which agrees with j on that subset. Moreover, if both M and N are transitive $\in$-structures and j has a critical point, we can additionally assume that* $\operatorname{crit}(j^*) = \operatorname{crit}(j)$. ⊣

Theorem 2.4 (Schindler, N.). *Every virtually measurable cardinal is strategic ω-Ramsey, and every strategic ω-Ramsey cardinal is virtually measurable in L.*

Proof. Let κ be virtually measurable, witnessed by a transitive M, a poset $\mathbb{P}$ and, in $V^{\mathbb{P}}$, an elementary embedding $\pi : H_{\kappa^+} \to M$. Fix a name $\dot\mu$ and a $\mathbb{P}$-condition p such that[2]

$$p \Vdash \ulcorner \dot\mu \text{ is a weakly amenable 0-good } \check H_{\kappa^+}\text{-normal } \check H_{\kappa^+}\text{-measure} \urcorner$$

We now define a strategy σ for player II in $\mathcal{G}_\omega(\kappa)$ as follows. Whenever player I plays a weak κ-model M_n, player II fixes $p_n \in \mathbb{P}$, an M_n-measure μ_n and a function $\pi_n : M_n \to V$ such that $p_0 \leqslant p$, $p_n \leqslant p_k$ for every $k \leqslant n$ and that

$$p_n \Vdash \ulcorner \dot\mu \cap \check M_n = \check\mu_n \wedge \check\pi_n = \dot\pi \restriction \check M_n \urcorner. \tag{1}$$

Note that by the Ancient Kunen Lemma 2.2 we get that $\pi \restriction M_n \in M \subseteq V$, so such π_n always exist in V. The μ_n's also always exist in V, by weak amenability of μ. Player II responds to M_n with μ_n. It's clear that the μ_n's are legal moves for player II, so it remains to show that $\mu_\omega := \bigcup_{n<\omega} \mu_n$ is 0-good. Assume it's not, so that we have a sequence $\langle g_n \mid n < \omega \rangle$ of functions $g_n : \kappa \to M_\omega := \bigcup_{n<\omega} M_n$ such that $g_n \in M_\omega$ and

$$X_{n+1} := \{\alpha < \kappa \mid g_{n+1}(\alpha) < g_n(\alpha)\} \in \mu_\omega. \tag{2}$$

Without loss of generality we can assume that $g_n, X_n \in M_n$. Then (2) implies that $p_{n+1} \Vdash \ulcorner \dot\pi(\check g_{n+1})(\check\kappa) < \dot\pi(\check g_n)(\check\kappa) \urcorner$, but by (1) this also means that

$$p_{n+1} \Vdash \ulcorner \check\pi_{n+1}(\check g_{n+1})(\check\kappa) < \check\pi_n(\check g_n)(\check\kappa) \urcorner, \tag{3}$$

[2]Recall that an M-measure μ is 0-**good** if $\operatorname{Ult}(M, \mu)$ is well-founded.

so defining, in V, the ordinals $\alpha_n := \pi_n(g_n)(\kappa)$, (3) implies that $\alpha_{n+1} < \alpha_n$ for all $n < \omega$, $\lightning$. So μ_ω is 0-good, making σ a winning strategy and thus therefore also making κ strategic ω-Ramsey.

Next, let κ be strategic ω-Ramsey and fix a winning strategy σ for player II in $\mathcal{G}_\omega(\kappa)$. Let $g \subseteq \mathrm{Col}(\omega, \kappa^{+L})$ be V-generic and in $V[g]$ fix an elementary chain $\langle L_{\kappa_n} \mid n < \omega\rangle$ of weak κ-models such that $H^l_{\kappa^+} \subseteq \bigcup_{n<\omega} L_{\kappa_n}$. Player II follows σ, resulting in a $H^L_{\kappa^+}$-normal $H^L_{\kappa^+}$-measure μ on κ.

Claim 2.4.1. $\mathrm{Ult}(H^L_{\kappa^+}, \mu)$ is well-founded.

Proof of claim. Assume for a contradiction that $\mathrm{Ult}(H^L_{\kappa^+}, \mu)$ is illfounded, witnessed by a sequence $\langle g_n \mid n < \omega\rangle$ of functions $g_n : \kappa \to \kappa$ such that $g_n \in H^L_{\kappa^+}$ and $\{\alpha < \kappa \mid g_{n+1}(\alpha) < g_n(\alpha)\} \in \mu$. Now, in V, define a tree $\mathcal{T}$ of triples (f, M_f, μ_f) such that $f : \kappa \to \kappa$, M_f is a weak κ-model, μ_f is an M_f-measure on κ and letting $f_0 <_{\mathcal{T}} \cdots <_{\mathcal{T}} f_n = f$ be the $\mathcal{T}$-predecessors of f,

- $\langle M_{f_0}, \mu_{f_0}, \dots, M_{f_n}, \mu_{f_n}\rangle$ is a partial play of $\mathcal{G}_\omega(\kappa)$ in which player II follows σ; and
- $\{\alpha < \kappa \mid f_{k+1}(\alpha) < f_k(\alpha)\} \in \mu_{k+1}$ for every $k < n$.

Now, the g_n's induce a cofinal branch through $\mathcal{T}$ in $V[g]$, so by absoluteness of well-foundedness there's a cofinal branch b through $\mathcal{T}$ in V as well. But b now gives us a play of $\mathcal{G}_\omega(\kappa)$ where player II is following σ but player I wins, a contradiction. Thus $\mathrm{Ult}(H^L_{\kappa^+}, \mu)$ is well-founded. ⊣

Let $j : H^L_{\kappa^+} \to \mathrm{Ult}(H^L_{\kappa^+}, \mu) \cong M$ be the ultrapower embedding followed by the transitive collapse, so that $M = L_\alpha$ for some α by elementarity. Let now $h \subseteq \mathrm{Col}(\omega, \kappa^{+L})^L$ be L-generic, so that $H^L_{\kappa^+}$ is countable in $L[h]$ and (trivially) $M \in L[h]$. By Lemma 2.3 we then get that there's an elementary embedding $j^* : H^L_{\kappa^+} \to M$ in $L[h]$ with critical point κ. Since we also have that $M \in L$ this makes κ virtually measurable in L. ■

We get the following immediate corollary.

Corollary 2.5 (Schindler, N.). *Strategic ω-Ramseys are downwards absolute to L, and the existence of a strategic ω-Ramsey cardinal is equiconsistent with the existence of a virtually measurable cardinal. Further, in L the two notions are equivalent.* ⊣

Theorem 2.6 (Schindler, N.). *Every virtually measurable cardinal is a limit of ω-Ramseys.*

Proof. Let κ be virtually measurable, and fix a transitive M, a forcing poset $\mathbb{P}$ and let $g \subseteq \mathbb{P}$ be V-generic such that, in $V[g]$, there's an elementary embedding $\pi : H_{\kappa^+} \to M$ with crit $\pi = \kappa$. We aim to show that $M \models \ulcorner\kappa$ is ω-Ramsey$\urcorner$.

Let $\sigma \in M$ be a strategy for player I in $\mathcal{G}_\omega(\kappa)^{\mathcal{M}}$. Now, whenever player I plays M_n let player II play $\mu_{\pi \restriction M_n}$, the derived measure of $\pi \restriction M_n$, which is an element of M by the Ancient Kunen Lemma 2.2. After ω moves we then get a play $\langle M_n, \mu_{\pi \restriction M_n} \mid n < \omega\rangle \in V[g]$ according to σ.

But now both $\langle M_n \mid n < \omega\rangle, \pi \restriction M_\omega \in M$, where $M_\omega := \bigcup_{n<\omega} M_n$, so the sequence $\langle \mu_{\pi \restriction M_n} \mid n < \omega\rangle$ is an element of M as well. This means that M sees the play, and it remains to show that the play is winning for player II — i.e. that $\mathrm{Ult}(M_\omega, \mu_\omega)$ is well-founded, where $\mu_\omega := \bigcup_{n<\omega} \mu_{\pi \restriction M_n}$.

Assume $\mathrm{Ult}(M_\omega, \mu_\omega)$ is ill-founded, giving us a sequence $\langle g_n \mid n < \omega\rangle \in V[g]$ of functions $g_n : \kappa \to \mathrm{On} \cap M_\omega$ such that $g_n \in M_\omega$ and

$$X_{n+1} := \{\alpha < \kappa \mid g_{n+1}(\alpha) < g_n(\alpha)\} \in \mu_\omega$$

for every $n < \omega$. Without loss of generality we may assume that $X_n \in \mu_{\pi \restriction M_n}$, meaning that, for all $n < \omega$,

$$(\pi \restriction M_{n+1})(g_{n+1})(\kappa) < (\pi \restriction M_{n+1})(g_n)(\kappa),$$

which yields a decreasing ω-sequence of ordinals in M, $\lightning$. So $\mathrm{Ult}(M_\omega, \mu_\omega)$ is well-founded, making the play winning for player II and hence making κ ω-Ramsey in M.

But consider now the factor map $k : \mathrm{Ult}(H^V_{\kappa^+}, \mu_\pi) \to M$ in $V[g]$, given by $k([f]_{\mu_\pi}) := \pi(f)(\kappa)$. By H_{κ^+}-normality of μ_π we get that $k(\kappa) = \pi([\mathrm{id}]_{\mu_\pi})(\kappa) = \kappa$, so that

$$\mathrm{Ult}(H^V_{\kappa^+}, \mu_\pi) \models \ulcorner\kappa \text{ is } \omega\text{-Ramsey}\urcorner.$$

Łoś' theorem now implies that κ is a limit of ω-Ramseys in $H^V_{\kappa^+}$. But since κ is inaccessible and the question whether λ is ω-Ramsey is absolute beween $H_{(2^\lambda)^+}$ and V by results in Holy and Schlicht (2018), we get that κ *is* a limit of ω-Ramsey cardinals. ■

Now the above Theorems 2.4 and 2.6 immediately imply the following.

Corollary 2.7. *Every strategic ω-Ramsey cardinal is a limit of ω-Ramseys in L.* ⊣

3 Strategic ω_1-Ramseys

In this section we present Schindler's argument that strategic ω_1-Ramseys are measurable in the core model K below a Woodin cardinal. This improves upon a result of Philip Welch, who showed it below $0^\P$, the sharp of a strong cardinal, using a slightly different argument. We will need the following special case of Corollary 3.1 from Schindler (2006).[3]

Theorem 3.1 (Schindler). *Assume that there exists no inner model with a Woodin cardinal, let μ be an measure on a cardinal κ, and let $\pi : V \to \text{Ult}(V, \mu) \cong N$ be the ultrapower embedding. Assume that N is closed under countable sequences. Write K^N for the core model constructed inside N. Then K^N is a normal iterate of K, i.e. there is a normal iteration tree $\mathcal{T}$ on K of successor length such that $\mathcal{M}^{\mathcal{T}}_\infty = K^N$. Moreover, we have that $\pi^{\mathcal{T}}_{0\infty} = \pi \upharpoonright K$.* ⊣

Theorem 3.2 (Schindler). *Assume there exists no inner model with a Woodin cardinal. Then every strategic ω_1-Ramsey cardinal is measurable in K.*

Proof. Fix a large regular $\theta \gg 2^\kappa$. Let κ be strategic ω_1-Ramsey and fix a winning strategy σ for player II in $\mathcal{G}_{\omega_1}(\kappa)$. Let $g \subseteq \text{Col}(\omega_1, 2^\kappa)$ be V-generic and in $V[g]$ fix an elementary chain $\langle M_\alpha \mid \alpha < \omega_1 \rangle$ of weak κ-models $M_\alpha \prec H^V_\theta$ such that $M_\alpha \in V$, ${}^\omega M_\alpha \subseteq M_{\alpha+1}$ and $H^V_{\kappa^+} \subseteq M_{\omega_1} := \bigcup_{\alpha<\omega_1} M_\alpha$.

Note that V and $V[g]$ have the same countable sequences since $\text{Col}(\omega_1, 2^\kappa)$ is $<\omega_1$-closed, so we can apply σ to the M_α's, resulting in an M_{ω_1}-measure μ on κ. Since we required that ${}^\omega M_\alpha \subseteq M_{\alpha+1}$ we get that $\mathcal{M}_{\omega_1}$ is closed under ω-sequences in $V[g]$, making μ countably complete in $V[g]$. As we also ensured that $H^V_{\kappa^+} \subseteq \mathcal{M}_{\omega_1}$ we can lift j to an ultrapower embedding $\pi : V \to \text{Ult}(V, \mu) \cong N$ with N transitive.

Since V is closed under ω-sequences in $V[g]$ we get by standard arguments that N is as well, which means that Theorem 3.1 applies, meaning that K^N is an iterate of K with the iteration map having critical point κ, making κ measurable in K. ∎

References

Gitman, V. (2011). Ramsey-like cardinals. *The Journal of Symbolic Logic*, 76(2):519–540.

[3]That paper assumes the existence of a measurable as well, but by Jensen and Steel (2013) we can omit that here.

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