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Philipp Schlicht: The open coloring axiom for definable subsets of generalized Baire spaces

Joint work with Dorottya Sziráki, preprint in preparation

Motivation: We study analogies between $L(\mathbb{R})$ and $L(\mathcal{P}(\kappa))$ for $\kappa = \omega_1, \omega_2, \ldots$

(a) Nice properties of definable subsets of ${}^\kappa\kappa$, e.g. analgues to the perfect set property, Baire property.

(b) Games of length κ.

This connects the topic of this talk with determinacy theory and large cardinals.

We always assume that $\kappa^{<\kappa} = \kappa$.

Definition: ${}^\kappa\kappa$ carries the bounded topology with basic open sets $N_t = \{x \in {}^\kappa\kappa \mid t \subseteq x\}$ for $t \in {}^{<\kappa}\kappa$.

Definition: Suppose that $X \subseteq {}^\kappa\kappa$ and (K_0, K_1) is a partition of $[X]^2$. We identify K_i with $\{(x,y) \in X^2 \mid \{x,y\} \in K_i\}$.
(K_0, K_1) is called an open partition if $K_0 \subseteq X^2$ is open in the (product of the) bounded topology.

Definition: Let $X \subseteq {}^\kappa\kappa$.

(a) $OCA(X)$: If (K_0, K_1) is an open partition of $[X]^2$, then either:
  (i) there is a K_0-homogeneous set of size $> \kappa$, or
  (ii) X is a union of κ many K_1-homogeneous sets.

(b) $OCA^*(X)$: If (K_0, K_1) is an open partition of $[X]^2$, then either:
  (i) there is a perfect K_0-homogeneous set, or
  (ii) X is a union of κ many K_1-homogeneous sets.

For $K_0 = [X]^2$, $OCA^*(X)$ implies the perfect set property for X.

<u>Theorem</u> (Feng) (1) $OCA^*(X)$ holds for all $\underset{\sim}{\Sigma}^1_1$ subsets of ${}^\omega\omega$.

(2) In Solovay's model, $OCA^*(X)$ holds for all subsets X of ${}^\omega\omega$.

<u>Theorem</u> (Sziraki 2017) In $Col(\kappa, <\mu)$-generic extensions, where $\mu > \kappa$ is inaccessible, $OCA^*(X)$ holds for all $\underset{\sim}{\Sigma}^1_1$ subsets of ${}^\kappa\kappa$.

$OCA^*(X)$ also has the following game characterization. The two cases are equivalent to the existence of winning strategies for each player.

<u>Proposition</u> (Sziraki 2017) $OCA^*(X)$ is equivalent to determinacy of the following game:

$$\begin{array}{lcccccccc} \mathrm{I} & & i_0 & & i_1 & & & i_\alpha & \\ \mathrm{II} & u_0^0, u_0^1 & & u_1^0, u_1^1 & & \cdots & u_\alpha^0, u_\alpha^1 & & \cdots \end{array}$$

such that

(a) $u_\alpha^{i_\alpha} \subseteq u_\beta^i$ for all $\alpha < \beta$ and $i < 2$

(b) $u_\alpha^0 \perp_{K_1} u_\alpha^1$ if $\alpha \in Succ \cap \kappa$ and

$u_\alpha^0 = u_\alpha^1$ if $\alpha \in Lim \cap \kappa$,

where $u \perp_{K_1} v :\Longleftrightarrow u \perp v$ and $(N_u \times N_v) \cap [X]^2 \subseteq K_0$.

Our main result:

<u>Theorem</u> (Sziraki-S. 2018) In $Col(\kappa, <\mu)$-generic extensions, where $\mu > \kappa$ is inaccessible, $OCA^*(X)$ holds for all subsets X of ${}^\kappa\kappa$ which are definable from subsets of κ.

In the rest of the talk, we sketch the proof of this result.

<u>Outline</u>: Let G be $Col(\kappa, <\mu)$-generic over V. In $V[G]$, let $X \subseteq {}^{\kappa}\kappa$ and (K_0, K_1) be an open partition of $[X]^2$ with X, K_i definable from some $z_0 \in {}^{\kappa}V$. We will assume that $z_0 \in V$.

Suppose that X is not covered by κ many K_1-homogeneous sets. Then it is not covered by the sets $[T]$, where $T \in V$ and $[T]$ is K_1-homogeneous. Let $x \in ({}^{\kappa}\kappa)^{V[G]}$ witness this. Let $x \in V[G_\alpha]$, where $\alpha < \kappa$ and $G_\alpha = G \cap Col(\kappa, <\alpha)$ and let ν be a $Col(\kappa, <\alpha)$-name for x.

One can reduce this to the case that ν is an $Add(\kappa, 1)$-name.

We will add a perfect tree $T \in V[G]$ by forcing over $V[G_\alpha]$ such that $\{\nu^b \mid b \in [T]\}$ is a perfect K_0-homogeneous subset of X.

<u>S-homogeneous trees</u>

Let $S \subseteq ({}^{<\kappa}\kappa)^2$ be a pruned tree with $K_1 = [S]$. Pruned means that every node is contained in a cofinal branch.

<u>Definition</u> (a) $t \perp_S u \iff ([t] \times [u]) \cap = \emptyset$, where $[t] = \{v \in {}^{<\kappa}\kappa \mid t \subseteq v\}$.
(b) A tree $T \subseteq {}^{<\kappa}\kappa$ is <u>S-homogeneous</u> if $\forall t, u \in T \;\; t \not\perp_S u$.

If T is S-homogeneous, then $[T]$ is $[S]$-homogeneous.
The converse also holds if T is pruned.

<u>Lemma</u> Suppose that $\mathbb{Q}$ is $<\kappa$-closed, $\mathbb{R}$ preserves ${}^{<\kappa}\kappa$, $G \times H$ is $\mathbb{Q} \times \mathbb{R}$-generic over V and $x \in V[H]$. If $x \notin [T]$ for all S-homogeneous trees $T \in V$, then this holds for all $T \in V[G]$.

<u>Proof</u>: Let τ be a $\mathbb{Q}$-name for an S-tree T and $x = \sigma^H$.
Let $\vec{u} = \langle u_\alpha \mid \alpha < \kappa \rangle$ enumerate ${}^{<\kappa}\kappa$.

We construct a decreasing sequence $\vec{q} = \langle q_\alpha \mid \alpha < \kappa \rangle$ in $\mathbb{Q}$ and approximations T_α, U_α to τ and its complement with

(a) $u_\alpha \in T_\alpha$ if $q_\alpha \Vdash_{\mathbb{Q}} u_\alpha \in \tau$ and

(b) $u_\alpha \in U_\alpha$ if $q_\alpha \Vdash_{\mathbb{Q}} u_\alpha \notin \tau$.

Then $T' = \bigcup_{\alpha < \kappa} T_\alpha \in V$ is S-homogeneous. So some $r \Vdash_{\mathbb{R}} \sigma \notin [T']$.

Find $r' \leq r$, $\alpha, \beta < \kappa$ with $r' \Vdash_{\mathbb{R}} \sigma \restriction \alpha = u_\beta \in U_\beta$. Then $(q_\beta, r') \Vdash_{\mathbb{Q} \times \mathbb{R}} \sigma \notin [T]$. $\square$

<u>The forcing</u> Let ν be an $\mathrm{Add}(\kappa,1)$-name for an element of ${}^{<\kappa}\kappa$ such that

$(*)$ $\mathbb{1} \Vdash_{\mathrm{Add}(\kappa,1)} \nu \notin [T]$ for all S-homogeneous trees $T \in V$.

<u>Definition</u> Let $\mathbb{P} = \mathbb{P}_\nu$ consist of the conditions $p = (t, s)$ such that

(i) t is a subtree of ${}^{<\kappa}\kappa$ with $|t| < \kappa$.

(ii) every node in t has at most 2 direct successors in t.

(iii) $s = \{u \in t \mid u$ has <u>exactly one</u> direct successor in $t\}$.

(iv) if $u \in t$ has two direct successors in t, then there are $u_0, u_1 \in t$ with

$u_0, u_1 \supseteq u$, $u_0 \perp u_1$ and

(a) if $u \subsetneq w \subsetneq u_i$, then $w \in S$

(b) there are $v_0 \perp_S v_1$ with $u_i \Vdash_{\mathrm{Add}(\kappa,1)} u_i \subseteq \nu$ for $i < 2$.

$\mathbb{P}$ is non-atomic, $<\kappa$-closed and $|\mathbb{P}| = \kappa$, so $\mathbb{P}$ is forcing equivalent to $\mathrm{Add}(\kappa,1)$.

If G is $\mathbb{P}$-generic over V, let $T_G = \bigcup_{(t,s) \in G} t$. Let $\dot{T}$ be a $\mathbb{P}$-name for T_G.

<u>Lemma</u> Let $p = (t,s) \in \mathbb{P}$ and $u \in {}^{<\kappa}\kappa$.

(1) If $l(u) \in \mathrm{Succ}$, then $p \Vdash_{\mathbb{P}} u \in \dot{T} \iff u \in t$.

(2) If $l(u) \in \mathrm{Lim}$, then $p \Vdash_{\mathbb{P}} u \in \dot{T} \iff \forall \alpha < l(u)\ u \restriction \alpha \in t$.

Lemma T_G is a $<\kappa$-closed perfect subtree of ${}^{<\kappa}\kappa$.

Proof To see that T_G is perfect, let $p=(t,s)\in\mathbb{P}$.

Suppose that u is an end node of t. We aim to find $q=(t',s')\leq p$ such that t' splits above u. Let $T^{(u)}=\{r\in{}^{<\kappa}\kappa \mid \exists u'\supseteq u \mid u' \Vdash_{Add(\kappa,1)} r\subseteq\nu\}$ be the "tree of possibilities" for ν below u.

Since $u\Vdash_{Add(\kappa,1)} \nu\in[T^{(u)}]$, $T^{(u)}$ cannot be S-homogeneous.

Find $r_0 \perp_S r_1$ in $T^{(u)}$ and $u_0,u_1\supseteq u$ with $u_i \Vdash_{Add(\kappa,1)} r_i\subseteq\nu$.

Find $q=(t',s')\leq q$ such that t' splits at $u_0\wedge u_1$. $\square$

Lemma Every branch in T_G is $Add(\kappa,1)$-generic over V.

Let $\mathbb{P}^*$ be the set of conditions $p=(t,s)$ such that

(a) $ht(t)\in Lim$

(b) p decides $\sigma\upharpoonright ht(t)$,

where σ is a fixed name for a branch in T. Then $\mathbb{P}^*$ is dense in $\mathbb{P}$.

Moreover, let $\sigma_p=\{(\alpha,\beta)\mid p\Vdash\sigma(\alpha)=\beta\}$. If $p\in\mathbb{P}^*$, then $\sigma_p\in{}^{<\kappa}\kappa$.

Quotient forcings

Let $\mathbb{Q}_0=\{(\sigma_p,1_{\mathbb{P}})\mid p\in\mathbb{P}^*\}$, $\mathbb{Q}_1=\{(\sigma_p,p)\mid p\in\mathbb{P}^*\}$ and $\mathbb{Q}=\mathbb{Q}_0\cup\mathbb{Q}_1$.

Then $\mathbb{Q}_1$ is isomorphic to $\mathbb{P}^*$ and dense in $\mathbb{Q}$.

Lemma $\pi:\mathbb{Q}\to\mathbb{Q}_0$, $\pi(\sigma_p,q)=(\sigma_p,1_{\mathbb{P}})$ is a projection.

Proof. Let $u\in\mathbb{Q}$ and $v\leq\pi(u)$ in $\mathbb{Q}_0$. We will find $w\leq u,v$. Then $\pi(w)\leq\pi(v)\leq v$.

We can assume that $u\in\mathbb{Q}_1$. Let $u=(\sigma_p,p)$ and $v=(\sigma_q,1_{\mathbb{P}})$. Then $\sigma_p\leq\sigma_q$.

Let $p=(t,s)$. Since $l(\sigma_p)=ht(t)$, we can extend p to p' with $\sigma_{p'}=\sigma_q$.

Then $w=(\sigma_{p'},p')$ and $v=(\sigma_q,1)$ are compatible. $\square$

<u>Lemma</u> $\mathbb{Q}_0$ is a complete subforcing of $\mathbb{Q}$.

<u>Proof</u> Let $A \subseteq \mathbb{Q}_0$ be a maximal antichain. We claim that A is maximal in $\mathbb{Q}$.
Take $u = (\sigma_p, p) \in \mathbb{Q}_1$. Let $v = (\sigma_q, 1) \in \mathbb{Q}_0$ with $v \leq \pi(u) = (\sigma_p, 1_{\mathbb{P}})$ and $v \leq a \in A$.
Since π is a projection, let $u' \leq u$ with $\pi(u') = v$.
Then $u' \leq \pi(u') \leq a$, so $u \| a$. □

<u>Lemma</u> In $V[G]$, σ^G has a quotient forcing that is equivalent to $\mathrm{Add}(\kappa, 1)$.

<u>Proof</u> Let $G_0 = G \cap \mathbb{Q}_0$. The quotient forcing for σ^G is
$$\{(\sigma_p, q) \in \mathbb{Q} \mid \underset{\underset{(\sigma_p, 1)}{\shortparallel}}{\pi(\sigma_p, q)} \in G_0\} = \{(\sigma_p, q) \in \mathbb{Q} \mid \sigma_p \leq \sigma^G\}.$$
This is $<\kappa$-closed. □

<u>The main step</u>

For any submodel M of $V[G]$, let
$$\mathcal{I}^M = \bigcup\{[T] \mid T \in M \text{ is } S\text{-homogeneous}\}$$

By our assumption, $X^{V[G]} \not\subseteq \mathcal{I}^V$. So there is some $\alpha < \kappa$ with $\alpha^{<\kappa} = \alpha$ and $X^{V[G_{\alpha+1}]} \not\subseteq \mathcal{I}^V$, where $G_{\alpha+1} = G \cap \mathrm{Col}(\kappa, <(\alpha+1))$.

Let $V[G] = V[G_{\alpha+1} \times g \times h]$, where $g \times h$ are $\mathrm{Add}(\kappa,1) \times \mathrm{Col}(\kappa, <\mu)$-generic over $V[G_{\alpha+1}]$.

<u>Lemma</u> $X^{V[G_{\alpha+1} \times g]} \not\subseteq \mathcal{I}^{V[G_{\alpha+1}]}$

<u>Proof</u> $V[G] = V[G_{\alpha+1} \times g \times h] = V[G_{\alpha+1} \times k \times h]$, where k is $\mathrm{Col}(\kappa, <(\alpha+1))$-generic. We have $X^{V[G_{\alpha+1}]} \not\subseteq \mathcal{I}^V \Rightarrow X^{V[k]} \not\subseteq \mathcal{I}^V \Rightarrow X^{V[k]} \not\subseteq \mathcal{I}^{V[G_{\alpha+1}]}$ by the Lemma on page 3 $\Rightarrow X^{V[G_{\alpha+1} \times g]} \not\subseteq \mathcal{I}^{V[G_{\alpha+1}]}$. □

In $V[G_{\alpha+1}]$, there is an $\mathrm{Add}(\kappa,1)$-name ν that satisfies the condition $(*)$ on page 4 and that is forced to be an element of X in all further $\mathrm{Col}(\kappa, <\mu)$-generic extensions.

Let T be $\mathbb{P}_D$-generic over $V[G_{\alpha+1}]$ with $V[G_{\alpha+1}][T] = V[G_{\alpha+1}][g]$.
Then $\{\nu^b \mid b \in [T]\}$ is a K_0-homogeneous subset of $X^{V[G]}$. □

<u>Question</u>: Can one formulate an interesting coloring version of the superperfect set property?