generic Vopěnka cardinals + models of ZF with few $\aleph_1$-Suslin sets.

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<u>main thm.</u> equicon. are:

(1) ZFC + $\exists$ gen. Vopěnka cardinals.

(2) ZF + DC + $S_{\aleph_1} = \underset{\sim}{\Sigma}^1_2 + \theta = \aleph_2$.

(3) ZF + DC + $2^{\aleph_1} \not\leq |S_{\aleph_1}|$.

no inj.

<u>outline.</u> con(1) $\Rightarrow$ con(2):

κ generic Vopěnka $\Rightarrow$ κ gen. Vopěnka in L

$\Rightarrow L(\mathbb{R})^{L^{Col(\omega, <\kappa)}} \models (2)$

(2) $\Rightarrow$ (3) easy.

con(3) $\Rightarrow$ con(1): (3) $\Rightarrow$ $\aleph_1^V$ is a gen. Vopěnka cardinal in L, in fact in every inn model of choice.

<u>note.</u> can remove DC.

can replace $S_{\aleph_1}$ by S_∞.

[S_κ = κ-Suslin sets.]

can replace $\not\leq$ by $\not\leq^*$: no surjection defn. (ZF)

$$S_{\aleph_1} = \{ A \subset {}^\omega\omega : A = pB \text{ for some } B \subset {}^\omega\omega \times \aleph_1^\omega,\ B \text{ closed} \}.$$

$$= \{ A \subset {}^\omega\omega : A = p[T] \text{ for some tree } T \text{ on } \omega \times \omega_1 \}.$$

Θ = least ordinal not a surj. image of $\mathbb{R}$

ZFC $\vdash$ $\Theta = \aleph_2 \leftrightarrow$ CH.

note. (a) $|S_{\aleph_1}| \leq^* 2^{\aleph_1}$, i.e. have a surj. $2^{\aleph_1} \to S_{\aleph_1}$.

(b) $\underset{\sim}{\Sigma}^1_2 \subset S_{\aleph_1}$ (Shoenfield)

what about the reverse?

in ZFC :

(a) fix $A \subset {}^\omega\omega$, $|A| = \aleph_1$. every $B \subset A$ is $\aleph_1$-Suslin. so $2^{\aleph_1} \leq |S_{\aleph_1}|$ + hence $2^{\aleph_1} = |S_{\aleph_1}|$.

(b) CH $\Rightarrow$ every set of reals is in $S_{\aleph_1}$.

$\Rightarrow S_{\aleph_1} \neq \underset{\sim}{\Sigma}^1_2$.

(c) $MA + \neg CH \Rightarrow \bigcup^{\aleph_1} \underset{\sim}{\Sigma}^1_2 \subset \underset{\sim}{\Sigma}^1_2$

(i.e., $\underset{\sim}{\Sigma}^1_2$ closed under unions of size $\aleph_1$)

(martin – solovay) $S_{\aleph_1}$

then use every $\underset{\sim}{\Sigma}^1_2$ set is the union

of $\aleph_1$ $\underset{\sim}{\Sigma}^1_1$ sets.

So in part., even $ZFC + S_{\aleph_1} = \underset{\sim}{\Sigma}^1_2$ doesn't have cons. strength. (cf. (2) of main thm.) *)

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in ZF,

$2^{\aleph_1} \not\leq |S_{\aleph_1}| \Rightarrow \aleph_1 \not\leq 2^{\aleph_0}$ by above

argt for (a)

$\Rightarrow \aleph_1^V$ is a strong limit in L (in fact, in every inner model of choice)

$\Rightarrow \aleph_1^V$ is inaccessible in L.

<u>def.</u> (ZFC) a <u>generic vopenka like cardinal</u> is

$\kappa > \aleph_0$ s.t. f.a. κ-sequences $(M_\alpha : \alpha < \kappa)$

of structures of card. $< \kappa$ in a common

1^{st} order language of size $< \kappa$,

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*) but don't know what happens with

$ZF + DC + S_\infty = \underset{\sim}{\Sigma}^1_2$

$\exists\, \alpha, \beta < \kappa$ distinct

$\exists\, \mathbb{P}$ $\quad V^{\mathbb{P}} \models \exists$ el. embedding $M_\alpha \longrightarrow M_\beta$.

note. WLOG assm $M_\alpha \in H_\kappa$ f.a. α.

WLOG $\quad \mathbb{P} = \mathrm{Col}(\omega, M_\alpha)$.

def. (ZFC) a gen. vopěnka cardinal is an inaccessible vopěnka-like cardinal.

note: (ZFC) κ gen. vopěnka-like $\Longrightarrow$ κ strg limit :

let $\alpha < \kappa$: consider $\mathcal{L}$ with α unary predicate symbols. have 2^α structures on $\{0\}$ with no generic e.e.

gen. vopěnka-like cardinals can be singular (equiconsistent with inf. many ω-erdős cardinals)

note. (ZFC) κ is generic vopěnka iff

$(V_\kappa\, ;\, V_{\kappa+1}\, ;\, \in) \models$ GBC + gen. vopěnka principle.

defined by bagaria – gitman – schindler

note. (ZFC) κ generic vopěnka $\Rightarrow$

κ is gen. vopěnka.

let $M, N \in L$. if there is an e.e. $M \to N$ in $V^{\mathbb{P}}$, then there is an e.e. $M \to N$ in $L^{\mathrm{Col}(\omega, M)}$.

(same for any inner model.)

definition. (ZFC) $A \subset H_\kappa$ is generically hereditary iff

① all elts. of A are structures for some 1st ord lang.

② A is downward closed under generic e.e.

(i.e., if $M \underset{\substack{\text{e.e.}\\ \text{in}\\ V^{\mathrm{Col}(\omega,M)}}}{\longrightarrow} N \in A$, then $M \in A$)

proposition. (ZFC) let κ be inaccessible. TFAE.

① κ is gen. vopěnka.

② every gen. hereditary $A \subset H_\kappa$ is $\underset{\sim}{\Sigma}_1^{H_\kappa}$.

③ there are $< 2^\kappa$ gen. hereditary subsets of H_κ.

proof ① ⇒ ② of proposition

κ gen. Vopenka.

let $A \subset H_\kappa$ gen. hereditary, set of $\mathcal{L}$-structures.

let $B = \{ M \in H_\kappa \setminus A : M$ an $\mathcal{L}$-structure$\}$,

B is upward closed.

claim. B is the upward closure of some

$B_0 \subset B$ with $\overline{\overline{B_0}} < \kappa$.

pf. of claim: if not, get $(M_\alpha : \alpha < \kappa)$

$\subset B$ s.t. $\forall \alpha < \beta < \kappa$ there is no

gen. e.e. $M_\alpha \longrightarrow M_\beta$.

let N_α code $M_\alpha \oplus (\alpha; \in)$. there is then

no pair $\alpha, \beta < \kappa$, $\alpha \neq \beta$ with a gen. e.e.

$N_\alpha \rightarrow N_\beta$. ⊣(claim)

$B_0 \in H_\kappa$. we show A is $\Sigma_1^{H_\kappa}(B_0)$.

"$\neg\exists$ gen. e.e. $M \longrightarrow N$" is Σ_1

Thm. (see Bagaria–Gitman–Schindler paper) ← in fact, folklore. TFAE.

(1) $\exists$ gen. e.e. $M \longrightarrow N$.

(2) II has a w.s. in $G(M, N)$:

I	x_0		x_1		$\dots$
II		y_0		y_1	$\dots$

II wins iff f.a. $n<\omega$,

$tp^M(x_0, \text{—}, x_n) = tp^N(y_0, \text{—}, y_n)$.

(3) I has no w.s.

So: $\neg\exists$ gen. e.e. $\Longleftrightarrow$

$\exists$ w.s. for I $\Longleftrightarrow$

$\exists$ rank function on the game tree

so this is Σ_1.

(2) $\Rightarrow$ (3) trivial.

(3) $\Rightarrow$ (1) assume κ is not gen. Vopěnka-like, witnessed by $(M_\alpha : \alpha<\kappa)$. for every $S \subset \kappa$ define $A_S = \{M_\alpha : \alpha \in S\}$

$A_S^* =$ downward closure of A_S with respect to gen. e.e.

claim. $S_0 \neq S_1 \implies A^*_{S_0} \neq A^*_{S_1}$.

proof. w.l.o.g., $\exists \alpha \in S_0 \setminus S_1$.

so $M_\alpha \in A_{S_0} \subset A^*_{S_0}$.

but $M_\alpha \notin A^*_{S_0}$: $M_\alpha \notin A^*_{S_0}$, and M_α doesn't e.e. into any other M_β

$\dashv$ (claim) (prop.)

note: proposition holds for reflecting cardinals with $\underset{\sim}{\Pi}_1^{H_\kappa}$ in place of $\underset{\sim}{\Sigma}_1^{H_\kappa}$.

to prove the main thm.:

lemma 1. (ZF) let $x \in \mathbb{R}$, $T \in L[x]$ be a tree on $\omega \times \lambda$, $\lambda \in OR$. assume $\aleph_1^V$ is gen. reflecting ~~in~~ in $L[x]$.

then $p[T]$ is $\underset{\sim}{\Sigma}^1_2$ in V.

proof: $\gamma = |\lambda|^{+L[x]}$, so $T \in L_\gamma[x]$.

claim. $\forall z \in \mathbb{R}$ TFAE.

① $z \in p[T]$

② $\exists$ ctbl. $\bar{\gamma}$ ~~&~~ $\exists \bar{T} \in L_{\bar{\gamma}}[x] \big(z \in p[\bar{T}]$

$+ \exists$ e.e. $\pi : (L_{\bar{\gamma}}[x]; \in, x, \bar{T}) \to (L_\gamma[x], \in, x, T)$

pf.: ① ⇒ ② $z \in p[T]$, say $(z, f) \in [T]$, $f \in {}^{\omega}\lambda$. take a skolem hull.

② ⇒ ①: $p[\bar{T}] \subset p[T]$. ⊣

note: $V \models \exists$ e.e. as in ② $\Longleftrightarrow$

$L[x] \models \exists$ gen. e.e. as in ②.

$\kappa = \aleph_1^V$.

in $L[x]$, let $A = \{ M \in H_\kappa : \exists$ gen. e.e.

$M \longrightarrow (L_\gamma[x]; \in, x, T) \}$

by the proposition, A is $\left(\underset{\sim}{\Sigma}_1^{H_\kappa}\right)^{L[x]} = \left(\underset{\sim}{\Sigma}_1\right)^{L_\kappa[x]}$, so A is $\underset{\sim}{\Sigma}_1^{HC}$.

by the claim, $p[T]$ is $\underset{\sim}{\Sigma}_1^{HC}$, so $\underset{\sim}{\Sigma}^1_2$. ⊣

(lemma 1)

<u>lea 2.</u> (ZF) assume $2^{\aleph_1} \not\leq |S_{\aleph_1}|$. | Q: does this give ω_1 is regular?

then $\aleph_1^V$ is gen. vopěnka-like in every inner model of ZFC.

proof: it suff. to show $\aleph_1$ has a "vopěnka-like" property in V:

claim. let $\mathcal{L}$ be a ctbl. language + let $(M_\alpha : \alpha < \omega_1)$ be a seq. of $\mathcal{L}$ structures, M_α is on $\mu_\alpha < \omega_1$ f.a. $\alpha < \omega_1$. then $\exists$ dist. $\alpha, \beta < \omega_1$ $\exists$ e.e. $M_\alpha \longrightarrow M_\beta$.

(gives lemma 2)

proof of claim: WLOG each M_α has dy. the skolem functions.

let $\mathcal{L}' = \mathcal{L} \cup \{ c_n : n < \omega \}$

$\uparrow$ constant symbols

for $\alpha < \omega_1$, $f \in {}^{\omega}\mu_\alpha$, expand M_α to $\mathcal{L}'$-structure M_α^f : $c_n^{M_\alpha^f} = f(n)$.

fix bijection $\mathcal{L}' \longrightarrow \omega$.

let $\mathrm{Code}(\mathrm{Th}(M_\alpha^f)) \in \mathbb{R}$ code $\mathrm{Th}(M_\alpha^f)$.

for $A \subset \aleph_1$ define

$$A^* = \{ \mathrm{Code}(\mathrm{Th}(M_\alpha^f)) : \alpha \in A \wedge f \in \mu_\alpha^\omega \}$$

note A^* is $\aleph_1$-Souslin.

because $2^{\aleph_1} \nleq |S_{\aleph_1}|$, we get $\overset{A_0}{A}, \overset{A_1}{B}$, $A \neq B$, with $A^* = B^*$.

w.l.o.g., $\exists \alpha \in A_0 \setminus A_1$.

take a surjection $f : \omega \to \mu_\alpha$ (universe of M_α).

$\mathrm{code}(\mathrm{Th}(M_\alpha^f)) \in A_0^*$. so

$\mathrm{code}(\mathrm{Th}(M_\alpha^f)) \in A_1^*$.

so $\mathrm{th}(M_\alpha^f) = \mathrm{th}(M_\beta^g)$, some $\beta \in A_1$, $g \in \mu_\beta^\omega$.

so $\beta \neq \alpha$.

let $F = \mathrm{Hull}^{M_\alpha}(\mathrm{ran}(f)) \prec M_\alpha$

$G = \mathrm{Hull}^{M_\beta}(\mathrm{ran}(g)) \prec M_\beta$

then $F \cong G \prec M_\beta$.

$F = M_\alpha$ (f surj.)

i.e.

$\exists$ e.e. $M_\alpha \to M_\beta$.

$\dashv$ (claim, lemma 2)

recall the main thm: equiconsistent are:

1. ZFC + $\exists$ gen. vopenka cardinal
2. ZF + DC + $S_{\aleph_1} = \undertilde{\Sigma}^1_2$ + $\theta = \aleph_2$
3. ZF + DC + $2^{\aleph_1} \not\leq |S_{\aleph_1}|$.

proof: con (1) $\Rightarrow$ con (2)

let κ be gen. Vopěnka in L. let $G \subset \mathrm{Col}(\omega, <\kappa)$ be L-gen. ~~ET~~ WTS $L(\mathbb{R})^{L[G]} \models$ (2).

$\aleph_1^{L(\mathbb{R})^{L[G]}} = \aleph_1^{L[G]} = \kappa$.

$\aleph_2^{L[G]} = \aleph_2^{L(\mathbb{R})^{L[G]}} = \kappa^+$.

$L[G] \models CH$, so $L(\mathbb{R})^{L[G]} \models \Theta = \aleph_2$.

let $A \in L(\mathbb{R})^{L[G]}$ be $\aleph_1$-suslin, or just λ-suslin. $A = p[T]$, some $T \in L(\mathbb{R})^{L[G]}$ on $\omega \times \lambda$, $\lambda \in OR$.

T $OD_x^{L(\mathbb{R})^{L[G]}}$, some $x \in \mathbb{R}^{L[G]}$.

WLOG, x codes $G \cap \mathrm{Col}(\omega, <\alpha+1)$, some $\alpha < \kappa$.

so $L[x] = L[G \cap \mathrm{Col}(\omega, <\alpha+1)]$.

by homogeneity, $T \in L[x]$.

κ gen. Vopěnka in $L[x]$ (preserved by small forcing)

by lem 1, $\underset{\substack{\shortparallel \\ A}}{p[T]}$ is $\utilde{\Sigma}^1_2$.

2 $\Rightarrow$ 3 easy.

Con (3) $\Rightarrow$ Con (1) : assume 3. by lem 3, $\aleph_1^V$ is generic ropeška - like in L. by DC, $\aleph_1^V$ is repla in L. so $\aleph_1^V$ is generic ropeška in L.

trevor 4.

recall: TFA equiconsistent.

(1) ZFC + $\exists$ gen. vopěnka cardinal

(2) ZF + DC + $2^{\aleph_1} \not\leq |S_{\aleph_1}|$

$$S_{\aleph_1} = \{A \subset {}^\omega\omega : A \text{ is } \omega_1\text{-Suslin}\}$$

recall: ZF + $2^{\aleph_1} \not\leq |S_{\aleph_1}| \Rightarrow$
$\aleph_1^V$ is gen. vopěnka-like in L.

defn. κ is gen. vopěnka-like ~~(gv)~~ (gv-like) if f.a. first order lang $\mathcal{L}$, $|\mathcal{L}| < \kappa$, and for every seq. $(M_\alpha : \alpha < \kappa)$ of $\mathcal{L}$-structures, $|M_\alpha| < \kappa$, then $\exists$ $\alpha < \beta < \kappa$ + a gen. e.e. $j : M_\alpha \longrightarrow M_\beta$.
(w.l.o.g., in $V^{Col(\omega, M_\alpha)}$).

recall. gv-like $\Rightarrow$ strong limit.

fact. gv-like $\Rightarrow$ $|V_\kappa| = \kappa$

so can replace $|M_\alpha| < \kappa$ by $M_\alpha \in V_\kappa$.

defn. κ is gen. vopěnka (gv) if it is gv-like + regular ($\Rightarrow$ inaccessible) we'll show for a singular cardinal κ, TFAE:

① κ is gr-like

② κ is a lim of ω-erdős cardinals

moreover, the least gr-like cardinal is regular (+ hence gr).

<u>def</u>. (baumgartner) η is ω-erdős if f.a. club $C \subset \eta$ + any regressive $f : [C]^{<\omega} \to \eta$,

$f(a) < \min(a)$

f has a homogeneous subset of otp ω. ($f \upharpoonright [C]^n$ is constant f.a. $n < \omega$).

<u>facts</u>. ① η ω-erdős $\Rightarrow$ inaccessible.

$\forall \alpha < \eta \quad \eta \to (\omega)^{<\omega}_{\alpha}$.

(for all $f : [\eta]^{<\omega} \to \alpha$, f has a hom. set of ~~otp~~ otp ω).

② $\forall \alpha \geq 2$ the least η s.t. $\eta \to (\omega)^{<\omega}_{\alpha}$ is ω-erdős (schmerl)

virtual large cardinal characterization.

<u>lem</u>. TFAE, for $2 \leq \alpha < \eta$.

① $\eta \to (\omega)^{<\omega}_{\alpha}$.

(2) $\forall A \subset V_\eta \ \exists$ gen. e.e. $j : (V_\eta ; \in, A) \to (V_\eta ; \in, A)$
with $\mathrm{crit}(j) > \alpha$.

proof (1)⇒(2): see Gitman–Schindler, "virtual large cardinal" for $A = \emptyset$.

(2)⇒(1): let $f : [\eta]^{<\omega} \to \alpha$.
by (2), $\exists$ gen. e.e. $j : (V_\eta ; \in, f) \to (V_\eta ; \in, f)$ with $\kappa = \mathrm{crit}(j) > \alpha$. $\kappa_n = j^n(\kappa)$.

$(\kappa_n : n < \omega)$ is hom. for f (silver).

example: $f(\kappa_0, \kappa_3) = f(\kappa_{10}, \kappa_{18})$.

$\forall \xi < \kappa_0 \quad f(\xi, \kappa_0) = f(\xi, \kappa_5)$ by elt. of j^5.

$\Rightarrow \quad f(\kappa_0, \kappa_3) = f(\kappa_0, \kappa_8)$ by elt. of j^3

$\qquad\qquad\qquad\qquad \parallel$

$\qquad f(\kappa_{10}, \kappa_{18})$ by elt. of j^{10}.

existence of hom. set for f of otp ω is absolute.

<u>proposition</u>. let λ be ω-erdős, or a lim of ω-erdős cardinals.

(1) λ is gr-like.

(2) λ is a lim of gr cardinals.

proof: hypothesis $\Longrightarrow$ $\forall \alpha < \lambda \quad \lambda \to (\omega)^{<\omega}_{\alpha}$.

(1) ~~...~~ for $\mathcal{L}$, $\vec{M}$ as in defn. of gv-like, by the lea, for gen. e.e. $j : (V_\lambda ; \in, \vec{M}) \to (V_\lambda ; \in, \vec{M})$, $\mathrm{crit}(j) > |\mathcal{L}|$. let $\kappa = \mathrm{crit}(j)$.

$$j \restriction M_\kappa : M_\kappa \Longrightarrow j(M_\kappa) = M_{j(\kappa)} \text{ is a}$$

gen. el. embedding.

(2) follows from lea + proof that ω-huge $\Rightarrow$ vopenka. (don't need A.) $\dashv$

<u>Corollary</u>. (1) least limit of ω-erdős cardinals is ~~a~~ generic vopenka-like and singular.

(2) least inaccessible limit of ω-erdős cardinals is gv but not mahlo.

$\Rightarrow$ ω-erdős cardinals are necessary here.

<u>defn.</u> κ is ~~...~~ <u>club gv</u> iff κ is gv-like, ~~...~~ witnessed by elt. embeddings with critical point in any given club.

(this is equivalent to $(V_\kappa ; V_{\kappa+1} ; \in) \models gVP^*$,

bagaria-gitman-sch

and also equivalent to κ is a virtual woodin cardinal.)

<u>note</u>. for $M_\alpha = (V_{\alpha+1}; \in, \{\alpha\})$, critical pt. of any gen. e.e. $M_\alpha \to M_\beta$ ex. and is weakly compact.

so: ① κ is gr-like $\Rightarrow$ κ is a limit of weakly compact cardinals.

② κ is club gr-like $\Rightarrow$ κ is a stat. limit of weakly compacts

$\Rightarrow$ κ is mahlo

$\Rightarrow$ κ is regular.

<u>th.</u> assume λ is gr-like, but not club gr. (e.g., λ is singular, or regular but not mahlo), then λ is a limit of ω-erdős cardinals.

proof: if not, then $\exists \alpha_0 < \lambda \; \forall \gamma < \lambda \; \gamma \not\to (\omega)^{<\omega}_{\alpha_0}$.
by lemma, can choose $A_\gamma \subset V_\gamma$ s.t.
$\neg\exists$ gen. e.e. $(V_\gamma; \in, A_\gamma) \longrightarrow (V_\gamma; \in, A_\gamma)$ with crit $> \alpha_0$. let $\vec{A} = (A_\gamma : \gamma < \lambda)$.

let $\mathcal{L}, \vec{M}$ + a club $C \subset \lambda$ witness λ is not club gr.

for $\alpha < \lambda$, let $\gamma_\alpha = \min(C \setminus \alpha)$

$$\rho_\alpha = \max(\gamma_\alpha, \text{rank } M_\alpha)$$

$$N_\alpha = \left(V_{\rho_\alpha + 1} ; \in, \{\alpha\}, \vec{M} \restriction \alpha+1, \left(C \cap (\gamma_\alpha + 1)\right), \vec{A} \restriction \gamma_\alpha + 1, (\bar{\xi})_{\xi \leq \max(\alpha_0, |\mathcal{L}|)} \right).$$

λ gr-like, so $\exists \alpha < \beta < \lambda \; \exists$ gen. e.e. $j : N_\alpha \to N_\beta$ with $\alpha_0 < \text{crit}(j) \leq \alpha$.

$j \restriction M_\alpha : M_\alpha \longrightarrow M_\beta$ is a gen. e.e., so $\text{crit}(j) \notin C$

let $\kappa = \text{crit}(j)$.

γ_κ = least elt. of $C \geq \kappa$ + here $> \kappa$.

γ_κ' = C-pred. of γ_κ. $\gamma_\kappa' < \kappa$.

$j(\gamma_\kappa') = \gamma_\kappa'$ + j pres. C.

so $j(\gamma_\kappa) = \gamma_\kappa$. so $j \restriction V_{\gamma_\kappa}$ is a gen. e.e.

$$(V_{\gamma_\kappa} ; \in, A_{\gamma_\kappa}) \longrightarrow (V_{\gamma_\kappa} ; \in, A_{\gamma_\kappa}).$$ Contradiction.