hugh woodin 1.

$V \subset V[G]$.

def. (hamkins) supp. δ is reg., $\delta > \omega$,

$N \subset V$ is an inner model of ZFC. (1) N has the δ-cover property iff f.a. $\sigma \subset N$, $|\sigma| < \delta$, there is $\tau \in N$, $\sigma \subset \tau$, $|\tau| < \delta$.

(2) N has the δ-approximation property iff f.a. $X \subset N$, if $X \cap \tau \in N$ for all $\tau \in N$, $|\tau| < \delta$, then $X \in N$.

definition. N is a weak extender model of δ is supercompact iff f.a. $\lambda > \delta$ there is a δ-complete normal fine ultrafilter $\mathcal{U}$ on $\mathcal{P}_\delta(\lambda)$ s.t.

(1) $\mathcal{P}_\delta(\lambda) \cap N \in \mathcal{U}$

(2) $\mathcal{U} \cap N \in N$.

~~(hamkins - jonas)~~

remark. supp. N is a inner model of V.

N has the ω cover-property.

N has the ω-appr. property iff $N = V$.

<u>lem</u>. (hawkins – jonas)

supp. N has the δ-cover + δ approximation property. then N has the κ-cover + κ-approximation property.

proof: N has the κ-appr. property. ✓

it suff. to show:

<u>claim</u>. for all cardinals $\lambda \geq \delta$, if $\sigma \subset N$, $|\sigma| = \lambda$, then there is $\tau \in N$, $\sigma \subset \tau$, $|\tau| = \lambda$.

prove the claim by induction on λ.

assume it holds f.a. cards. $< \lambda$.

fix a set $x \subset N$, $|x| = \lambda$. fix a large V_θ s.t. $|V_\theta| = \theta$, $x \in V_\theta$.

we have that f.a. $Y \subset N$, if $|Y| < \lambda$, then $\exists Z \in N$ s.t. $Y \subset Z$, $|Z| < \lambda$.

let $\lambda_0 = cf(\lambda)$.

build an elementary chain, $\langle Z_\alpha : \alpha < \lambda_0 \rangle$, s.t. $|Z_\alpha| < \lambda$, $Z_\alpha \prec V_\theta$, $x \in Z_0$.

and s.t. f.a. $\alpha < \lambda_0$,

$\exists Y_\alpha \in N$ s.t. $Z_\alpha \cap N \subset Y_\alpha \subset Z_{\alpha+1}$,

$|Y_\alpha| < \lambda$.

uses $<\lambda$-cov. property.

$Z = \bigcup_{\alpha<\lambda_0} Z_\alpha$. can also arrange $x \subset Z$.

first arrange $\lambda \subset Z$.

$Z \cap N = \bigcup_{\alpha<\lambda_0}$

<u>case 1.</u> $\lambda_0 \geq \delta$.

then $\bigcup_{\alpha<\lambda_0} Y_\alpha \in N$ by δ-appr. property:

$\sigma \in N$, $|\sigma| < \delta \Rightarrow \sigma \cap Y = \sigma \cap Y_\alpha$ f.a.

suff. large $\alpha < \lambda_0$.

<u>case 2.</u> $\lambda_0 < \delta$.

consider $(Y_\alpha : \alpha < \lambda_0)$.

$(Y = \bigcup Y_\alpha$, $Y_\alpha \subset Y_\beta$ $\forall \alpha < \lambda_0$.

$X \subset Y$.)

by δ-cover, have $\tau \in N$ s.t. $Y_\alpha \in \tau$ $\forall \alpha < \lambda$

and $|\tau| < \delta$.

let $A = \bigcup\{B \in \tau : |B| < \lambda\}$.

$A \in N$, $|A| \not\leq \lambda$, and $Y \subset A$.

thus $X \subset Y$, so $X \subset A$. $\dashv$

corollary. Supp. N has the δ-cover + δ-appr.

1. supp. $\gamma > \delta$ is a singular cardinal, then γ is singular in N, and $\gamma^+ = \gamma^{+N}$.

2. supp. $\gamma \geq \delta$ is a regular cardinal. then either $\gamma^+ = \gamma^{+N}$ or else $cf(\gamma^{+N}) = \gamma$.

3. supp. $\gamma \geq \delta$ is a reg. cardinal of N. then $cf^V(\gamma) = |\gamma|^V$.

theorem (hamkins) supp. N has the δ-cover + δ-appr. properties. supp. $\mathcal{U}$ is a δ-complete u.f. on some $I \in N$. then $\mathcal{U} \cap N \in N$.

proof: by δ approximation, it suff. to show ~~that~~ $\mathcal{U} \cap \sigma \in N$ f.a. $\sigma \in N$, $|\sigma| < \delta$.

may assume $A \subset I$, $A \in \sigma \Rightarrow I \setminus A \in \sigma$.

$\mathcal{U}$ is δ-complete. so $\bigcap \mathcal{U} \cap \delta \in \mathcal{U}$.

choose $a \in \bigcap \mathcal{U} \cap \delta$ and

$\mathcal{U} \cap \sigma = \{ \cancel{A \in} \; A \subset I : A \in \sigma \wedge a \in A \}$.

so $\mathcal{U} \cap \sigma \in N$. $\dashv$

<u>corollary</u>. supp. N has the δ-cover + δ-appr. properties. supp. $j : N \longrightarrow M$ is an elementary embedding with $\mathrm{crit}(j) \geq \delta$. then j is "close" to N. (i.e., all ultrafilters of derived extenders are in N.)

the proof just uses that $\mathcal{U}$ is an N-ultrafilter which is N-δ-complete.

<u>corollary (hamkins)</u> supp. N has δ-cover + δ-approximation properties. supp. $\kappa \geq \delta$ is strongly compact. then κ is strongly compact in N.

pf.: fix $\lambda > \delta$, and let $I = \mathcal{P}_\kappa(\lambda) \cap N$. let F be the filter gen. by the sets

$A_\sigma = \{ \tau \in I : \sigma \subset \tau \}$, $\sigma \in \mathcal{P}_\kappa(\lambda)$.

by κ-core, F is a κ-complete filter. since κ is strongly compact, F can be extended to a κ-complete ultrafilter $\mathcal{U}$. so $\mathcal{U} \cap N \in N$. thus κ is λ-strongly compact in N.

th. (universality, version 1). Supp. N has the δ-cover and δ-appr. properties. Supp. E is a N-extender of length η s.t.

① $\kappa_E = \text{crit}(E) \geq \delta$,

② η is a cardinal of $N_E = \text{ult}(N; E)$.

the following are equivalent.

① $E \in N$.

② $j_E(A) \cap \eta \in N$ f.a. $A \subset OR$, $A \in N$.

③ $N_E \subset N$.

proof: it suff. to show ② $\Rightarrow$ ①.

~~[illegible]~~

let γ be least s.t. $j_E(\gamma) \geq \eta$.

define F as follows.

$$\mathrm{dom}(F) = \mathcal{P}([\gamma]^{<\omega}) \cap N$$

$$F(A) = j_E(A) \cap [\eta]^{<\omega}.$$

suff. to show $F \in N$.

<u>claim</u>. $F \subset N$.

pf.: fix $A \subset [\gamma]^{<\omega}$, $A \in N$.

we must show $F(A) \in N$.

γ is an N-cardinal.

choose $B \subset \gamma$ s.t. f.a. cardinals $\varepsilon \leq \gamma$,

$A \cap [\varepsilon]^{<\omega} \in L[B \cap \varepsilon]$.

by (2), $j_E(B) \cap \eta \in N$.

so since η is an N_E-cardinal,

$j_E(A) \cap [\eta]^{<\omega} \in L[j_E(B) \cap \eta] \subset N$. ⊣(claim)

now to show $F \in N$, it suff. by δ-appr. to show $F \cap \sigma \in N$ for all $\sigma \in N$ with $|\sigma| < \delta$.

fix a bij. $\rho : |\sigma|^N \to \sigma$, $\rho \in N$.

choose $A \in \mathcal{P}([\delta]^{<\omega}) \cap N$ s.t.

f.a. $\xi \in |\sigma|^N$, if $\rho(\xi) \in \sigma \cap \mathrm{dom}(F)$,

then $\{\rho^\frown s : s \in \rho(\xi)\} \in A$.

(so $A \in \mathcal{P}([\delta]^{<\omega}) \cap N$ and A codes nicely $\mathrm{dom}(F) \cap \sigma$.)

but we have $j_E(A) \cap [\eta]^{<\omega} \in N$.

further, $\mathrm{crit}(j_E) \geq \delta > |\sigma|^N = \mathrm{dom}(\rho)$.

so now it follows

$F''(\sigma \cap \mathrm{dom}(F))$ can be computed in N

from $j_E(A) \cap [\eta]^{<\omega}$.

$\mathrm{dom}(j_E(\rho)) = \mathrm{dom}(\rho)$.

$j_E(\sigma)(\xi) = j_E(\sigma(\xi))$. $\dashv$

<u>thm.</u> (uniqueness)

supp. N_0, N_1 each have the δ-core, δ-approx. property.

Supp. $N_0 \cap H_{\delta^+} = N_1 \cap H_{\delta^+}$.

then $N_0 = N_1$.

pf.: it suff. to show $\mathcal{P}(\alpha) \cap N_0 = \mathcal{P}(\alpha) \cap N_1$.

fix α, and supp. $\mathcal{P}(\beta) \cap N_0 = \mathcal{P}(\beta) \cap N_1$ f.a. $\beta < \alpha$. ~~the~~

~~then~~ fix $X \in \mathcal{P}(\alpha) \cap N_0$. we prove $X \in N_1$.

let $\mathbb{M} = (\alpha, \bigcup \{\mathcal{P}(\beta) \cap N_0 : \beta < \alpha\})$.

$\mathbb{M} \in N_0 \cap N_1$.

since N_1 has the δ-appr. property we can reduce to the case $cf(\alpha) < \delta$.

build an elt. chain ~~...~~ $(\sigma_\alpha : \alpha < \delta)$ of el. substructures of $\mathbb{M}$ s.t.

(1) $\sigma_0 \in N_0$, $\sup(\sigma_0 \cap \alpha) = \alpha$,

$X \cap \beta \in \sigma_0$ $\forall \beta \in \sigma_0 \cap \alpha$, $|\sigma_0| < \delta$.

② $\cancel{\sigma_{\rho_1}}$ $\sigma_{\rho_0} \subset \sigma_{\rho_1}$, $\rho_0 < \rho_1$.

③ $|\sigma_\rho| < \delta$ $\forall \rho < \delta$

④ $\{\rho < \delta : \sigma_\rho \in N_0\}$ is cofinal in δ

⑤ $\{\rho < \delta : \sigma_\rho \in N_1\}$ — " — .

let $Z = \bigcup_{\rho < \delta} \sigma_\rho \prec \mathbb{M}$.

by δ-approx. + ④ + ⑤ , $Z \in N_0 \cap N_1$.

let M be the tr. collapse of ~~α~~ Z,

let $\bar\alpha$ be the image of α under the tr. coll.

let $\bar x \subset \bar\alpha$ be the image of x under the tr. coll. (so $\bar x \in N_0$ since ~~α~~ $x \in N_0$).

$\bar\alpha < \delta^+ \Rightarrow \mathcal{P}(\bar\alpha) \cap N_0 = \mathcal{P}(\bar\alpha) \cap N_1$.

$\Rightarrow \bar x \in N_1$.

let $\bar\alpha \longrightarrow \alpha$

$\pi_1 : M \longrightarrow Z \prec \mathbb{M}$ be the uncoll. map .

$$X = \bigcup_{\beta < \bar{\alpha}} \{ u \in Z : u \cap \pi''\beta = \pi''(\bar{x} \cap \beta) \}.$$

so $X \in N_1$. $\dashv$

<u>cor.</u> to the proof.

supp. N has δ-cover, δ-appr. Supp. $\gamma > \delta$ is a strong lim cardinal.

then $N \cap H_\gamma$ is uniformly def. in H_γ from $N \cap H_{\delta^+}$.

$\Rightarrow$ N is Σ_2 def. the in V from $N \cap H_{\delta^+}$.

<u>thm.</u> (universality, version 2)

supp. δ has δ-cover, and δ-appr. Supp. $\kappa > \delta$ is extendible. then κ is extendible in N.

proof: fix $\lambda > \delta$ s.t. $cf(\lambda) > \omega$, $|V_\lambda| = \lambda$. κ is extendible, so

$$\exists\, \pi : V_{\lambda+1} \longrightarrow V_{\pi(\lambda)+1},$$

$$\operatorname{crit}(\pi) = \kappa, \quad \pi(\kappa) > \lambda.$$

let E be the V-extender of length $\pi(\lambda)$ given by π. let $j_E : V \longrightarrow M_E = \operatorname{ult}(V; E)$

so $j_E(V_\lambda) = \pi(V_\lambda) = V_{\pi(\lambda)}$.

$j_E \restriction V_\lambda = \pi \restriction V_\lambda$.

<u>claim</u>: $j_E(N) \cap V_{\pi(\lambda)} = N \cap V_{\cancel{j_E}\cdot\pi(\lambda)}$.

$\operatorname{crit}(\cancel{j}\,\pi) > \delta$.

$j_E(N) \cap H_{\delta^+} = \cancel{j_E}\; N \cap H_{\delta^+}$.

N has the δ-cover + δ appr. property in V,

hence ~~...~~ in $V_{\cancel{j}\,\pi(\lambda)}$.

further by elt., $j_E(N) \cap V_{\pi(\lambda)}$ has the δ-cover + δ-appr. property in $V_{\pi(\lambda)}$.

$\Rightarrow$ $j_E(N) \cap V_{\pi(\lambda)} = N \cap V_{\pi(\lambda)}$.

thus f.a. cardinals $\theta < j(\lambda)$,

$$(E|\theta) \cap N \in N.$$

assume further $cf(\lambda) \geq \delta$, so $cf(j(\lambda)) \geq \delta$.

$\Rightarrow$ $E \cap N \in N$.

$\Rightarrow$ witness α is λ-extendible in V.

<u>thm.</u> (universality, version 3, hamkins)

supp. N has δ-cover, δ appr. supp.

$j : V \to M$ is an elt. embedding s.t.

$crit(j) > \delta$ + ${}^{\delta}M \subset M$. then f.a. $\gamma \in OR$,

$$j \upharpoonright (N \cap V_\delta) \in N.$$

(i.e., $j \upharpoonright N$ is amenable to N).

in fact, if j is given by some extender E, then $j \upharpoonright N$ is given by an internal N-ultrapower by some extender in N.

<u>defin.</u> supp. $\mathcal{U}$ is an ultrafilter on the index set $\mathcal{P}(\lambda)$.

$\mathcal{U}$ is <u>fine</u> iff f.a. $\alpha < \lambda$,

$\{A \subset \lambda : \alpha \in A\} \in \mathcal{U}$

$\mathcal{U}$ is <u>normal</u> iff for all $F : \mathcal{P}(\lambda) \to \lambda$,
if $\{A : F(A) \in A\} \in \mathcal{U}$, then
$\exists \alpha \ \{A : F(A) = \alpha\} \in \mathcal{U}$.

<u>cor.</u> supp. N has δ-cover, δ-appr. supp. $\lambda > \delta$, $\mathcal{U}$ is a δ^+-complete normal fine ultrafilter on $\mathcal{P}(\lambda)$, then:

① $N \cap \mathcal{P}(\lambda) \in \mathcal{U}$.

② $\mathcal{U} \cap N \in N$.

pf.: let $j_\mathcal{U} : V \to M_\mathcal{U} = \mathrm{ult}(V; \mathcal{U})$ be the ultrapower embedding. then

① $(M_\mathcal{U})^\delta \subset M_\mathcal{U}$, $\mathrm{crit}(j_\mathcal{U}) > \delta$

② $j_\mathcal{U}{}'' \lambda \in M_\mathcal{U} \Rightarrow M_\mathcal{U}{}^\lambda \subset M_\mathcal{U}$.

thus by hamkins' thm., $j_\mathcal{U} \restriction N_\gamma \in N$ for all γ.

it suff. to show $j_\mathcal{U}{}'' \lambda \in j_\mathcal{U}(N)$.

$M_\mathcal{U} \models$ "$j_\mathcal{U}(N)$ has the δ-appr. property."

let $\sigma \subset j_u(\lambda)$ be a set in $j_u(N)$,

$|\sigma| < \delta$. we need to see

$$j''\lambda \cap \sigma \in j_u(N) .$$

$j_u(N) \subset N$, so $\sigma \in N$.

$\sigma \cap j''\lambda \in N$, since $j_u \restriction N$ is amenable to N.

but $\mathcal{P}(\sigma) \cap N = \mathcal{P}(\sigma) \cap j_u(N)$

$\Rightarrow$ ~~j~~ $j''\lambda \cap \sigma \in j_u(N)$. $\dashv$

hugh woodin, july 25

th. (universality, version 3: hashing)

supp. N has δ-cover + δ-approx. supp.

$j_\kappa : V \longrightarrow M_\kappa$, $\underset{\underset{j_\kappa}{\shortparallel}}{\mathrm{crit}(j)} > \delta$, ${}^{\delta}M_\kappa \subset M_\kappa$.

then ① $j_\kappa(N) \subset N$

② $j \upharpoonright (N \cap V_\gamma) \in N \quad \forall \gamma$

~~to~~ note: it suff. to prove ①, since ① $\Rightarrow$ ②, by univ. th., version 1.

th. (uniqueness th., version 2)

supp. N has δ-cover, δ-approx. supp. $M \subset V$, ${}^{\delta}M \subset M$. supp. $N_0 \subset M$ and N_0 has δ-cover, δ-appr. in M.

supp. $N_0 \cap H_{\delta^+} \subset N$. then $N_0 \subset N$.

[gives unif. th., version 1, by taking $M = V$.

also gives th. above by taking $N_0 = j(N)$.]

proof: it suff. to show that f.a. α, $\mathcal{P}(\alpha) \cap N_0 \subset N$.

for $\alpha \leq \delta^+$ this is immediate, since $N_0 \cap H_{\delta^+} \subset N$ and since N has the δ-appr.

fix $X \subset \alpha$, $X \in N_0$, and supp. $X \cap \beta \in N$ $\forall \beta < \alpha$.

clearly we can reduce to the case $cf(\alpha) < \delta$, as N has the δ-approx. property.

note since ${}^{\delta}M \subset M$, N_0 has the δ-cover property in V. fix γ very large, $|V_\gamma| = \gamma$, $X \in V_\gamma$. build an iter.

$$(\tau_\eta : \eta < \delta)$$

of elt. substructures of $(V_\gamma ; N_0, N)$ together with $((\tau_\eta^N, \tau_\eta^{N_0}) : \eta < \delta)$ s.t for all η:

(1) $X \in \tau_0$, $\tau_0 \cap \alpha$ is cofinal in α

(2) $|\tau_\eta| < \delta$

(3) $\tau_\eta^N \in N$,

$\tau_\eta^N \prec N \cap V_\gamma$ and

$\tau_\eta^N \prec N \cap V_\gamma$ and

$\tau_\eta \cap N \subset \tau_\eta^N \subset \tau_{\eta+1}^N$

(4) $\tau_\eta^{N_0} \in N_0$, $\tau_\eta^{N_0} \prec N_0 \cap V_\gamma$,

$\tau_\eta \cap N_0 \subset \tau_\eta^{N_0} \subset \tau_{\eta+1}$.

let $\tau = \bigcup_{\eta<\delta} \tau_\eta$, $\tau_N = \tau \cap N = \bigcup_{\eta<\delta} \tau_\eta^N$

$\tau_{N_0} = \tau \cap N_0 = \bigcup_{\eta<\delta} \tau_\eta^{N_0}$

by δ-appr., $\tau_N \in N$.

$(\tau_\eta^{N_0} : \eta < \delta) \in M$, since ${}^\delta M \subset M$.

N_0 has δ-appr. in M. so $\tau_{N_0} \in N_0$.

let M_τ be the tr. collapse of τ.

let α_τ be the image of α,

X_τ be the image of X.

$\alpha_\tau < \delta^+$, $X_\tau \subset \alpha_\tau$.

let M_{τ_N} = coll. of τ_N

$M_{\tau_{N_0}}$ = coll. of τ_{N_0}

so $X_\tau \in N_0$ $\quad (X_\tau \in M_{\tau_{N_0}})$

$N_0 \cap H_{\delta^+} \subset N$, so $X_\tau \in N$.

for all $\beta \in \tau \cap \alpha$, $X \cap \beta \in \tau_N$.

further $\tau \cap \alpha$ is cofinal in α. let

$\pi : M_{\tau_N} \longrightarrow \tau_N$ be the uncollapse.

for all $\xi < \alpha_\tau$, $X_\tau \cap \xi \in M_{\tau_N}$,

so $X = \cup \{ \pi(X_\tau \cap \xi) : \xi < \alpha_\tau \}$

$$\underset{\displaystyle X \cap \pi(\xi)}{\overset{\pi(X_\tau \cap \xi)}{\|}}$$

$\Longrightarrow X \in N$. $\quad \dashv$

def. Supp $\delta > \omega$ is a regular cardinal. supp. $N \subset V$ is an in model of ZFC with $OR \subset N$, then N has the δ-genericity property if $\forall A \subset \alpha < \delta$, A is N-gen. for some $\mathbb{P} \in N$, $|\mathbb{P}| < \delta$.

we are going to consider N with all three properties: δ-cover, δ-approx., δ-genericity.

th. Supp. N is a weak extender model of δ is supercompact. then N has the δ-cover + δ-appr.

lem. (magidor) Supp. δ is inacc. TFAE.

① δ is supercompact.

② f.a. $\lambda > \delta$ $\exists$ $\bar{\delta} < \bar{\lambda} < \delta$ and an el. $\pi : V_{\bar{\lambda}+1} \longrightarrow V_{\lambda+1}$ $\mathrm{crit}(\pi) = \bar{\delta}$, $\pi(\bar{\delta}) = \delta$.

<u>th.</u> supp. N is a weak extender model for δ is supercompact.

then f.a. $\lambda > \delta$, f.a. $A \in V_\lambda$

$\exists\ \bar\delta < \bar\lambda$ $\bar A \in V_{\bar\lambda}$ and an elt. emb.

$\pi : V_{\bar\lambda+1} \longrightarrow V_{\lambda+1}$ s.t.

(1) $\operatorname{crit}(\pi) = \bar\delta$, $\pi(\bar\delta) = \delta$, $\pi(\bar A) = A$.

(2) $\pi(N \cap V_{\bar\lambda}) = N \cap V_\lambda$,

(3) $\pi \restriction N \cap V_{\bar\lambda} \in N$.

proof: attempt 1.

assume $|V_\lambda| = \lambda$. let $\mathcal{U}$ be a δ-complete normal fine ~~measure~~ ultrafilter on $\mathcal{P}_\delta(2^\lambda)$ s.t. $\mathcal{P}_\delta(2^\lambda) \cap N \in N$, $\mathcal{U} \cap N \in N$.

let $j : V \longrightarrow M_u = \operatorname{ult}(V; u)$,

$\operatorname{crit}(j) = \delta$, $j(\delta) > \lambda$

${}^{2^\lambda}M_u \subset M_u$, so ${}^{V_{\lambda+1}}M_u \subset M_u$.

so $j \restriction V_{\lambda+1}$ witnesses existence of π at $j(\delta)$, $j(\lambda+1)$.

<u>problem</u>: is $j(N) \cap V_\lambda = N \cap V_\lambda$?

<u>question</u>: Supp. $|V_\lambda| = \lambda$, $\mathcal{U}$ is a

δ-complete normal fine ultrafilter on $\mathcal{P}_\delta(\lambda)$.

$N \cap \mathcal{P}_\delta(\lambda) \in \mathcal{U}$, and $\mathcal{U} \cap N \in N$.

must $j_{\mathcal{U}}(N) \cap V_\lambda = N \cap V_\lambda$.

choose a bijection $\rho : \lambda \leftrightarrow N \cap V_\lambda$, $\rho \in N$.

need to show

$X = \{ \sigma \in \mathcal{P}_\delta(\lambda) :$ ① $\rho''\sigma \prec N \cap V_\lambda$

② the trans. coll. of $\rho''\sigma$ is ~~not~~ a rank initial seg. of N $\}$

$\in \mathcal{U}$.

but $\mathcal{U} \cap N \in N$. $X \cap N \in N$, since $\rho \in N$.

So if $X \notin \mathcal{U}$, then

$N \models$ " $X \cap N \notin \mathcal{U} \cap N$ " $\S$. $\dashv$

So answer to <u>question</u> is "yes."

<u>thm.</u> supp. N is a weak extender model of δ supercompact. then N has the δ-cover + δ-appr. properties.

pf.: δ-cover is immediate.

to show the δ-appr. prop.:

fix $X \subset N$ s.t. $X \cap \sigma \in N$ f.a. $\sigma \in N$, $|\sigma| < \delta$. choose a large λ with $X \in V_\lambda$. then ex. $\bar\delta < \bar\lambda < \delta$, $\bar X \in V_{\bar\lambda}$ and an elt. emb. $\pi : V_{\bar\lambda+1} \longrightarrow V_{\lambda+1}$ s.t.

① $\mathrm{crit}(\pi) = \bar\delta$, $\pi(\bar\delta) = \delta$, $\pi(\bar X) = X$,

② $\pi(V_{\bar\lambda} \cap N) = V_\lambda \cap N$.

③ $\pi \upharpoonright V_{\bar\lambda} \cap N \in N$.

let $\sigma = \pi'' N \cap V_{\bar\lambda}$, $\in N$, and $|\sigma| < \delta$. so $X \cap \sigma \in N$.

but $\bar X = \{ u \in V_{\bar\lambda} \cap N : \pi(u) \in X \cap \sigma \}$.

so $\bar X \in N$. so $\bar X \in N \cap V_{\bar\lambda}$, so $\underset{X}{\underset{\shortparallel}{\pi(\bar X)}} \in N$. ⊣

<u>Th.</u> (Goldberg) Sup. δ is supercompact.
then $\exists$ ile model N with
δ – cover + δ – appr. s.t.
$\{\kappa < \delta : \kappa$ is measurable in $N\}$
is not stationary (and maybe not stat.).

Suitable extender models are weak extender models of δ is supercompact for which the following holds at δ_N when δ_N is the least δ s.t. N is a weak extender model ~~at~~ of δ is supercompact.
~~but~~ then ex. a seq. $(E_\alpha : \alpha < \delta_N)$ of short extenders s.t.

① $(E_\alpha \cap N : \alpha < \delta_N) \in N$

② $(E_\alpha \cap N : \alpha < \delta_N)$ witnesses in N that δ_N is woodin.

$\Longrightarrow$ δ – genericity.

Questions.

① Supp. N is a weak extender model of δ-supercompact. must N have the δ-gen. property ?

② assume large cardinals and supp. N is a weak extender model of δ-supercompact, must there ex. κ s.t. N is a weak ext. model of κ is supercompact and s.t. N has the κ-gen. property ?

Thm. Supp. δ is supercompact. let $\kappa < \delta$ be measurable, and let $\mathcal{U}$ be a normal measure on κ. let $N = \mathrm{Ult}(V; \mathcal{U})$.

then ① N is a weak extender model of δ is supercompact.

② $\mathcal{U}$ is not set gen. over N.

proof: ② must hold:

consider $N[\mathcal{U}]$. ${}^{\kappa}N \subset N$.

$\Rightarrow$ $j_u \upharpoonright OR$ is amenable to $N[u]$.

$\Rightarrow$ $\mathcal{P}(\alpha) \subset N[u]$ $\forall \alpha$.

$\Rightarrow$ $N[u] = V$:

$A \subset \alpha$.

$A = \{\eta < \alpha :$

$j_u(\eta) \in j_u(A)\}$.

Th. ~~Supp. LN is a weak~~ Supp. δ is supercompact.

① (version 1) Supp. $E \in V_\delta$ is an extender. then $ult(V; E)$ is a weak extender model of δ is supercompact.

② (version 2) ~~Supp.~~ Same with $E \in V[G]_\delta$, where g is V-gen., some $\mathbb{P} \in V_\delta$. then $ult(V; E)$ is a weak extender model of δ is supercompact in $V[G]$.

③ Supp. N is a weak ext. model of δ is supercompact. Supp. N satisfies δ-genericity and $E \in V_\delta$ is a N-extender.

then ult$(N; E)$ is a weak extender model of δ is supercompact.

applicatn. Supp. δ is supercompact and $\mathcal{U}$ is a normal (uniform) measure.

⋮

M_ω is a weak ext. of δ s.c. Let $\langle \kappa_i : i < \omega \rangle$ be the crit seq. of $j_\mathcal{U} : V \to M_1 = \mathrm{ult}(V; \mathcal{U})$. Let $j_{0\omega} : V \to M_\omega$ be the iterat ext. $\langle \kappa_i : i < \omega \rangle$ M_ω-gen. for prikry forcing.

$M_\omega[\langle \kappa_i : i < \omega \rangle] = \bigcap M_n$.

$M_i^\lambda \subset M_i \ \forall i < \omega$.

So ${}^\kappa N \subset N$.

N is a weak extender model of δ is supercompact. N does not have δ-ȷn. property ($N[\mathcal{U}] = V$).

note : $j_u(N) = N$.

so we have a weak extender model, N, of δ is supercompact for ${}^{\omega}N \subset N$ (and much more) ~~the~~ and $\exists j : N \longrightarrow N$

(shows $\mathrm{crit}(E) \geq \delta$ is nec. in universality thm.)

question : Supp. N is a weak extender model for δ is supercompact. Supp. N also has δ-gen. & can there ex. $j : N \longrightarrow N$?

related question : Supp. δ is extendible.

HOD hypothesis holds $\Longleftrightarrow$

HOD is a weak extender model of δ is supercompact.

but HOD has the δ-gen. princ.

question : is $j : \mathrm{HOD} \longrightarrow \mathrm{HOD}$ possible ?

proof of version 1. fix E. let $N = \mathrm{ult}(V; E)$.

it suff. to show that. f.a. $\lambda > \delta$

there ex. $\bar{\delta} < \bar{\lambda} < \delta$ s.t. $\mathrm{crit}(\pi) = \bar{\delta}$,

$\pi(\bar{\delta}) = \delta$, $\pi(N \cap V_{\bar{\lambda}}) = N \cap V_\lambda$, $\pi \restriction N \cap V_{\bar{\lambda}} \in N$.

fix $\lambda > \delta$, $|V_\lambda| = \lambda$, $cf(\lambda) \geq \delta$.

choose $\bar{\delta} < \bar{\lambda} < \delta$ and $\pi : V_{\bar{\lambda}+1} \longrightarrow V_{\lambda+1}$ s.t.

$\mathrm{crit}(\pi) = \bar{\delta}$, $\pi(\bar{\delta}) = \delta$, $E \in V_\delta$.

let $F =$ the ext. of length λ given by π.

$\mathrm{ult}(V; F)$ is well-fdd.

consider $j_F : V \longrightarrow M_F = \mathrm{ult}(V; F)$.

<u>note</u>: since $E \in V_\delta$, $N = \mathrm{ult}(V; E)$,

we must have

$\pi(N \cap V_{\bar{\lambda}}) = N \cap V_\lambda$.

<u>basic claim</u>: $j_E : V \longrightarrow N = \mathrm{ult}(V; E)$

$j_E(j_F) \restriction N = j_F$

~~question: supp. N has δ–cov + δ–appr. (maybe δ–gen.) supp. δ is strongly inaccessible.~~

let δ be str. inacc., let $N = \mathrm{ult}(V; E)$ for some extender $E \in V_\delta$. must N have δ–cov. + δ–appr.

<u>lem.</u> supp. δ is supercompact. let $\kappa < \delta$ be measurable and let $\mathcal{U}$ be a unique normal ultrafilter on κ. then $\exists$ gen. extension $V[G]$ s.t.

① ${}^{\delta}V \subset V$ in $V[G]$.

② in $V[G]$, let $N = \mathrm{ult}(V[G], \mathcal{U})$. then N does not have δ–appr. property.

<u>proof:</u> let $\lambda > \delta$ be a str. limit cardinal with $\mathrm{cf}(\lambda) = \kappa$. let $(\lambda_\alpha : \alpha < \kappa)$ be an incr. cont. seq. cof. in λ.

for each $\alpha < \kappa$, let $\mathbb{P}_\alpha$ = the forcing for adding a gen. subset of λ_α^+ by initial segm. let $\mathbb{P} = \prod \mathbb{P}_\alpha / \mathcal{U}$.

clm. $\mathbb{P}$ is (λ,∞)-distr.

proof. since λ is sing., it suff. to show $\mathbb{P}$ is (γ,∞)-distr. $\forall \gamma < \lambda$.

fix $\gamma < \lambda$. by repl. $(\lambda_\alpha : \alpha < \kappa)$ with $(\lambda_{\alpha_0+\alpha} : \alpha < \kappa)$ we can reduce to the case that $\gamma < \lambda_{\alpha_0}$. each $\mathbb{P}_\alpha$ is γ^+-closed.

consider $\prod_\kappa \mathbb{P}_\alpha$. this is γ^+-closed.

fix $(D_\eta : \eta \leq \gamma)$ open den in $\mathbb{P}$.

build a pt.wise incr. seq.

$$(f_\eta : \eta \leq \gamma) \qquad f_\eta \in \prod_\alpha \mathbb{P}_\alpha$$

$$f_\eta / \mathcal{U} \in D_\eta.$$

let $G \subset \mathbb{P}$ V-g. ${}^\lambda V \subset V$ in $V[G]$.

$H_\lambda^V = H_\lambda^{V[G]}$.

$\mathrm{ult}_0(H_\lambda ; \mathcal{U})^V = \mathrm{ult}_0(H_\lambda ; \mathcal{U})^{V[G]}$

let $M_\mathcal{U} = \mathrm{ult}(V[G] ; \mathcal{U})$

$$\lambda^{+V[G]} = \lambda^{+V} = \lambda^{+M_\mathcal{U}},$$

$$\sup j_\mathcal{U}{}'' \lambda = \lambda.$$

so G gives a M_u-gen. subset of $(\lambda^+)^{M_u}$.

<u>question.</u> supp. δ is supercompact and N has δ-cover + δ-appr + δ-gen. ~~must~~ ~~ult~~ supp. E is an N-extender. must $ult(N; E)$ have the δ-appr. property?

~~then supp. N has δ cover + δ appr.~~

<u>defn.</u> supp. E is an extender, δ reg.

① E is <u>δ-closed</u> iff ${}^{\delta}ult(E) \subset ult(V; E)$.

② E is <u>strongly δ-closed</u> if ${}^{\delta}ult(V; E) \subset ult(V; E)$.

<u>thm.</u> (universality thm., final version)

supp. N has δ-cover + δ-appr. supp. E, F are stgly δ-closed with crit's $\geq \delta$. let $M_E = ult(V; E)$, $j_E : V \longrightarrow M_E$ be the ultrapow emb., $M_F = ult(V; F)$,

$j_F : V \to M_F$ be the ult. ext.

let $\pi : M_E \to M_F$ be a elt. ext.

s.t. $j_F = \pi \circ j_E$.

Then ~~j_E~~ then the ext. derived

for π is in N.

th. δ s.c. N weak ext. model

$G \subset \mathrm{Col}(\omega, <\delta)$ – gen. $/V$

$\mathbb{R}_G = \mathbb{R}^{V[G]}$

$\Gamma^\infty_G = (\Gamma^\infty)^{V[G]}$

$\Gamma^\infty_G \subset V(\mathbb{R}_G)$

$N(\mathbb{R}_G)$ sym. ext. of N, by $\mathrm{Col}(\omega, <\delta)$

H be $N(\mathbb{R}_G)$-gen. for $\mathrm{Col}(\omega_1, \mathbb{R})^{N(\mathbb{R}_G)}$

$$\Gamma^\infty_G = (\Gamma^\infty_G)^{N(\mathbb{R}_G)[H]} = (\Gamma^\infty_G)^{N(\mathbb{R}_G)}.$$

$\Rightarrow$ N is corr. for Ω-logic.