$\mathsf{HOD}^{L(\mathbb{R})}$, Bukowský, Vopěnka, and Woodin's extender algebra

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Right after the discovery of forcing by P. Cohen in 1963, B. Balcar, L. Bukowský, P. Vopěnka, and others produced impressive results in the pure theory of forcing. Whereas Vopěnka's result (see [12]) according to which every set of ordinals is generic over HOD has always been well-known (see e.g. [2, Theorem 26]), Bukowský's criterion (see [1]) on when some $W \supset V$ is a generic extension of V got somewhat forgotten but saw a forceful revival through recent work of T. Usuba (see [10] and [11]).

A forcing which plays a fundamental role in descriptive inner model theory is Woodin's extender algebra (see e.g. [7, pp. 1657ff.]). In [6] we showed that the extender algebra and a version of Bukowský's forcing to establish his criterion from [1] may be naturally presented in a uniform fashion, so that these two forcings may be construed as two sides of one and the same coin.

The current paper further explores the tight connections between the respective forcings of Bukowský, Vopěnka, and Woodin. We show that in an important case, namely over $\mathsf{HOD}^{L(\mathbb{R})}$, the existence of large cardinals proves that Vopěnka's forcing is equal to Bukowský's forcing which in turn is equal to a forcing which naturally arises from Woodin's extender algebra in the context of the analysis of $\mathsf{HOD}^{L(\mathbb{R})}$ via a direct limt system based on M_ω, the least iterable inner model with infinitely many Woodin cardinals.[1] See Theorems 0.3 and 0.4 below.

Throughout our paper we shall assume at least that M_ω exists and is fully iterable, and we shall build upon [9] which presents the analysis of $\mathsf{HOD}^{L(\mathbb{R})}$. We shall assume that the reader is familiar with the relevant portions of [9] and we shall refer to [9] frequently. Vopěnka's forcing is used to prove a theorem of Woodin according to which $\mathsf{HOD}^{L(\mathbb{R})} = L[P]$ for some

[1] Our results generalize to larger determinacy models, but we decided that this paper would be more transparent and accessible if we restrict attention to $\mathsf{HOD}^{L(\mathbb{R})}$.

$P \subset \Theta^{L(\mathbb{R})}$, see [8, Theorem 3.1]. We shall bring in Woodin's extender algebra to show this result, see Theorem 0.8 below.

The author would like to thank Gabriel Fernandes for drawing the beautiful diagram below.

We start with a minor comment. F. Schlutzenberg was the first to point out that Claim 6.52 of [9] is wrong but can be easily fixed by replacing Card, the class of all cardinals, with the class of all $\mathbb{R}$–indiscernibles. This works als none of the maps under consideration move the $\mathbb{R}$–indiscernibles, which takes an argument, thouch, see [5].

The arguments on p. 321 of [9] may be (re-)organized as follows.

First, define a function $\alpha \mapsto \alpha^*$ as on p. 320 of [9]: $\alpha^* = \alpha_\alpha^{\mathcal{A}}$, the interpretation of the constant α_α inside the (transitivized) term model $\mathcal{A}$ (see [9, p. 318]). We may also represent $\alpha \mapsto \alpha^*$ as follows. Let us look at the following diagram, cf. [9, p. 318 ff].

$$\begin{tikzcd}
N_0^\omega \arrow[r, dashed] & N_k^\omega \arrow[r, "j_k^\omega = j_{k,k+1}^\omega"] \arrow[rr, bend left, "j_{k,\omega}^\omega"] & N_{k+1}^\omega \arrow[r, dashed] & N_\omega^\omega \\
N_0 = M \arrow[u, dashed, "j_0^\omega"] \arrow[r, dashed] & N_k \arrow[u, "i_k"] \arrow[r, "\substack{j_{k,k+1} \\ \| \\ j_k}"'] \arrow[rr, bend right, "j_{k,\omega}"'] & N_{k+1} \arrow[u, "i_{k+1}"] \arrow[r, dashed] & N_\infty \cong \mathcal{A}^* \arrow[u, dashed, "\sigma"]
\end{tikzcd}$$

Let α be any ordinal. There is some k_0 such that $j_k^\omega(\alpha) = \alpha$ for all $k \geq k_0$. We may also assume that $N_k^- = (N_k^\omega)^-$ is strongly $\{\alpha\}$–iterable for all $k \geq k_0$.

By the argument for [9, Claim 6.52], $\alpha \in \mathrm{Hull}^{N_{k_0}^\omega}(\delta_0^{N_{k_0}^\omega} \cup I)$, where $\delta_0^{N_{k_0}^\omega}$ is the bottom Woodin cardinal of $N_{k_0}^\omega$ and I is the class of $\mathbb{R}$–indiscernibles. This gives that $\alpha \in \mathrm{ran}(i_{k_0})$, say $\alpha = i_{k_0}(\overline{\alpha})$, and we may define

$$\alpha^* = j_{k_0,\omega}(\overline{\alpha}).$$

This definition works: It is not sensitive to the choice of k_0, because if (say) $k \geq k_0$ and $\alpha = \tau^{N_{k_0}^\omega}(\vec{\eta}, \vec{\iota})$, where τ is a Skolem term, $\vec{\eta} < \delta_0^{N_{k_0}^\omega}$, and $\vec{\iota} \in I$, then $\alpha = j^{\omega}_{k_0,k}(\alpha) = \tau^{N^{\omega}_k}(j^{\omega}_{k_0,k}(\vec{\eta}), \vec{\iota})$, and hence

$$\begin{aligned} j_{k_0,k}(i^{-1}_{k_0}(\alpha)) &= i^{-1}_k(j^{\omega}_{k_0,k}(\alpha)) \\ &= i^{-1}_k(\alpha). \end{aligned}$$

Also,

$$\begin{aligned} \sigma(\alpha^{\mathcal{A}}_{\alpha}) &= j^{\omega}_{k,0,\omega}(\alpha) \text{ (see [9, p. 319])} \\ &= J_{k_0,\omega}(i^{-1}_{k_0}(\alpha)). \end{aligned}$$

This shows that indeed $\alpha^* = j_{k_0,\omega}(i^{-1}_{k_0}(\alpha))$. We then get, see [9, p. 321], that if $\vec{\alpha}$ are ordinals and φ is a formula,

$$\begin{aligned} & L(\mathbb{R}) \models \varphi(\vec{\alpha}) \iff \\ & \Vdash^{\mathrm{Col}(\omega,<\omega_1^{\nu})}_{N^{\omega}_k} L(\mathbb{R}^*) \models \varphi(\vec{\alpha}) \iff \\ & \Vdash^{\mathrm{Col}(\omega,<\sup \text{ Wds})}_{N_k} L(\mathbb{R}^*) \models \varphi(i^{-1}_k(\vec{\alpha})) \text{ for } k \text{ large enough} \\ & \iff \Vdash^{\mathrm{Col}(\omega,<\sup \text{ Wds})}_{\mathcal{M}^+_{\infty}} L(\mathbb{R}^*) \models \varphi(\vec{\alpha}^*), \end{aligned}$$

where "sup Wds" refers to the supremum of the Woodin cardinals of the respective models. This shows that $HOD^{L(\mathbb{R})} \subset L[\mathcal{M}_{\infty}, \alpha \mapsto \alpha^*]$, see [9, Corollary 6.54]. This finishes our minor comment.

Let us now turn towards Woodin's theorem [8, Theorem 3.1] which states that $\mathsf{HOD}^{L(\mathbb{R})} = L[P]$ for some $P \subset \Theta^{L(\mathbb{R})}$. Traditional proofs of this, in particular the proof given on [8, pp. 182ff.], make use of a variant of Vopeňka's forcing. Instead of making use of Vopeňka, though, we may use the forcing from [4] to show that $HOD^{L(\mathbb{R})} = L[P]$, some $P \subset \Theta^{L(\mathbb{R})}$, as follows.

Let M be an inner model, say, with a Woodin cardinal, δ, and let $\mathbb{B} = \mathbb{B}_{\delta}$ be M's extender algebra at δ (given by some fixed collection $\mathcal{E} \in M$ of M–extenders witnessing that δ is Woodin in M). Let $\vec{\varphi} = (\varphi_i : i < \delta) \in M$ be a sequence of formulae associated with $\mathbb{B}$, and let $\kappa < \delta$ be $\vec{\varphi}$–strong in M. Let

$$j \colon M \to \mathrm{ult}(M; F),$$

where F an extender with $\mathrm{crit}(F) = \kappa$ such that $j(\vec{\varphi})_{\kappa} = \varphi_{\kappa}$. Let $U = \{X \in P(\kappa) \cap M : \kappa \in j(X)\}$ be the normal measure derived from j. We claim that

$$\{\xi < \kappa : \varphi_{\xi} || \varphi_{\kappa} \text{ in } \mathbb{B}\} \in U. \tag{1}$$

Otherwise $X = \{\xi < \kappa : \varphi_{\xi} \perp \varphi_{\kappa}\} \in U$. Let $\xi_0 = \min(X)$, and define $(\varphi'_{\xi} : \xi < \kappa)$ by:

$$\varphi'_{\xi} = \begin{cases} \varphi_{\xi} \text{ if } \xi \in X \\ \varphi_{\xi_0} \text{ if } \xi \notin X \end{cases}$$

Then $\varphi_\kappa = j((\varphi'_\xi : \xi < \kappa))_\kappa$, as

$$\{\xi < \kappa : \underbrace{(\varphi'_{\bar\xi} : \bar\xi < \kappa)_\xi}_{\varphi'_\xi} = \underbrace{(\varphi'_{\bar\xi} : \xi < \kappa)_\xi}_{\varphi_\xi}\} \in U$$

yields that

$$j((\varphi'_\xi : \xi < \kappa))_\kappa = j((\varphi_\xi : \xi < \kappa))_\kappa = j(\vec\varphi)_\kappa = \varphi_\kappa.$$

But also $\varphi_\kappa \to \bigvee_{\xi<\kappa} \varphi'_\xi$, as this is an axiom associated with $\mathbb{B}$, so that $\varphi_\kappa || \varphi'_\xi$ for some $\xi < \kappa$. But then $\varphi_\kappa || \varphi_\xi$ for some $\xi \in X$. Contradiction!

We have verified (1).

This may be used e.g. to show G. Hjorth's theorem according to which the finite support product $\mathbb{B}^\omega_{\rm fin}$ of ω copies of $\mathbb{B}$ has the δ–c.c. in M. For if $\vec\varphi = (\vec\varphi_i : i < \delta) \in {}^\delta(\mathbb{B}^\omega_{\rm fin}) \cap M$, and if κ is $\vec\varphi$–strong in M, then there is

$$j : M \to {\rm ult}(M; F),$$

F an extender with ${\rm crit}(F) = \kappa$ such that $j(\vec\varphi)_\kappa = \vec\varphi_\kappa$. If $U = \{X \in P(\kappa) \cap M : \kappa \in j(X)\}$, then for each $n < \omega$, $\{\xi < \kappa : \vec\varphi_\xi(n) || \vec\varphi_\kappa(n)\} \in U$, so that there is one $\xi < \kappa$ such that for all $n < \omega$, $\vec\varphi_\xi(n) || \vec\varphi_\kappa(n)$ in $\mathbb{B}$ and hence $\vec\varphi_\xi || \vec\varphi_\kappa$ in $\mathbb{B}^\omega_{\rm fin}$.

We know that $\mathsf{HOD}^{L(\mathbb{R})} = L[\mathcal{M}_\infty, \alpha \mapsto \alpha^*]$, see [9, Corollary 6.54]. Let us now define a forcing $\mathbb{B} \in L[\mathcal{M}_\infty, \alpha \mapsto \alpha^*] = HOD^{L(\mathbb{R})}$ as follows.

Let $\mathcal{L}^+$ be the language with atomic formulae"$\check{n} \in \dot{a}$," $n < \omega$, $\dot{a}$ a fixed constant symbol. A formula of $\mathcal{L}^+$ is obtained from atomic formulae by closing under negation and infinite conjunctions in $\mathcal{M}_\infty$ (equivalently, in $L[\mathcal{M}_\infty, \alpha \mapsto \alpha^*] = HOD^{L(\mathbb{R})}$) of length $< \delta_0^{\mathcal{M}_\infty} = \Theta^{L(\mathbb{R})}$. We let $\varphi \in \mathcal{L}$ iff $\varphi \in \mathcal{L}^+$ and

$$\exists p \in \mathbb{B}^{\mathcal{M}_\infty}_{\Theta^{L(\mathbb{R})}}\, p \Vdash^{\mathbb{B}^{\mathcal{M}_\infty}_{\Theta^{L(\mathbb{R})}}}_{\mathcal{M}_\infty} \dot{g} \models \varphi^*.$$

Here, $\mathbb{B}^{\mathcal{M}_\infty}_{\Theta^{L(\mathbb{R})}}$ is $\mathcal{M}_\infty$'s extender algebra at $\delta_0^{\mathcal{M}_\infty}$ which adds a real (a subset of ω), and $\dot{g}$ is the canonical name for this real. φ^* denotes the image of φ under the map $\alpha \mapsto \alpha^*$, extended to act on all of $\mathcal{M}_\infty$ (i.e., $\varphi^* = \pi^-_\infty(\varphi)$ for the map π^-_∞ as in [9, Lemma 6.60]).

We let $\varphi \leq \psi$ in $\mathcal{L}$ iff

$$\forall p \in \mathbb{B}^{\mathcal{M}_\infty}_{\Theta^{L(\mathbb{R})}}\, p \Vdash^{\mathbb{B}^{\mathcal{M}_\infty}_{\Theta^{L(\mathbb{R})}}}_{\mathcal{M}_\infty} \dot{g} \models \varphi \to \psi.$$

We know that $V_{\Theta^{L(\mathbb{R})}}^{\mathsf{HOD}^{L(\mathbb{R})}} = V_{\Theta^{L(\mathbb{R})}}^{\mathcal{M}_\infty}$ and $\Theta^{L(\mathbb{R})}$ is a Woodin cardinal in $\mathsf{HOD}^{L(\mathbb{R})}$ (see [9, Lemma 6.36]), so that by Hjorth's theorem, $\mathbb{B}^\omega_{\rm fin}$ has the $\delta^{\mathcal{M}_\infty}$–c.c. in $L[\mathcal{M}_\infty, \alpha \mapsto \alpha^*] = \mathsf{HOD}^{L(\mathbb{R})}$.

We may then use this to show that $(\mathcal{L}; \leq)$ has the $\delta^{\mathcal{M}_\infty}$–c.c. in $L[\mathcal{M}_\infty, \alpha \mapsto \alpha^*]$, and in fact $(\mathcal{L}; \leq)^\omega_{\rm fin}$, the finite support product of $(\mathcal{L}; \leq)$, also has the $\delta^{\mathcal{M}_\infty}$–c.c. in $L[\mathcal{M}_\infty, \alpha \mapsto \alpha^*]$, by the following argument. Working in $L[\mathcal{M}_\infty, \alpha \mapsto \alpha^*]$, let $\{(\varphi^i_n : n < \omega) : i < \delta^{\mathcal{M}_\infty}\} \subset (\mathcal{L}; \leq)^\omega_{\rm fin}$, and pick, for each $i < \delta^{\mathcal{M}_\infty}$, $(p^i_n : n < \omega) \in \mathbb{B}^\omega_{\rm fin}$ such that $p^i_n \Vdash \dot{\rm g} \models (\varphi^i_n)^*$ for all $n < \omega$. There are $i < j < \delta^{\mathcal{M}_\infty}$ with $p^i_n || p^j_n$ for all $n < \omega$, say $q_n \leq p^i_n, p^j_n$. Then $q_n \Vdash \dot{\rm g} \models (\varphi^i_n \wedge \varphi^j_n)^*$ for all $n < \omega$, so that $(\varphi^i_n : n < \omega) || (\varphi^j_n : n < \omega)$ in $(\mathcal{L}; \leq)^\omega_{\rm fin}$. Therefore, $\{(\varphi^i_n : n < \omega) : i < \delta^{\mathcal{M}_\infty}\}$ is not an antichain, and $(\mathcal{L}; \leq)$ has indeed the $\delta^{\mathcal{M}_\infty}$–c.c. in $L[\mathcal{M}_\infty, \alpha \mapsto \alpha^*]$.

It seems that it is impossible to fully analyze the effect of $(\mathcal{L}; \leq)^\omega_{\rm fin}$ without showing that $(\mathcal{L}; \leq)$ is forcing equivalent to Vopěnka forcing, see Theorem 0.3 below.

Claim 0.1 *Let $\varphi \in \mathcal{L}^+$. Then $\varphi \in \mathcal{L}$ iff there is some $x \in \mathcal{P}(\omega) \cap V$ with $x \models \varphi$.*

Proof. "$\Leftarrow$" Let N be a model from the system giving rise to $\mathcal{M}^+_\infty$ ($= \mathcal{M}_\infty$) such that x is generic over N at its bottom Woodin cardinal, δ^N_0. As $x \models \varphi$, $p \Vdash^{\mathbb{B}^N_{\delta^N_0}}_N \dot{\rm g} \models \varphi$, some $p \in \mathbb{B}^N_{\delta^N_0}$. We may have picked N in such a way that $\varphi^* = j_{N,\mathcal{M}^+_\infty}(\varphi)$. By elementarity of $j_{N,\mathcal{M}^+_\infty}$, there is then some $p \in \mathbb{B}^{\mathcal{M}_\infty}_{\delta^{\mathcal{M}_\infty}_0}$ with $p \Vdash^{\mathbb{B}^{\mathcal{M}_\infty}_{\delta^{\mathcal{M}_\infty}_0}}_{\mathcal{M}_\infty} \dot{\rm g} \models \varphi^*$, i.e., $\varphi \in \mathcal{L}$.

"$\Rightarrow$" Let N be a model from the system giving rise to $\mathcal{M}^+_\infty$ ($= \mathcal{M}_\infty$) such that $\varphi^* = j_{N,\mathcal{M}^*_\infty}(\varphi)$. As $\varphi \in \mathcal{L}$, the elementarity of $j_{N,\mathcal{M}^*_\infty}$ gives that there is some $p \in \mathbb{B}^N_{\delta^N_0}$ such that $p \Vdash^{\mathbb{B}^N_{\delta^N_0}}_N \dot{\rm g} \models \varphi$. In V, pick some g which is $\mathbb{B}^N_{\delta^N_0}$–generic over N and such that $p \in g$, and let $x \in \mathcal{P}(\omega)$ be the real described by g. Then $x \models \varphi$. $\square$ (Claim 0.1)

Let $\mathbb{V} \in \mathsf{HOD}^{L(\mathbb{R})}$ be Vopěnka's forcing to add a real (or rather, a subset of ω), i.e. $\mathbb{V} \cong (\mathcal{O}; \subset)$, where $\mathcal{O}$ is the collection of all non–empty $OD^{L(\mathbb{R})}$ subsets of $\mathcal{P}(\omega) \cap V$. By a theorem of Woodin, every $OD^{L(\mathbb{R})}$ set of reals has an $OD^{L(\mathbb{R})}$ ∞–Borel code. (See [13, Lemma 9.5], see also e.g. [3, Corollary 1.3].) In $L(\mathbb{R})$, if $(A_i : i < \Theta)$ is a sequence of sets of reals, then there is some $\gamma < \Theta$ with $\bigcup_{i<\Theta} A_i = \bigcup_{i<\gamma} A_i$. (This just follows from the definition of Θ and the regularity of Θ.) This buys us the first statement of the following claim, the second one being trivial.

Claim 0.2 *If $A \subset \mathcal{P}(\omega) \cap V$ be $OD^{L(\mathbb{R})}$, then there is some $\varphi \in \mathcal{L}^+$ with $A = \{x : x \models \varphi\}$. Conversely, if $\varphi \in \mathcal{L}^+$, then $\{x \subset \omega : x \models \varphi\} \in OD^{L(\mathbb{R})}$.*

We now get the following. For $\varphi \in \mathcal{L}^+$, write $A_\varphi = \{x : x \models \varphi\}$. Then for every $\varphi \in \mathcal{L}^+$,

$$\begin{aligned} \varphi \in \mathcal{L} \quad & \text{iff } \exists x \in \mathcal{P}(\omega) \cap V\, x \models \varphi \quad \text{by Claim 0.1} \\ & \text{iff } A_\varphi \neq \emptyset, \text{ i.e. } A_\varphi \in \mathcal{O}. \end{aligned}$$

For $\varphi, \psi \in \mathcal{L}$ we get, by the proof of Claim 1,

$$\begin{aligned} \varphi \leq_{(\mathcal{L};\leq)} \psi \quad & \text{iff } \forall x \in \mathcal{P}(\omega) \cap V (x \models \varphi \to x \models \psi) \\ & \text{iff } A_\varphi \subset A_\psi. \end{aligned}$$

Therefore,

Theorem 0.3 *$(\mathcal{L}, \leq)$ and $\mathbb{V}$ are forcing equivalent, in fact $\mathbb{V}$ is isomorphic to $(\mathcal{L}; \leq)$ if we identify two elements φ, ψ of $\mathcal{L}$ in case $\varphi \leq \psi \leq \varphi$.*

This gives a characterization of the Vopěnka forcing $\mathbb{V}$ in terms of Woodin's extender algebra.

Theorem 0.4 *Let $\sigma \colon \mathbb{V} \cong (\mathcal{O}; \subset)$. Then $\sigma^{-1}(A) \in \mathbb{V}$ (equivalently, $A \in \mathcal{O}$) if and only if for some/all $\varphi \in \mathcal{L}^+$ with $A = A_\varphi$, $\exists p \in \mathbb{B}^{\mathcal{M}_\infty}_{\theta^{L(\mathbb{R})}}\; p \Vdash^{\mathbb{B}^{\mathcal{M}_\infty}_{\Theta^{L(\mathbb{R})}}}_{\mathcal{M}_\infty} \dot{g} \models \varphi^*$.*

Here, we may think of φ with $A = A_\varphi$ as an ∞–Borel code for A.

We now need a variant of $(\mathcal{L}; \leq)^\omega_{\text{fin}}$ to show that $HOD^{L(\mathbb{R})} = L(P)$, some $P \subset \Theta^{L(\mathbb{R})}$. The problem with $(\mathcal{L}; \leq)^\omega_{\text{fin}}$ is that the reals it adds are mutually generic.

Working inside $L[\mathcal{M}_\infty, \alpha \mapsto \alpha^*]$, let $\mathcal{L}^{\omega,+}$ be the language with atomic formulae "$\check{n} \in \dot{a}_k$" for $n, k < \omega$ (i.e., $\{\check{n} : n < \omega\} \cup \{\dot{a}_k : k < \omega\}$ is the set of constants of $\mathcal{L}^{\omega,+}$). A formula of $\mathcal{L}^{\omega,+}$ is obtained from atomic formulae by closing under negation and infinite conjunctions in $\mathcal{M}_\infty$ (equivalently, in $L[\mathcal{M}_\infty, \alpha \mapsto \alpha^*]$) of lenght $< \delta_0^{\mathcal{M}_\infty} = \Theta^{L(\mathbb{R})}$.

$\mathcal{L}^{\omega,+}$ canonically gives rise to a version of the extender algebra of $\mathcal{M}_\infty$ at $\Theta^{L(\mathbb{R})}$ which adds a sequence of reals (rather than just a single real). Let us write $\mathbb{B}^{\mathcal{M}_\infty}_{\theta^{L(\mathbb{R})},\omega}$ for this version of the extender algebra.

We let $\varphi \in \mathcal{L}^\omega$ iff $\varphi \in \mathcal{L}^{\omega,+}$, there is some $m < k$ such that if $\dot{a}_k$ occurs in φ, then $k < m$, and ther is some $p \in \mathbb{B}^{\mathcal{M}_\infty}_{\theta^{L(\mathbb{R})},\omega}$ with

$$p \Vdash^{\mathbb{B}^{\mathcal{M}_\infty}_{\Theta^{L(\mathbb{R})},\omega}}_{\mathcal{M}_\infty} \dot{g} \models \varphi^*,$$

where $\dot{g}$ is the name for the sequence of reals added by $\mathbb{B}^{\mathcal{M}_\infty}_{\theta^{L(\mathbb{R})},\omega}$.

We let $\varphi \leq \psi$ in $\mathcal{L}^\omega$ iff for all $p \in \mathbb{B}^{\mathcal{M}_\infty}_{\theta^{L(\mathbb{R})},\omega}$, $p \Vdash^{\mathbb{B}^{\mathcal{M}_\infty}_{\Theta^{L(\mathbb{R})},\omega}}_{\mathcal{M}_\infty} \dot{g} \models \varphi \rightarrow \psi$.

By arguments that were given before, $(\mathcal{L}^\omega; \leq)$ has the δ–c.c. in $L[\mathcal{M}_\infty, \alpha \mapsto \alpha^*]$.

We may think of a $\varphi \in \mathcal{L}^\omega$ as an ∞–Borel code (in $HOD^{L(\mathbb{R})}$) for a nonempty ($OD^{L(\mathbb{R})}$–) subset A_φ of ${}^m\mathcal{P}(\omega)$, some $m < \omega$. The first two of the following three claims are proven in much the same way as before.

Claim 0.5 *Let $\varphi \in \mathcal{L}^{\omega,+}$. Then $\varphi \in \mathcal{L}^\omega$ iff $\exists (x_n : n < m) \in V (x_n : n < m) \models \varphi$ (for the right m), and $\varphi \leq \psi$ iff $A_\varphi \subset A_\psi$.*

Claim 0.6 *If $A \subset {}^m\mathcal{P}(\omega) \cap V$, some $m < \omega$, is $OD^{L(\mathbb{R})}$, then there is some $\varphi \in \mathcal{L}^{\omega,+}$ with $A = \{s : s \models \varphi\}$. Conversely, if $\varphi \in \mathcal{L}^{\omega,+}$, then (for the right m) $\{s \in {}^m\mathcal{P}(\omega) : s \models \varphi\} \in OD^{L(\mathbb{R})}$.*

Claim 0.7 *If g is $\mathrm{Col}(\omega, \mathbb{R})$–generic over V and g gives rise to $(x_n : n < \omega)$, then $\{\varphi \in \mathcal{L}^\omega : (x_n : n < \omega) \models \varphi\}$ is $(\mathcal{L}^\omega; \leq)$–generic over $L[\mathcal{M}_\infty, \alpha \mapsto \alpha^*]$.*

Proof. Let $D \in L[\mathcal{M}_\infty, \alpha \mapsto \alpha^*]$, D being dense in $(\mathcal{L}^\omega; \leq)$. We aim to see that for every $s = (x_n : n < m) \in \mathrm{Col}(\omega, \mathbb{R})$ there is some $t \supset s$, $t = (x_n : n < m') \in \mathrm{Col}(\omega, \mathbb{R})$ such that for some $\varphi \in D$, $(x_n : n < m') \models \varphi$. Otherwise there is some $m < \omega$ such that $D^* =$

$$\begin{aligned} \{s = \ & (x_n : n < m) \in \mathrm{Col}(\omega, \mathbb{R}) : \text{ there is } \mathbf{no} \\ & t \supset s,\ t = (x_n : n < m') \in \mathrm{Col}(\omega, \mathbb{R}), \text{ some } m' \geq m, \\ & \text{such that for some } \varphi \in D,\ t \models \varphi\} \end{aligned}$$

is nonempty. As $D^* \in OD^{L(\mathbb{R})}$, there is some $\psi \in \mathcal{L}^\omega$ with $D^* = A_\psi$. As D is dense in $\mathcal{L}^\omega$, there is some $\varphi \leq \psi$ in $\mathcal{L}^\omega$, $\varphi \in D$. Let $(x_n : n < m') \in V$ be such that $(x_n : n < m') \models \varphi$. Then $(x_n : n < m) \models \psi$, i.e., $(x_n : n < m) \in D^*$. However, this is not true as being witessed by the existence of $(x_n : n < m')$. $\square$ (Claim 0.7)

Theorem 0.8 *$HOD^{L(\mathbb{R})} = L[(\mathcal{L}^\omega; \leq)]$. In particular, as $(\mathcal{L}^\omega; \leq)$ can be coded inside $HOD^{L(\mathbb{R})}$ by a subset of $\Theta^{L(\mathbb{R})}$, $HOD^{L(\mathbb{R})} = L[P]$ for some $P \subset \Theta^{L(\mathbb{R})}$.*

Proof. It obviously suffices to prove that if $\vec{\alpha} \in OR$ and φ is a formula, then

$$(*) \begin{cases} L(\mathbb{R}) \models \varphi(\vec{\alpha}) \text{ iff} \\ \Vdash^{(\mathcal{L}^\omega; \leq)}_{L[(\mathcal{L}^\omega; \leq)]} L(\dot{\mathbb{R}}^*) \models \varphi(\vec{\alpha}), \end{cases}$$

where $\dot{\mathbb{R}}^*$ is the canonical name for the unordered collection of all reals added by $(\mathcal{L}^\omega; \leq)$.

(*), "$\Rightarrow$": Say $L(\mathbb{R}) \models \varphi(\vec{\alpha})$. Let $\psi \in \mathcal{L}^\omega$ be arbitrary. Suppose for contradiction that $\psi \Vdash^{\mathcal{L}^\omega}_{L[(\mathcal{L}^\omega)]} L(\dot{\mathbb{R}}^*) \models \neg\varphi(\vec{\alpha})$.

Let $(x_n : n < m) \in V$ be such that $(x_n : n < m) \models \psi$. Let $(x_n : n \geq m)$ be such that $(x_n : n < \omega)$ is given by some $\mathrm{Col}(\omega, \mathbb{R})$–generic over V. By Claim 0.7, $h = \{\varphi' \in \mathcal{L}^\omega : (x_n : n < \omega) \models \varphi'\}$ is $(\mathcal{L}^\omega; \leq)$–generic over $L[\mathcal{M}_\infty, \alpha \mapsto \alpha^*]$. As $\psi \in h, L(\mathbb{R}^V) = L(\dot{\mathbb{R}}^{*h}) \models \neg\varphi(\vec{\alpha})$. Contradiction!

(*), "$\Leftarrow$": Apply (*), "$\Rightarrow$" to $\neg\varphi(\vec{\alpha})$. $\square$ (Theorem 0.8)

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