Homogeneously Souslin sets in small inner models

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Abstract

We prove that every homogeneously Souslin set is coanalytic provided that either (a) 0^{long} does not exists, or else (b) $V = K$, where K is the core model below a μ-measurable cardinal.

1 Homogeneously Souslin sets.

In this paper we shall deal with homogeneously Souslin sets of reals, or rather with sets of reals which admit an ω-closed embedding normal form.

Definition 1.1 *(Cf. [4, p. 92].) Let $A \subset {}^\omega\omega$. Let $\alpha \in \mathrm{OR}$. We say that A has an* α-closed embedding normal form *if and only if the following holds true. There is a commutative system*

$$((M_s : s \in {}^{<\omega}\omega), (\pi_{st} : s, t \in {}^{<\omega}\omega, s \subset t))$$

such that $M_0 = V$, each M_s is an inner model of ZFC *with ${}^\alpha M_s \subset M_s$, each $\pi_{st} : M_s \to M_t$ is an elementary embedding, and if $x \in {}^\omega\omega$ and $(M_x, (\pi_{x\restriction n,x} : n < \omega))$ is the direct limit of $((M_{x\restriction n} : n < \omega), (\pi_{x\restriction n,x\restriction m} : n \leq m < \omega))$ then*

$$x \in A \;\Leftrightarrow M_x \text{ is wellfounded.}$$

As we shall not need it here, we do not repeat the definition of the concept of being homogeneously Souslin in this paper (cf. [4, p. 87]). We just remind the reader of the following facts.

Lemma 1.2 *Let $A \subset {}^\omega\omega$.*

(1) If A is coanalytic and if κ is a measurable cardinal then A is κ-homogeneously Souslin (cf. [3], [4, Theorem 2.2]).

(2) If A is κ-homogeneously Souslin, where κ is a (measurable) cardinal, then A is determined (cf. [4, Theorem 2.3]) and has a κ-closed embedding normal form (cf. [4, p. 92]).

(3) If A has a $2^{\aleph_0}$-closed embedding normal form then A is homogeneously Souslin (cf. [7, Lemma 2.5], [1, Theorem 5.2]).

Our aim is to prove a converse to Lemma 1.2 (1) under appropriate anti-large cardinal hypotheses.

Definition 1.3 *A cardinal* κ *is called* μ-measurable *if there is an embedding* $\pi\colon V \to M$ *such that* M *is transitive,* $\kappa = \mathrm{crit}(\pi)$*, and* $\{X \subset \kappa | \kappa \in \pi(X)\} \in M$ *(cf. [5]).*

We say that $0^\P$ *does not exist if for every iterable premouse* $\mathcal{M}$*, if* $E_\nu^\mathcal{M} \neq \emptyset$ *then* $\mathcal{M}||\mathrm{crit}(E_\nu^\mathcal{M})$ *is a model of "there is no strong cardinal (as being witnessed by the extenders from the* $\mathcal{M}$*-sequence)" (cf. [8]).*

Suppose that $0^\P$ *does not exist, and let* K *denote the core model (cf. [8, Chap. 8]). We say that* K *is* below a μ-measurable cardinal *if* $K \models$ *"there is no* μ*-measurable cardinal."*

If K is below a μ-measurable cardinal then every total extender on the K-sequence has exactly one generator.

Definition 1.4 *We say that* 0^{long} *does not exist if for every iterable premouse* $\mathcal{M}$*, if we let* A *be the set of critical points of the total measures from the* $\mathcal{M}$*-sequence then* $A = \emptyset$ *or else* $\mathrm{otp}(A) < \min(A)$ *(cf. [1]).*

We can now state the main results of our paper.

Theorem 1.5 *Suppose that* $0^\P$ *does not exist, and* K *is below a* μ*-measurable cardinal. Suppose that* $V = K$*. Let* $A \subset {}^\omega\omega$ *have an* ω*-closed embedding normal form. Then* A *is coanalytic.*

Theorem 1.6 *Suppose that* 0^{long} *does not exist. Let* $A \subset {}^\omega\omega$ *have an* ω*-closed embedding normal form. Then* A *is coanalytic.*

Our main technical tool will be the concept of a "shift map." Shift maps will be defined in the next section, where we shall also show that if K is below a μ-measurable cardinal then any elementary embedding from one universal weasel into another one is a shift map. The final section will prove Theorems 1.5 and 1.6.

As to prerequisites, we shall assume familiarity with the core model theory as presented in [8, Chap. 8].

2 Shift maps.

In this section we shall prove a result on shift maps. This result is more general than what we would need in order to prove the main theorems, but it might be interesting in its own right.

Convention 2.1 *Let $\zeta \leq \theta$, and let $\varphi\colon \zeta+1 \to \theta+1$ be strictly monotone, i.e., $\varphi(\alpha') > \varphi(\alpha)$ for $\alpha' > \alpha$. We then let φ^- denote the partial map from $\theta+1$ to $\zeta+1$ which is defined by $\varphi^-(\beta) =$ the least α such that $\varphi(\alpha) \geq \beta$.*

Notice that in this situation, φ^- is total (i.e., $\mathrm{dom}(\varphi^-) = \zeta+1$) if and only if $\varphi(\zeta) = \theta$.

Definition 2.2 *Let $\pi\colon W \to W'$ be an elementary embedding, where both W and W' are weasels. We say that π is a* shift map *provided the following holds true.*

There is a weasel W_0, there are non-dropping iterations $\mathcal{T}$ and $\mathcal{U}$ of W_0 with $\mathrm{lh}(\mathcal{T}) = \zeta+1$ and $\mathrm{lh}(\mathcal{U}) = \theta+1$, where $\zeta \leq \theta \leq \mathrm{OR}$, there is a strictly monotone $\varphi\colon \zeta+1 \to \theta+1$ with $\varphi(\zeta) = \theta$, and there are elementary embeddings $\pi_\beta\colon \mathcal{M}^{\mathcal{T}}_{\varphi^-(\beta)} \to \mathcal{M}^{\mathcal{U}}_\beta$ for $\beta \leq \theta$ such that:

(a) $\pi = \pi_\theta$,

(b) $\pi_{\beta'} \circ \pi^{\mathcal{T}}_{\varphi^-(\beta)\varphi^-(\beta')} = \pi^{\mathcal{U}}_{\beta\beta'} \circ \pi_\beta$ whenever $\beta \leq \beta' \leq \theta$,

(c) $\pi_{\varphi(\alpha)+1} \restriction \mathrm{lh}(E^{\mathcal{T}}_\alpha) = \pi_{\varphi(\alpha)} \restriction \mathrm{lh}(E^{\mathcal{T}}_\alpha)$ whenever $\alpha+1 \leq \zeta$, and

(d) $E^{\mathcal{U}}_{\varphi(\alpha)} = \pi_{\varphi(\alpha)}(E^{\mathcal{T}}_\alpha)$ whenever $\alpha+1 \leq \zeta$.

We shall prove:

Theorem 2.3 *Suppose that $0^\P$ does not exist, and K is below a μ-measurable cardinal. Let $\pi\colon W \to W'$ be an elementary embedding, where both W and W' are universal weasels. Then π is a shift map.*

Let us state some terminology, assuming that $0^\P$ does not exist, before commencing with the proof of Theorem 2.3. Let W be a weasel and let $\Gamma \subset \mathrm{OR}$ be thick in W (cf. [8, p. 214f.]). Let α be an ordinal. We shall write $H^W_\Gamma(\alpha)$ for the set of all $\tau^W(\vec{\gamma}, \vec{\epsilon})$, where τ is a Skolem term, $\vec{\gamma} \in [\alpha]^{<\omega}$, and $\vec{\epsilon} \in [\Gamma]^{<\omega}$. We shall write $H^W(\alpha)$ for the intersection of all $H^W_\Gamma(\alpha)$ where Γ is thick in W. W has the *definability property at* α just in case that $\alpha \in H^W(\alpha)$ (cf. [6, Definition 4.4]).

Let $\mathcal{M}$ be a premouse, and let $\alpha \leq \mathcal{M} \cap \mathrm{OR}$. We say that $\mathcal{M}$ is α*-very sound* if there is a weasel W such that $\mathcal{M} \triangleleft W$ and $\mathcal{M} \subset H^W(\alpha)$. In this case, W is called an α*-very soundness witness* for $\mathcal{M}$.

A premouse $\mathcal{M}$ is *strong* if there is a universal weasel W with $\mathcal{M} \triangleleft W$ (cf. [8, p. 212]).

Let $\mathcal{M}$ be strong, and let $W \triangleright \mathcal{M}$ be a witness to this. By [8, Theorem 7.4.9 and p. 275] there is a normal non-dropping iteration $\mathcal{T}^*$ of K such that $W = \mathcal{M}^{\mathcal{T}^*}_\infty$. Therefore, if we let $(\mathcal{U}, \mathcal{T})$ denote the coiteration of $\mathcal{M}$ with K then $\mathcal{U}$ is trivial and $\mathcal{T}$ is simple. The following is part of the folklore.

Lemma 2.4 *Suppose that $0^\P$ does not exist, and K is below a μ-measurable cardinal. Let W be a universal weasel, let κ be a cardinal of W, and let $\mathcal{M} = W||\kappa^{(+3)W}$. Then $\mathcal{M}$ is $(\kappa+1)$-very sound. Moreover, $\mathcal{M}$ is κ-very sound if and only if the following holds true: if $\mathcal{T}$ is the normal non-dropping iteration of K which arises from the comparison with $\mathcal{M}$ then no $E^{\mathcal{T}}_\alpha$ has critical point κ.*

Lemma 2.4 is no longer true if K is not assumed to be below a μ-measurable cardinal (cf. [6, p. 29, Example 4.3]).

Lemma 2.5 *Suppose that $0^\P$ does not exist, and K is below a μ-measurable cardinal. Let both W and W' be universal weasels. Let $W||\mu = W'||\mu$, where μ is either a limit cardinal or else a double successor cardinal in both W and W'. Then $W||\mu^{+W} = W'||\mu^{+W'}$.*

The following will be used towards the end of the proof of Theorem 2.3.

Lemma 2.6 *Suppose that $0^\P$ does not exist, and K is below a μ-measurable cardinal. Let W, W' be weasels, and let $\pi \colon W \to W'$ be elementary. Let $E^W_\alpha \neq \emptyset$ with $\kappa = \mathrm{crit}(E^W_\alpha)$. Let π be continuous at κ^{+W}, and let $E^{W'}_\beta \neq \emptyset$ be such that*

$$\pi\text{"} E^W_\alpha \subset E^{W'}_\beta.$$

Then $\beta = \pi(\alpha)$, i.e., $E^{W'}_\beta = \pi(E^W_\alpha)$.

Proof of Lemma 2.6. Set $\lambda = \pi(\kappa)$. We must have $\lambda = \mathrm{crit}(E^{W'}_\beta)$. Let $X \in E^{W'}_\beta$. We aim to prove that $X \in E^{W'}_{\pi(\alpha)}$.

As π is continuous at κ^{+W} we may pick some $\gamma < \kappa^{+W}$ such that $X \in W'||\pi(\gamma)$. Let $(X_i \colon i < \kappa) \in W$ be such that $E^W_\alpha \cap W||\gamma = \{X_i \colon i < \kappa\}$. Let $Y = \Delta_{i<\kappa} X_i$ be the diagonal intersection. As E^W_α is normal, $Y \in E^W_\alpha$, and thus $\pi(Y) \in E^{W'}_\beta$ by $\pi\text{"} E^W_\alpha \subset E^{W'}_\beta$. Moreover, for each $Z \in \mathcal{P}(\kappa) \cap W||\gamma$ there is some $\xi < \kappa$ such that $Y \setminus \xi \subset Z$ or $Y \setminus \xi \subset \kappa \setminus Z$ (we'll have the former if and only if $Z \in E^W_\alpha$). By elementarity, hence, for each $Z \in \mathcal{P}(\lambda) \cap W'||\pi(\gamma)$ there is some $\xi < \lambda$ such that $\pi(Y) \setminus \xi \subset Z$ or $\pi(Y) \setminus \xi \subset \lambda \setminus Z$.

In particular, there is some $\xi < \lambda$ such that $\pi(Y) \setminus \xi \subset X$ or $\pi(Y) \setminus \xi \subset \lambda \setminus X$. As $\pi(Y) \in E^{W'}_\beta$ and $X \in E^{W'}_\beta$ we cannot have that $\pi(Y) \setminus \xi \subset \lambda \setminus X$. Therefore, $\pi(Y) \setminus \xi \subset X$. But because $\pi(Y) \in E^{W'}_{\pi(\alpha)}$ we then get $X \in E^{W'}_{\pi(\alpha)}$, as desired.

$\square$ (Lemma 2.6)

We are now ready to prove the key result of this section.

Proof of Theorem 2.3. As W and W' are universal, there are non-dropping iterations $\mathcal{T}$ and $\mathcal{U}$ of K with $\mathrm{lh}(\mathcal{T}) = \zeta + 1$ and $\mathrm{lh}(\mathcal{U}) = \theta + 1$ for some ζ, $\theta \leq \mathrm{OR}$ such that $W = \mathcal{M}^{\mathcal{T}}_\zeta$ and $W' = \mathcal{M}^{\mathcal{U}}_\theta$ (cf. [8, Theorem 7.4.9 and p. 275]).

Claim 2.7 *Let* $\alpha + 1 \leq \zeta$, $\kappa = \mathrm{crit}(E^{\mathcal{T}}_\alpha)$, *and* $\lambda = \pi(\kappa)$. *Then* $\lambda = \mathrm{crit}(E^{\mathcal{U}}_\beta)$ *for some* β *with* $\beta + 1 \leq \theta$.

PROOF of Claim 2.7. Set $\nu = \kappa^{(+3)W}$. Let Q and Γ be such that $Q \triangleright W||\nu$ is a universal weasel, Γ is thick in Q, and $\kappa \notin H^Q_\Gamma(\kappa)$. In particular, Γ witnesses that Q does not have the definability property at κ. Let $E = E_{\pi \restriction (W||\nu)}$ be the long extender derived from $\pi \restriction (W||\nu)$, and let

$$\tilde{\pi}\colon Q \to_E \tilde{Q} = Ult(Q; E)$$

be the ultrapower of Q by E. Of course, $\pi \restriction (W||\nu) = \tilde{\pi} \restriction (W||\nu)$. Because $\tilde{Q}$ is universal, $\tilde{\pi}(\nu) = \pi(\nu)$ by Lemma 2.5. Thus $W'||\pi(\nu) \triangleleft \tilde{Q}$. Let

$$\tilde{\Gamma} = \Gamma \cap \{\epsilon\colon \tilde{\pi}(\epsilon) = \epsilon\}.$$

We'll have that $\tilde{\Gamma}$ is thick in both Q and $\tilde{Q}$.

Now suppose that there is no β with $\beta + 1 \leq \theta$ and $\lambda = \mathrm{crit}(E^{\mathcal{U}}_\beta)$. Then $\tilde{Q}$ has the definability property at λ, and there is a Skolem term τ and there are $\vec{\gamma} \in \lambda$ and $\vec{\epsilon} \in \tilde{\Gamma}$ such that $\lambda = \tau^{\tilde{Q}}(\vec{\gamma}, \vec{\epsilon})$. That is,

$$\tilde{Q} \models \exists \vec{\bar{\gamma}} < \lambda \; \lambda = \tau(\vec{\bar{\gamma}}, \vec{\epsilon}).$$

Therefore, by using the map $\tilde{\pi}$,

$$Q \models \exists \vec{\bar{\gamma}} < \kappa \; \kappa = \tau(\vec{\bar{\gamma}}, \vec{\epsilon}),$$

which contradicts $\kappa \notin H^Q_\Gamma(\kappa)$.

$\square$ (Claim 2.7)

We may now define a strictly monotone $\varphi\colon \zeta + 1 \to \theta + 1$ as follows. For $\alpha + 1 \leq \zeta$, we let $\varphi(\alpha)$ be the unique β such that $\mathrm{crit}(E^{\mathcal{U}}_\beta) = \pi(\mathrm{crit}(E^{\mathcal{T}}_\alpha))$. Moreover, we set $\varphi(\zeta) = \theta$.

We now wish to define, for $\beta \leq \theta$, elementary embeddings $\pi_\beta\colon \mathcal{M}^{\mathcal{T}}_{\varphi^-(\beta)} \to \mathcal{M}^{\mathcal{U}}_\beta$ by

$$\pi_\beta = (\pi^{\mathcal{U}}_{\beta\theta})^{-1} \circ \pi \circ \pi^{\mathcal{T}}_{\varphi^-(\beta)\zeta}.$$

In order to see that these maps are well-defined it suffices to prove the following.

Claim 2.8 *Let* $\beta \leq \theta$, *and set* $\alpha = \varphi^-(\beta)$. *Then* $\mathrm{ran}(\pi \circ \pi^{\mathcal{T}}_{\alpha\zeta}) \subset \mathrm{ran}(\pi^{\mathcal{U}}_{\beta\theta})$.

PROOF of Claim 2.8. Let $x \in \mathcal{M}_\alpha^\mathcal{T}$. Let $\nu = \mu^{(+3)W}$ for some μ such that $\pi_{\alpha\zeta}^\mathcal{T}(x) \in W||\nu$. Let Q and Γ be such that $Q \triangleright W||\nu$ is a universal weasel and Γ is thick in Q. Let $E = E_{\pi \restriction (W||\nu)}$ be the long extender derived from $\pi \restriction (W||\nu)$, and let

$$\tilde{\pi}: Q \to_E \tilde{Q} = Ult(Q; E)$$

be the ultrapower of Q by E. Again, $\pi \restriction (W||\nu) = \tilde{\pi} \restriction (W||\nu)$, $\tilde{Q}$ is universal, $\tilde{\pi}(\nu) = \pi(\nu)$ by Lemma 2.5, and $W'||\pi(\nu) \triangleleft \tilde{Q}$. Let

$$\tilde{\Gamma} = \Gamma \cap \{\epsilon : \tilde{\pi}(\epsilon) = \epsilon\}.$$

We'll have that $\tilde{\Gamma}$ is thick in both Q and $\tilde{Q}$.

Now let $\pi_{\alpha\zeta}^\mathcal{T}(x) = \tau^Q(\vec{\gamma}, \vec{\epsilon})$, where τ is a Skolem term, $\pi_{\alpha\zeta}^\mathcal{T} \restriction \vec{\gamma} = \text{id}$, and $\vec{\epsilon} \in \tilde{\Gamma}$.

Set $y = \pi \circ \pi_{\alpha\zeta}^\mathcal{T}(x)$. Notice that $y \in W'||\pi(\nu)$. Let β' be least with $\pi_{\beta'\theta}^\mathcal{U} \restriction \pi(\nu) = \text{id}$. There is then an elementary embedding

$$\sigma: \mathcal{M}_{\beta'}^\mathcal{U} \to \tilde{Q}$$

with $\sigma \restriction \pi(\nu) = \text{id}$. We now get that $y = \tilde{\pi} \circ \pi_{\alpha\zeta}^\mathcal{T}(x) = \tau^{\tilde{Q}}(\pi(\vec{\gamma}), \vec{\epsilon})$, where $\pi_{\beta\theta}^\mathcal{U} \restriction \pi(\vec{\gamma}) = \text{id}$. In particular, $y \in \sigma \circ \text{ran}(\pi_{\beta\beta'}^\mathcal{U})$. But $\sigma \restriction \text{TC}(\{y\}) = id$, and hence $y \in \text{ran}(\pi_{\beta\beta'}^\mathcal{U})$. But $\pi_{\beta'\theta}^\mathcal{U} \restriction \text{TC}(\{y\}) = \text{id}$, and therefore $y \in \text{ran}(\pi_{\beta\theta}^\mathcal{U})$.

$\square$ (Claim 2.8)

We are now left with having to verify that (a) through (d) as in the statement of Theorem 2.2 hold. Note that (a) and (b) are trivial. Let us show (c) and (d). It is easy to see that the following suffices to establish (c).

Claim 2.9 *Let $\alpha + 1 \leq \zeta$, and set $\kappa = \text{crit}(E_\alpha^\mathcal{T})$. Then $\pi_{\varphi(\alpha)+1}(\kappa) = \pi_{\varphi(\alpha)}(\kappa)$.*

PROOF of Claim 2.9. Let us write $\beta = \varphi(\alpha)$. We know that $\pi_{\beta+1}(\kappa) = \pi(\kappa) = \text{crit}(E_\beta^\mathcal{U})$. Setting $\lambda = \pi(\kappa)$, we must therefore prove that

$$\pi_{\beta\theta}^\mathcal{U}(\lambda) = \pi(\pi_{\alpha\zeta}^\mathcal{T}(\kappa)). \tag{1}$$

Let $\alpha' \in \text{OR}$ be such that $\pi_{\alpha'\zeta}^\mathcal{T} \restriction (\pi_{\alpha\alpha'}^\mathcal{T}(\kappa) + 1) = \text{id}$. Let $\beta' \in \text{OR}$ be such that $\pi_{\beta'\theta}^\mathcal{U} \restriction (\pi_{\beta\beta'}^\mathcal{U}(\lambda) + 1) = \text{id}$ and $\pi_{\beta'\theta}^\mathcal{U} \restriction (\pi(\pi_{\alpha\alpha'}^\mathcal{T}(\kappa)) + 1) = \text{id}$. (If $\zeta \in \text{OR}$ rather than $\zeta = \text{OR}$ then we may just let $\alpha' = \zeta$; similarly, if $\theta \in \text{OR}$ rather than $\theta = \text{OR}$ then we may just let $\beta' = \theta$.) Let $\nu = \mu^{(+3)K}$ for some μ be such that $\mathcal{T} \restriction (\alpha' + 1)$ as well as $\mathcal{U} \restriction (\beta' + 1)$ both "live on" $K||\nu$, i.e., such that $\text{lh}(E_\gamma^\mathcal{T}) < \pi_{0\gamma}^\mathcal{T}(\nu)$ whenever $\gamma + 1 \leq \alpha'$ and $\text{lh}(E_\gamma^\mathcal{U}) < \pi_{0\gamma}^\mathcal{U}(\nu)$ whenever $\gamma + 1 \leq \beta'$. We have that

$$\pi_{0\alpha'}^\mathcal{T}(\nu) > \pi_{\alpha\alpha'}^\mathcal{T}(\kappa) = \pi_{\alpha\zeta}^\mathcal{T}(\kappa) \quad \text{and}$$

$$\pi^{\mathcal{U}}_{0\beta'}(\nu) > \pi^{\mathcal{U}}_{\beta\beta'}(\lambda) = \pi^{\mathcal{U}}_{\beta\lambda}(\lambda).$$

We may and shall in fact assume that

$$\pi^{\mathcal{T}}_{\alpha'\zeta} \restriction \pi^{\mathcal{T}}_{0\alpha'}(\nu) = \mathrm{id} \quad \text{and}$$

$$\pi^{\mathcal{U}}_{\beta'\theta} \restriction \pi^{\mathcal{U}}_{0\beta'}(\nu) = \mathrm{id}.$$

Let $Q \trianglerighteq K||\nu$ be a very soundness witness for $K||\nu$. We may construe $\mathcal{T} \restriction (\alpha'+1)$ and $\mathcal{U} \restriction (\beta'+1)$ as iteration trees acting on Q rather than on K. More precisely, we let $\mathcal{T}^*$ be the iteration of Q which is such that $\mathrm{lh}(\mathcal{T}^*) = \alpha'+1$ and $E^{\mathcal{T}^*}_\gamma = E^{\mathcal{T}}_\gamma$ whenever $\gamma < \alpha'+1$, and we let $\mathcal{U}^*$ be the iteration of Q which is such that $\mathrm{lh}(\mathcal{U}^*) = \beta'+1$ and $E^{\mathcal{U}^*}_\gamma = E^{\mathcal{U}}_\gamma$ whenever $\gamma < \beta'+1$. We let

$$\tilde{\pi}\colon \mathcal{M}^{\mathcal{T}^*}_{\alpha'} \to \mathcal{R} = Ult(\mathcal{M}^{\mathcal{T}^*}_{\alpha'}; \pi \restriction (W||\pi_{0\alpha'}(\nu)).$$

We'll have that $\pi \restriction (W||\pi_{0\alpha'}(\nu)) = \tilde{\pi} \restriction (W||\pi_{0\alpha'}(\nu))$, $\mathcal{R}$ is a universal weasel, $\tilde{\pi}(\pi_{0\alpha'}(\nu)) = \pi(\pi_{0\alpha'}(\nu))$ by Lemma 2.5, and $W'||\pi(\pi_{0\alpha'}(\nu)) \triangleleft \mathcal{R}$. Let $\tilde{\mathcal{R}}$ be the common co-iterate of $\mathcal{R}$ and $\mathcal{M}^{\mathcal{U}^*}_{\beta'}$, and let

$$i\colon \mathcal{R} \to \tilde{\mathcal{R}} \quad \text{and}$$

$$j\colon \mathcal{M}^{\mathcal{U}^*}_{\beta'} \to \tilde{\mathcal{R}}$$

be the maps arising from the coiteration. Notice that

$$i \restriction (\pi(\pi^{\mathcal{T}}_{\alpha\alpha'}(\kappa)+1) = \mathrm{id} \quad \text{and}$$

$$j \restriction (\pi^{\mathcal{U}}_{\beta\beta'}(\lambda)+1) = \mathrm{id}.$$

Finally, let Γ be thick in $\mathcal{M}^{\mathcal{T}^*}_{\alpha}$, $\mathcal{M}^{\mathcal{T}^*}_{\alpha'}$, $\mathcal{R}$, $\mathcal{M}^{\mathcal{U}^*}_{\beta}$, $\mathcal{M}^{\mathcal{U}^*}_{\beta'}$, and $\tilde{\mathcal{R}}$ and such that for all $\epsilon \in \Gamma$ we have that

$$\pi^{\mathcal{T}^*}_{\alpha\alpha'}(\epsilon) = \tilde{\pi}(\epsilon) = i(\epsilon) = \pi^{\mathcal{U}^*}_{\beta\beta'}(\epsilon) = j(\epsilon) = \epsilon.$$

Now let $\kappa = \tau^{\mathcal{M}^{\mathcal{T}^*}_\alpha}(\vec{\gamma}, \vec{\epsilon})$, where $\vec{\gamma} < \kappa$ and $\vec{\epsilon} \in \Gamma$. In order to show (1) it suffices to prove that

(2) $$\lambda = \tau^{\mathcal{M}^{\mathcal{U}^*}_\beta}(\pi(\vec{\gamma}), \vec{\epsilon}),$$

because then $\pi(\pi^{\mathcal{T}}_{\alpha\zeta}(\kappa)) = \pi(\pi^{\mathcal{T}}_{\alpha\alpha'}(\kappa)) = \tilde{\pi}(\pi^{\mathcal{T}^*}_{\alpha\alpha'}(\kappa)) = i(\tilde{\pi}(\pi^{\mathcal{T}^*}_{\alpha\alpha'}(\kappa))) = \tau^{\tilde{\mathcal{R}}}(\pi(\vec{\gamma}), \vec{\epsilon}) = j(\pi^{\mathcal{U}^*}_{\beta\beta'}(\tau^{\mathcal{M}^{\mathcal{U}^*}_\beta}(\pi(\vec{\gamma}), \vec{\epsilon}))) = j(\pi^{\mathcal{U}^*}_{\beta\beta'}(\lambda)) = j(\pi^{\mathcal{U}}_{\beta\beta'}(\lambda)) = \pi^{\mathcal{U}}_{\beta\beta'}(\lambda) = \pi^{\mathcal{U}}_{\beta\theta}(\lambda)$.

Write $\lambda = \bar{\tau}^{\mathcal{M}^{\mathcal{U}^*}_\beta}(\vec{\gamma}_0, \vec{\epsilon}_0)$, where $\vec{\gamma}_0 < \lambda$ and $\vec{\epsilon}_0 \in \Gamma$.

Suppose that $\lambda < \tau^{\mathcal{M}_\beta^{\mathcal{U}^*}}(\pi(\vec{\gamma}),\vec{\epsilon})$. Then, using $j \circ \pi^{\mathcal{U}^*}_{\beta\beta'}$, $\lambda \leq \bar{\tau}^{\tilde{\mathcal{R}}}(\vec{\gamma}_0,\vec{\epsilon}_0) < \tau^{\tilde{\mathcal{R}}}(\pi(\vec{\gamma}),\vec{\epsilon})$, and therefore

$$\tilde{\mathcal{R}} \models \exists \vec{\bar{\gamma}} < \lambda\ (\lambda \leq \bar{\tau}(\vec{\bar{\gamma}},\vec{\epsilon}_0) < \tau(\pi(\vec{\gamma}),\vec{\epsilon})). \tag{3}$$

Using $i \circ \tilde{\pi}$, hence,

$$\mathcal{M}^{\mathcal{T}^*}_{\alpha'} \models \exists \vec{\bar{\gamma}} < \kappa\ (\kappa \leq \bar{\tau}(\vec{\bar{\gamma}},\vec{\epsilon}_0) < \tau(\vec{\gamma},\vec{\epsilon})). \tag{4}$$

But (4) is wrong, as $\tau^{\mathcal{M}^{\mathcal{T}^*}_{\alpha'}}(\vec{\gamma},\vec{\epsilon}) = \pi^{\mathcal{T}}_{\alpha\zeta}(\kappa)$ is the least ordinal in $H^{\mathcal{M}^{\mathcal{T}^*}_{\alpha'}}(\kappa) \setminus \kappa$. Therefore, $\lambda \geq \tau^{\mathcal{M}^{\mathcal{U}^*}_\beta}(\pi(\vec{\gamma}),\vec{\epsilon})$.

However, if $\lambda > \tau^{\mathcal{M}^{\mathcal{U}^*}_\beta}(\pi(\vec{\gamma}),\vec{\epsilon})$ then $\tau^{\tilde{\mathcal{R}}}(\pi(\vec{\gamma}),\vec{\epsilon}) = \tau^{\mathcal{M}^{\mathcal{U}^*}_\beta}(\pi(\vec{\gamma}),\vec{\epsilon}) < \lambda$, and thus by using $i \circ \tilde{\pi}$, $\kappa = \tau^{\mathcal{M}^{\mathcal{T}^*}_\alpha}(\vec{\gamma},\vec{\epsilon}) < \kappa$, which is nonsense. We have therefore established that (2) holds.

$\square$ (Claim 2.9)

Claim 2.10 *Let $\alpha + 1 \leq \zeta$, and set $\beta = \varphi(\alpha)$. Then π_β" $E^{\mathcal{T}}_\alpha \subset E^{\mathcal{U}}_\beta$. In fact, $\pi_\beta(E^{\mathcal{T}}_\alpha) = E^{\mathcal{U}}_\beta$.*

Proof of Claim 2.10. Set $\kappa = \mathrm{crit}(E^{\mathcal{T}}_\alpha)$, and $\lambda = \mathrm{crit}(E^{\mathcal{U}}_\beta) = \pi(\kappa)$. Fix $X \in E^{\mathcal{T}}_\alpha$. We then get $\kappa \in \pi^{\mathcal{T}}_{\alpha\zeta}(X)$, hence $\lambda = \pi(\kappa) \in \pi(\pi^{\mathcal{T}}_{\alpha\zeta}(X)) = \pi^{\mathcal{U}}_{\beta\theta}(\pi_\beta(X))$. But this means that $\pi_\beta(X) \in E^{\mathcal{U}}_\beta$. This shows that π_β" $E^{\mathcal{T}}_\alpha \subset E^{\mathcal{U}}_\beta$.

However, we may now apply Lemma 2.6 and deduce that in fact $\pi_\beta(E^{\mathcal{T}}_\alpha) = E^{\mathcal{U}}_\beta$.

$\square$ (Lemma 2.10)

This finishes the proof of Theorem 2.3.

$\square$ (Theorem 2.3)

3 The main results.

In order to obtain our main results we shall make use of the following criterion.

Lemma 3.1 *Let $A \subset {}^\omega\omega$. Then A is coanalytic if and only if there is a function $n\colon {}^{<\omega}\omega \to \omega$ with $n(t) \geq n(s) \geq \mathrm{lh}(s)$ for $s \subset t \in {}^{<\omega}\omega$ and a system $(\sigma_{st}\colon s \subset t \in {}^{<\omega}\omega)$ such that for $s \subset t \in {}^{<\omega}\omega$, $\sigma_{st}\colon n(s) \to n(t)$ is order-preserving, and for all $x \in {}^\omega\omega$,*

$$x \in A \iff \lim \mathrm{dir}((n(s)\colon s \subset x), (\sigma_{st}\colon s \subset t \subset x)) \text{ is wellfounded.}$$

Proof of Theorem 1.5. Let $A \subset {}^\omega\omega$, and let

$$((M_s\colon s \in {}^{<\omega}\omega), (\pi_{st}\colon s \subset t \in {}^{<\omega}\omega))$$

be an ω-closed embedding normal form for A. We have that $M_0 = V = K$. For an arbitrary $s \in {}^{<\omega}\omega$, M_s must be a *finite* iterate of K, because ${}^{\omega}M_s \subset M_s$. Let $\mathcal{T}_s = (E_i^{\mathcal{T}_s} : i < n(s))$ denote the iteration from K to M_s. In particular, $n(s)$ is the number of critical points of this iteration from K to M_s. Set $\kappa_i^s = \text{crit}(E_i^{\mathcal{T}_s})$ for $i < n(s)$. By Theorem 2.3, we know that in particular

$$\pi_{st}"\{\kappa_0^s, ..., \kappa_{n(s)-1}^s\} \subset \{\kappa_0^t, ..., \kappa_{n(t)-1}^t\}$$

whenever $s \subset t \in {}^{<\omega}\omega$. Hence if $s \subset t \in {}^{<\omega}\omega$ then π_{st} induces an order-preserving map σ_{st} such that $\pi_{st}(\kappa_i^s) = \kappa_{\sigma_{st}(i)}^t$ for all $i < n(s)$.

We claim that $((n(s) : s \in {}^{<\omega}\omega), (\sigma_{st} : s \subset t \in {}^{<\omega}\omega))$ witnesses that A is coanalytic. The non-trivial part here is to show that if the direct limit of $((n(s) : s \subset x), (\sigma_{st} : s \subset t \subset x))$ is wellfounded then the direct limit of $((M_s : s \subset x), (\pi_{st} : s \subset t \subset x))$ is also wellfounded.

Let $x \in {}^{\omega}\omega$ be such that the direct limit of $((n(s) : s \subset x), (\sigma_{st} : s \subset t \subset x))$ is wellfounded, and let $\theta < \omega_1$ be its ordertype. The point is that we may now use Theorem 2.3 to construct an iteration $\mathcal{T}$ of K of length $\theta + 1$ together with commuting elementary embeddings from the models $\mathcal{M}_i^{\mathcal{T}_x \restriction n}$ into the models $\mathcal{M}_j^{\mathcal{T}}$. This construction will in particular give embeddings $\pi_{sx} : M_s \to \mathcal{M}_\theta^{\mathcal{T}}$ such that $\pi_{tx} \circ \pi_{st} = \pi_{sx}$ whenever $s \subset t \subset x$. We leave the straightforward details of this construction to the reader. But then the direct limit of $((M_s : s \subset x), (\pi_{st} : s \subset t \subset x))$ can be embedded into (in fact, is equal to!) $\mathcal{M}_\theta^{\mathcal{T}}$, so that it must be wellfounded.

$\square$ (Theorem 1.5)

Lemma 3.2 ($\neg\, 0^{\text{long}}$) *Let $\pi : V \to M$, where M is transitive and ${}^{\omega}M \subset M$. Then K^M is a finite iterate of K.*

Proof. By [1, Theorem 3.23] (cf. also [5, Theorem 1.3] and the remark right after it), there is a maximal Prikry system $\mathbb{C}$ for K. Let us construe $\mathbb{C}$ as a set of ordinals; we shall have that if ν denotes the order type of the measurable cardinals of K then $\mathbb{C}$ will be of order type at most $\omega \cdot \nu$. By $\neg\, 0^{\text{long}}$, ν is less than the least measurable cardinal of K, which in turn is less than or equal to $\text{crit}(\pi)$. The covering lemma (cf. [1, Theorem 3.23]) says that for each set X of ordinals there is some function $f \in K$ and some $\alpha < \aleph_2 \cdot \text{Card}(X)^+$ such that $X \subset f"(\alpha \cup \mathbb{C})$.

By the Dodd-Jensen Lemma, K^M is universal and hence a non-dropping iterate of K. Now suppose that K^M would be an infinite iterate of K. Let A be the set of the first ω many critical points of the iteration from K to K^M. By elementarity, there is some function $f \in K^M$ and some $\alpha < \aleph_2$ such that $A \subset f"(\alpha \cup \pi(\mathbb{C}))$. However, $\pi(\mathbb{C}) = \pi"\mathbb{C}$, as $\text{otp}(\mathbb{C}) < \text{crit}(\pi)$. That is, $A \subset f"(\alpha \cup \pi"\mathbb{C})$.

Let $f = \pi(\bar{f})(a)$, where $\bar{f} \in K$ and a is a finite set of critical points of the iterations from K to K^M. Let $\kappa \in A \setminus a$. Then $\kappa = (\pi(\bar{f})(a))(\xi)$, where $\xi \in \alpha \cup \pi``\mathbb{C}$, so that in particular κ is in the hull of $\mathrm{ran}(\pi) \cup a$ formed inside K^M. But this is a contradiction, as κ is one of the critical points of the iteration from K to K^M.

$\square$ (Lemma 3.2)

Proof of Theorem 1.6. This can now be shown exactly as Theorem 1.5 above, using Lemma 3.2.

$\square$ (Theorem 1.6)

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