derived model 1

thm. (Woodin) let λ be a lim of Woodin cardinals, and let g $Col(\omega, <\lambda)$ gen. $/V$.

$\mathbb{R}^* = \bigcup_{\alpha<\lambda} \mathbb{R} \cap V[g\restriction\alpha]$.

$\Gamma_\alpha = \{ A \in \mathcal{P}(\mathbb{R})^{V[G\restriction\alpha]} : V[G\restriction\alpha] \models$ "A is $<\lambda$-universally baire" $\}$

$\Gamma^* = \{ A^* \subset \mathbb{R}^* : \exists \alpha < \lambda \ A \in \Gamma_\alpha \}$

$\uparrow$

union of versions of A in $V[g\restriction\beta]$, $\alpha \leq \beta < \lambda$.

then $L(\mathbb{R}^*, \Gamma^*) \models ZF + AD^+$.

let M be a ctbl. ω_1+1 iterable premouse (by Σ). Say $M \models$ "λ is a lim of woodins." ($cf(\lambda)$ not meas in M)

let $(x_n : n<\omega) = \mathbb{R}^V$ (in $V^{Col(\omega, \mathbb{R})}$)

let $(\delta_n : n<\omega)$ be s.t. $\sup_n \delta_n = \lambda$,
each δ_n is wood' in M.

build $\mathcal{I} = (M_n, \pi_{mn} : m \leq n < \omega)$
s.t.

π_{mn} results from a nontrivial dropping it. tree
in V on M_m living in $(\pi_{0m}(\delta_{m-1}), \pi_{0m}(\delta_{n-1}))$

$g_n \subset \mathrm{Col}(\omega, \pi_{0n+1}(\delta_n))$ gen. / M_{n+1},
$x_n \in M_{n+1}[g_n]$.

$M_\omega = \text{dir lim}(M_n, \pi_{mn} : m \leq n < \omega)$.
$g = \bigcup_n g_n \subset \mathrm{Col}(\omega, <\pi_{0\omega}(\lambda))$-g. $/ M_\omega$.

as being given by woodin's thm

$$D^{\mathcal{I}}(M, \lambda) = L(\mathbb{R}^V, \Gamma_g^{\pi(\lambda), M_\omega[g]}).$$

$$D^{\mathcal{I}}(M,\lambda) \models \text{"}AD^+ + ZF + \mathbb{R}^{D^{\mathcal{I}}(M,\lambda)} = \mathbb{R}^V\text{"}.$$

don't know that $D^{\mathcal{I}}(M,\lambda) \in V$.

forcing $\mathbb{P}$:

$p = (M_n^p, I_n^p, \delta_n^p, g_n^p : m \leq n < lh(p))$

$\pi_m^p = \ldots,$

M^p = lan model

$\pi_j^p = \ldots ,$

ordered by end-extension.

if $G \subset \mathbb{P}$, then I_G generic iteration.

can we have that $\Gamma^{I_G, \lambda}$ be independent for G.

<u>lemma</u>. (Steel) if M is sound + has a good iteration strategy, then $\Gamma^{I_G, \lambda} \in V$ $\forall G$.

<u>proof</u> (idea) :

$$\begin{tikzcd} I_G & M_p[g_p] \arrow[l, dotted, "i_1"'] \arrow[r, dotted, "i_0"] \arrow[d, phantom, "\ni"] & {} \\ M \arrow[ur] & \mathcal{T}, \mathcal{U} & \end{tikzcd}$$

$x \in p[i_1(\mathcal{T})]$ $\qquad\qquad$ $x \in p[i_0(\mathcal{U})]$

may compare $i_1(M_p)$, $i_0(M_p)$ back together into one model. gives $\subsetneq$.

So $\Gamma^{I_{G,\lambda}} \subset V$.

<u>def.</u> a ctbl. pm M is a sharp premouse if $M = (M; \in, \vec{E}, F)$, and

$$\text{crit}(F) \text{ is } = \min\{\alpha : \exists E \in \vec{E}^\frown F\ [\alpha = \text{crit}(E) \wedge M\|\alpha \models \varphi]\},$$

where φ is a sentence in the language of passive premice.

let M be a premouse, $a \in M$, $\kappa < M \cap OR$.

[ex. $M_\omega^\#$ $\quad \delta_1 \quad \delta_0 \quad$ $M_{wsws\ldots}$]

let $\mathbb{L}^M_\kappa(a)$ be the local K^c constr.
with +1 certificates for the M-
sequence + crit. $> \kappa$ over a.

<u>def.</u> a sharp premouse M rebuilds itself
below λ iff f.a. non-dropping iterates
$i : M \longrightarrow N$, $\forall \kappa < i(\lambda)$
we have $\mathbb{C}\left((\mathbb{L}^W_\kappa)^\# \right) = M$.

M_{refl} = least ω-proj. sound active
mouse with λ limit of woodins
+ $< \lambda$-strongs and $\kappa < \lambda$
which reflects the set of $< \lambda$ strongs.

<u>Thm.</u> (Steel) Let M be ω-proj. sound sharp mouse with a good ω_1+1 itrn strategy, $\lambda \in M$ a lim of M-woodins which reconducts itself below λ.

Then $D^{I_G(M;\lambda)}$ does not depend on G.

pf.: sketch:

$$\sigma''\mathcal{E}_s(\alpha) \subset \sigma(\mathcal{E}_s(\alpha)) \longrightarrow \mathbb{L}^{M_{p''}}$$

$$\begin{tikzcd}
 & \mathbb{L}^* \arrow[r, dashed] & M_{p'}[g_{p'}] \arrow[r] & M_{p''}[g_{p''}] \\
 & \mathbb{L}^{M_p}_{\kappa} \subset M_p[g_p] \arrow[u] \arrow[ur] & & \\
M_q[g_q] \arrow[uur, dotted] & & & \\
 & M \arrow[ul] \arrow[uu, dotted] \arrow[uur] & &
\end{tikzcd}$$

$\mathbb{L}^{M_{p''}} \cap M_{p''}[g_{p''}]$

$g_{p''}$ codes $\sigma''(\mathcal{E}_s(\alpha) : s \in {}^{<\omega}\omega)$

$T, \mathcal{U}$

$\cap$

$\wr$

$(\mathcal{E}_s(\alpha) : s \in {}^{<\omega}\omega)$ extender normal for $T^*, \mathcal{U}^*$

$\wr$

$\uparrow$

add. of extenders in round

let that $L(\Gamma, \mathbb{R}) \models AD^+$.

define $(\theta_\alpha : \alpha < \beta)$ s.t.

$$\theta_0 = \sup \{ \gamma : \exists f \in OD \quad f : \mathbb{R} \to \gamma \text{ onto} \}$$

$$\theta_{\alpha+1} = \sup \{ \gamma : \exists\ OD_A \quad f : \mathbb{R} \to \gamma \text{ onto} \}$$

$$\theta_\alpha < \theta, \quad \|A\|_w = \theta_\alpha .$$

$$\theta_\lambda = \sup_{\alpha < \lambda} \theta_\alpha \quad \text{if } \lambda \text{ is a lim}$$

Thm. (Woodin, + Steel: ③ left to right)

① Con (ZF + AD^+ + $\theta = \theta_0$) $\iff$

Con (ZFC + $\exists \lambda$ λ a lim of woodins)

② Con (ZF + AD^+ + $\theta = \theta_1$)

$\iff$ Con (ZFC + $\exists \lambda$ λ is a lim of woodins + $k < \lambda$ is $< \lambda$-strong)

③ Con (ZF + AD^+ + $\theta = \theta_\omega$) $\iff$

Con (ZFC + $\exists \lambda$ lim of woodins + $< \lambda$-strongs)

<u>lem.</u> let Σ be a (λ, λ)-strategy with hull condensation

s.t. for some $\alpha < \lambda$,

Σ extends uniquely to a $< \lambda$-u.b.

strategy on $M^{Col(\omega, \alpha)}$.

then f.a. $a \in M \| \lambda$,

$$L_p^{\Sigma \restriction D(M,\lambda)}(a) = \mathbb{F}^{M,\Sigma}_{rank(a)}(a) \| a^{+\mathbb{F}^{M,\Sigma}_{rank(a)}}$$

pf.: "$\subset$" maximality.

"$\supset$" $M \lhd \mathbb{F}^{M,\Sigma}(a)$, proj. to a.