Jensen:

making Δ_1 determined again

(joint work with [illegible])

I	$x(0)$		$x(2)$
II		$x(1)$	

...

we study games of length ω_1.

def: let φ be a formula. the closed game G_φ is the game

I	$f(0)$	
II		$f(1)$

$i < \omega_1$

I wins iff $\forall \alpha < \omega_1$

$$(H_{\omega_1}; f\restriction\alpha, \in) \models \varphi(f \restriction \alpha).$$

thm. (neeman) let M be a ctbl. tr. model of ZFC and ~~&~~ $\delta \in M$ be s.t.

$M \models$ "δ is a measurable woodin cardinal"

then for any φ, ~~G~~ G_φ is determined.

rmk: (neeman) one actually needs less.

defn: let φ be a fmla. the game G_φ $= \{ f \in 2^{\omega_1} : \varphi(f, \omega_1) \}$.

thm. (woodin) let δ be a woodin cardinal + a measurable cardinal which is a limit of woodin cardinals. then there is a model $L[A]$ for $A \subset \omega_1$ s.t. if φ is a fmla, then

$G_\varphi^{L[A]}$ is defined $L[A]$.

defn. a game of the form G_φ is provably Δ_1 if ~~the~~ φ is Σ_1

and there is a Σ_1 formula φ s.t.

$$ZFC \vdash G_\varphi = {}^{\omega_1}2 \setminus G_\varphi .$$

__th.__ assume $\exists$ ctb. il. model with a measurable woodin.

all provably Δ_1 games are determined.

__quest:__ suppose that all closed games of length ω_1 are determined. are all provably Δ_1-games determined?

__quest:__ supp. $\exists$ proper class of woodin limits of woodins + that closed ω_1-det. holds in all gen. extensions. does provably Δ_1-det. hold (in all gen. ext.)?

the proof of the thm. uses:

__therm:__ ~~supp. $\exists$ ctb. it. model with a~~

~~meas~~ supp. $\exists$ proper class of woodin cardinals, CH holds, for every u.b.

set of reals A there is a "nice" (w.r.t. A) model with a measurable cardinal.

let $A \subset \mathbb{R}$ be u.b.

then f.a. games $\{ f \in {}^{\omega_1}2 : \varphi(f, A) \}$ there is a strategy Σ for I or II s.t. ~~every~~ every play by σ is winning in a gen. ext. of M, N, s.t. $\omega_1^N = \omega_1$, $N \in V$.

proof uses the techniques of neeman heavily (you'll see).

<u>defn.</u> $A \subset \mathbb{R}$ is u.b. ($A \in \Gamma^\infty$) iff f.a. κ there are trees T, S on $\omega \times \lambda$ (some λ) s.t. $\Vdash_{\mathrm{Col}(\omega,\kappa)} p[T] = \mathbb{R} \setminus p[S]$, $A = p[T]$ in V.

<u>defn.</u> let φ be a formula. we write $\mathrm{ZFC} \models_\Omega \varphi$ iff for all g gen. $/V$,

$V_\kappa[g] \models ZFC$ implies $V_\kappa[g] \models \varphi$.

let A be u.b. a model M of ZFC is A closed if whenever $\mathbb{P} \in M$, $g \subset \mathbb{P}$ M-gen., then $A_g \cap M[g] \in M[g]$, where $A_g = \bigcup \{ p[T]^{V[g]} : p[T] = A, T \in V \}$

alternatively, if f.a. $b \in M$, $\tau_A^\infty \cap b \in M$, where $\tau_A^\infty = \{ (\gamma, p, \sigma) : p \Vdash_{Col(\omega,\gamma)} \sigma \in A_g \}$.

def. assume there is a proper class of woodin cardinals. then we write $ZFC \vdash_\Omega \varphi$ if there is $A \in \Gamma^\infty$ s.t. for all A-closed M, $M \models$ "$ZFC \vdash_\Omega \varphi$."

def. a game G_φ is Ω-provably $\Delta_1(A)$ definable if $\varphi(A,-)$ is Σ_1 and there is a Σ_1 formula $\psi(A,-)$ s.t. $ZFC \vdash_\Omega G_\varphi = {}^{\omega_1} \setminus G_\psi$.

<u>th.</u> Supp. th is a proper class of wood. cardinals, CH holds, and that f.a. $A \in \Gamma^\infty$ th is a model M that is A-closed, has a measurable woodin cardinal which is ctbl in V, and a it. strategy $\Sigma \in \Gamma^\infty$ s.t. if $j: M \to N$ is an it. by Σ, then

- N is A-closed,
- $j(\tau_A^\infty \cap (V_\kappa)^M) = \tau_A^\infty \cap (V_{j(\kappa)})^N$ f.a. κ

then every Ω-provably-$\Delta_1(A)$ is determined, f.a. $A \in \Gamma^\infty$.

proof sketch:

<u>def.</u> assume hypo. of th. we say that a u.b. set <u>satisfies condensation</u> iff $(N, \tau) \prec (V_\kappa, \tau_A^\infty)$ for κ inacc.

~~then~~ and (N^*, τ^*) is the transitive collapse, then $\tau^* = \tau_A^\infty \cap N^*$.

<u>lem</u> (Woodin) (same hypo) let $A \in \Gamma^\infty$ then there is $B \in \Gamma^\infty$ s.t. B satisfies condensation and A is continuously reducible to B.

so may assume A satisfies condensation.

$G_\varphi^{L,A}$

I	$f(0)$	...
II	$f(1)$	...

$(i < \omega_1)$

player I wins iff there is $\kappa > \omega_1$ and $g \in {}^{\omega_1}2$ s.t. $L_\kappa[f][g][\tau_A^\infty] \models ZFC$ and $\left(L_\kappa[f][g][\tau_A^\infty];\ f, g, A \cap L_\kappa[f][g][\tau_A^\infty]\right) \models \varphi$, where φ is Ω-provably $\Delta_1(A)$ as witnessed by some ψ.

~~$ZFC \vdash_\Omega G_\varphi^{L,A} \leftrightarrow {}^\omega 2 \not\models G_\psi^{L,A}$~~

<u>lem.</u> TFAE under our hypotheses.

- if $ZFC \vdash_{\Omega} G_{\varphi}^{L,A} = {}^{\omega_1}2 \setminus G_{\psi}^{L,A}$, then $G_{\varphi}^{L,A}$ is determined.
- all provably $-\Delta_1(A)-$ games are det.

if so, then let A^* ~~th~~ $\in \Gamma^{\infty}$ coding A and a witness for the fact that

$$ZFC \vdash_{\Omega} G_{\varphi}^{L,A} = {}^{\omega_1}2 \setminus G_{\psi}^{L,A}.$$

and let M be a model in a mouse / Woodin s.t. M is nice w.r.t. A^*. apply the 2^{nd} thm (the black box) to obtain:

<u>Case 1.</u> a strategy for player I s.t. every play by σ is winning in a gen. extension (in V) of an iterate of M that computes ω_1 correctly.

rest of the pf.: exercise.