Omer I

general quest.: to what extent do compactness (LC) properties imply the ex. of a diamond seq.?

quest: does κ weakly compact imply $\diamondsuit_\kappa$?

key test quest: ~~[illegible]~~ let $L[\mathcal{U}]$ be a min. model with a coherent sequence $\mathcal{U}$ satisfying WRP. does $\mathcal{U}$ satisfy the LRP ?

(RP) $\exists \rho < o(\kappa)\ \forall b \in m\mathcal{F}\quad [b]_{\mathcal{U}_{\kappa,\rho}} \in \mathcal{F}_\kappa$

(LRP) $\forall b \in m\mathcal{F}\ \exists \rho < o(\kappa)\quad [b]_{\mathcal{U}_{\kappa,\rho}} \in \mathcal{F}_\kappa$

(WRP) $\forall b \in m\mathcal{F}\ \exists W$ satisfying $\circledast$ $[b]_W \in \mathcal{F}_\kappa$

to be explained p. 7 f.

history.

(folklore) if κ is measurable, then $\diamondsuit_\kappa$.

(jensen–kunen) if κ is subtle, then $\diamondsuit_\kappa$.

κ is subtle iff for any sequence $(a_\alpha : \alpha < \kappa)$ s.t. $a_\alpha \subset \alpha$ f.a. α and for all clubs $C \subset \kappa$, there are $\alpha < \beta$ in C s.t. $a_\alpha = a_\beta \cap \alpha$.

proof of jen–kunen: constr. a seq. $((S_\alpha, C_\alpha) : \alpha < \kappa)$ by ind.

at lim β, if $(S_\alpha : \alpha < \beta)$ is not a $\diamondsuit_\beta$ – seq, take $(S_\beta, C_\beta) =$ a counterexample, i.e., $S_\beta \cap \alpha \neq S_\alpha$ $\forall \alpha \in C_\beta$.

Sps. o.w. and take (S, C) a counterexample. have limits ~~sth.~~ $\alpha < \beta$ of C s.t. $S_\alpha = S_\beta \cap \alpha$ + $C_\alpha = C_\beta \cap \alpha$.

$\alpha \in C \Rightarrow C_\alpha \subset \alpha$ is club,

$C_\alpha = C_\beta \cap \alpha \Rightarrow \alpha \in C_\beta$.

$\Rightarrow S_\beta \cap \alpha \neq S_\alpha$. $\lightning$ $\dashv$

actually, κ subtle $\Rightarrow$

$$\underbrace{\exists \diamondsuit_\kappa (\mathrm{Reg})}_{\text{guesses on regular sets}} - \text{seq}.$$

natural question: which LC properties imply $\diamondsuit(\mathrm{Reg})$.

(woodin) con($\neg \diamondsuit_\kappa(\mathrm{Reg})$ + κ is weakly compact)

(hauser) $\forall n \forall m$ con($\neg \diamondsuit_\kappa(\mathrm{Reg})$ + κ is Π^n_m indescribable)

(dzamonja - hamkins)

con($\neg \diamondsuit_\kappa(\mathrm{Reg})$ + κ is strongly unfoldable)

$$\frac{\text{strongly unfoldable}}{\text{strong}} = \frac{\text{weakly compact}}{\text{measurable}}$$

remark. all $\neg\Diamond(\mathrm{Reg})$ results are obtained for optimal assumptions. e.g.

$\mathrm{con}(\neg\Diamond_\kappa(\mathrm{Reg}) + \kappa \text{ w.c.})$ requires $\mathrm{con}(\kappa \text{ w.c.})$.

(jensen) $\neg\Diamond_\kappa + \kappa$ mahlo $\Rightarrow \exists\, 0^\#$.

(zeman) $\neg\Diamond_\kappa + \kappa$ mahlo $\Rightarrow$

$\forall \tau < \kappa \ \{\alpha < \kappa : o^K(\alpha) \geq \tau\}$ is stationary.

(woodin) from certain hypermeasurability assumptions

$\mathrm{con}(\neg\Diamond_\kappa + \kappa \text{ inacc.})$ for $o(\kappa) = \kappa^{++} + \kappa^{+}$

$\mathrm{con}(\neg\Diamond_\kappa + \kappa \text{ mahlo})$ for $o(\kappa) = \kappa^{++} + (\kappa^{+})^2$

$\mathrm{con}(\neg\Diamond_\kappa + \kappa \text{ greatly mahlo})$

$o(\kappa) = \kappa^{++} + (\kappa^{+})^{\kappa^{+}}$.

con($\neg \square_\kappa$ + κ reflects every stat. set)

?

in L, this is eq. to κ being weakly compact

for $o(\kappa) = \kappa^{+3} + \kappa^{+2}$.

<u>def.</u> let $\mathcal{U} =$ $(\mathcal{U}_{\alpha,\tau} : \alpha \leq \kappa,\ \tau < o^{\mathcal{U}}(\alpha))$ be a coherent seq. of normal measures.

here, $\mathcal{U}_{\beta,\tau}$ = a normal measure on β.

if $j_{\beta,\tau} : V \longrightarrow M_{\beta,\tau} = \mathrm{ult}(V; \mathcal{U}_{\beta,\tau})$.

$j_{\beta,\tau}(\vec{\mathcal{U}}) \upharpoonright \beta+1 = \vec{\mathcal{U}} \upharpoonright \beta \frown (\mathcal{U}_{\beta,\tau'} : \tau' < \tau)$.

in part., $o^{j_{\beta,\tau}(\mathcal{U})}(\beta) = \tau$.

<u>def.</u> for every $\alpha \leq \kappa$ define

$$\mathcal{F}_\alpha = \begin{cases} \bigcap_{\tau < o(\alpha)} \mathcal{U}_{\alpha,\tau} & \text{if } o(\alpha) > 0 \\ \{0\} & \text{o.w.} \end{cases}$$

<u>example</u> ("length" properties)

- if $o^{\mathcal{U}}(\kappa) = \kappa^+$, then κ remains regular in $R(\mathcal{U})$
  radial forcing ass. with $\mathcal{U}$.

- for $\tau \leq \kappa^+$, if $o^{\mathcal{U}}(\kappa) = (\kappa^+)^\tau \Rightarrow$
  κ is τ-mahlo in $R(\mathcal{U})$.

- if $o^{\mathcal{U}}(\kappa) = \kappa^{++}$, then κ reflects stat. sets.

<u>thm.</u> (woodin) if (in V) $2^\kappa > o^{\mathcal{U}}(\kappa)$, then $\neg \diamondsuit_\kappa$ fails in an $R(\mathcal{U})$ extension.

<u>defn</u> (mitchell) let $\rho < o^{\mathcal{U}}(\kappa)$. we say $\mathcal{U}_{\kappa,\rho}$ is a <u>repeat point</u> (r.p.) if

$$\mathcal{F}_\kappa = \bigcap_{\tau < o(\kappa)} \mathcal{U}_{\kappa,\tau} = \bigcap_{\tau < \rho} \mathcal{U}_{\kappa,\tau} = \mathcal{F}_\kappa^{M_{\kappa,\rho}}$$

[mitchell actually called this "weak repeat pt."]

thm. κ is measurable in a $R(\mathcal{U})$ extension iff $\mathcal{U}$ has a r.p.

def. Say $\mathcal{U}$ satisfies the repeat property (RP) if it has a repeat point.

clearly, $2^\kappa > o^{\mathcal{U}}(\kappa)$ + ~~BR~~ RP $\rightarrow$ $o = 1$.

another formulation of RP.

def. a function $b : \kappa \longrightarrow \mathcal{P}_\kappa(\kappa)$ is a measure function if $b(\alpha) \in \mathcal{F}_\alpha$ for all $\alpha < \kappa$.

fix $\rho < o^{\mathcal{U}}(\kappa)$. $[b]_{\mathcal{U}_{\kappa,\rho}} = j_{\kappa,\rho}(b)(\kappa)$

$\in \mathcal{F}_\kappa^{M_{\kappa,\rho}}$.

if $\mathcal{U}_{\kappa,\rho}$ is a r.p., then $[b]_{\mathcal{U}_{\kappa,\rho}} \in \mathcal{F}_\kappa$.

def. $\mathfrak{M}\mathcal{F}$ = set of all measure fcts b.

(RP) $\Leftarrow$ $\exists \rho < o(\kappa)$ $\forall b \in \mathfrak{M}\mathcal{F}$

$$[b]_{\mathcal{U}_{\kappa,\rho}} \in \mathcal{F}_\kappa .$$

& def. (local repeat property)

$$\forall b \in \mathfrak{M}\mathcal{F} \ \exists \rho < o(\kappa) \ [b]_{\mathcal{U}_{\kappa,\rho}} \in \mathcal{F}_\kappa .$$

fact. if $\mathcal{U}$ satisfies LRP, then κ is weakly compact in a $R(\mathcal{U})$ extension.

(don't know if this an "iff.")

def. weak repeat property, WRP.

$\forall b \in \mathfrak{M}\mathcal{F} \ \exists W$ satisfying $\circledast$ s.t. $[b]_W \in \mathcal{F}_\kappa$.

$\circledast$ means : $\circledast_1$ W is a κ-complete filter on κ s.t. $W \subset \bigcup_{\tau < o(\kappa)} \mathcal{U}_{\kappa,\tau}$

$\circledast_2$ $[b]_W$ is well-defined, i.e., for any $\beta < \kappa$,

$$X_\beta = \{\alpha < \kappa : \beta \in b(\alpha)\}, \text{ then}$$

either $X_\beta \in W$ or $X_\beta \setminus \kappa \in W$.

theorem. $\mathcal{U}$ satisfies WRP $\Leftrightarrow$ κ is w.c. in a $R(\mathcal{U})$-extension.

key question: can $2^\kappa > o^{\mathcal{U}}(\kappa)$ + WRP be consistent?

claim. $2^\kappa > o^{\mathcal{U}}(\kappa)$ + LRP $\Rightarrow$ $\longrightarrow$ $0 = 1$.