The ordinal u_2 and a thin Δ^1_3 equivalence relation

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4th Münster conference on inner model theory
July 27th, 2017

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More generally, if x is a set of ordinals we say that $x^\#$ exists iff there is a non-trivial elementary embedding $j : L[x] \to L[x]$ that does no move ordinals up to $\sup(x)$.

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Theorem (folklore)

The property "For every set of ordinals x, $x^\#$ exists" is preserved by any forcing.

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Theorem (R. David)

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However, if we impose some conditions over the forcing notion some positive results hold:

Theorem (Schlicht)

Suppose that $\mathbb{P}$ is a provably Σ^1_2-definable c.c.c. forcing notion. Then, $\mathbb{P}$ preserves the property "every real has a sharp".

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Theorem (C.-Schlicht)

Suppose $\mathbb{P} \in \{\mathbb{S}, \mathbb{M}, \mathbb{V}, \mathbb{L}, \mathbb{ML}\}$ and let $n \in \omega$. Then $\mathbb{P}$ preserves the property

$$\text{“}M_n^\#(x) \text{ exists for every real } x\text{”}.$$

Therefore, projective determinacy is preserved by $\mathbb{P}$.

Arboreal forcing notions

"Organic" illustrations of binary trees.

Arboreal forcing notions

Definition

A partial order $\mathbb{P}$ is arboreal if its conditions are perfect trees on ω or 2 ordered by inclusion. A partial order $\mathbb{P}$ is strongly arboreal if it is arboreal and for all $T \in \mathbb{P}$, if $t \in T$, $T_t = \{s \in T : \text{either } s \subseteq t \text{ or } t \subseteq s\} \in \mathbb{P}$.

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If $\mathbb{P}$ is strongly arboreal, we can code generic objects by reals in the standard way: if G is $\mathbb{P}$-generic over V, then $x_G = \bigcup\{\text{Stem}(T) : T \in G\} = \bigcap\{[T] : T \in G\}$ is a real and $G = \{T \in \mathbb{P} : x_G \in [T]\}$

Proper forcing and names for reals

Proposition

Let $\mathbb{P} \subseteq \mathbb{R}$ be a proper forcing notion, G a $\mathbb{P}$-generic filter over V. If $x \in V[G] \cap \mathbb{R}$, then there exists a name $\sigma \in \mathrm{H}(\omega_1)$ such that $\sigma^G = x$.

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Proof.

Suppose $\tau = \{\langle\langle n, m\rangle\check{}, p\rangle : n, m \in \omega, p \in A_n,\ A_n \text{ is an antichain}\}$ is a $\mathbb{P}$-name for x. This means that $p \Vdash_{\mathbb{P}} \dot{x}(n) = m$.

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Note that X is countable in $V[G]$, so, by properness of $\mathbb{P}$ there exists a countable set $Y \in V$ such that $X \subset Y$.

Take $\sigma = \tau \cap Y$. Then σ is countable and $\sigma^G = \tau^G$.

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Suppose that $S \in \mathbb{S}$. We define:

$$\mathbb{A}_{\mathbb{S},S} = \{t \subseteq S : t \text{ is a finite subtree of } S \text{ isomorphic to some } {}^{n}2\ \}$$

ordered by end-extension, i.e. $t \leq s$ if and only if $t \supseteq s$ and $t\restriction_{|s|} = s$.

Given $S \in \mathbb{S}$, let $\pi_S : \mathrm{Split}(S) \to^{<\omega} 2$ be the natural order isomorphism.

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Lemma

Suppose that G is $\mathbb{A}_{\mathbb{S},S}$-generic over V. Then:

- $T_G = \bigcup G$ is a perfect subtree of S.
- For every $x \in [T_G]$, $\pi_S(x) := \bigcup_{n<\omega} \pi_S(x\restriction_n)$ is Cohen-generic over V.

Lemma

Suppose that $\forall x \in \mathbb{R}(x^{\#}$ exists) and let $\sigma \in \mathrm{H}(\omega_1)$. Let $\dot{x}$ a name for the $\mathbb{S}$-generic real. For every $S \in \mathbb{S}$, there is some $T \leq S$ such that

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Proof

Since $(\sigma, S)^\#$ exists, we have that $|\wp(\mathbb{A}_{\mathbb{S},S})^{L[\sigma,S]}| < \omega_1$ so there is a $\mathbb{A}_{\mathbb{S},S}$-generic T in V over $L[\sigma, S]$. By the lemma above, every branch in T is $\mathbb{C}$-generic over $L[\sigma, S]$ modulo π_S and $T \leq S$.

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In particular, as $\dot{x}$ is a Sacks real, if $T \in G$ we have

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i.e., $T \Vdash \dot{x}$ is $\mathbb{C}$-generic over $L[\sigma, S]$ modulo π_S.

Lemma

Suppose that V is closed under sharps for reals. Suppose that $r \in \mathbb{R}$. Then, for every $\mathbb{S}$-generic real x over V, there exists some real $y \in V$ such that x is equivalent to a $\mathbb{C}$-generic over $L[r, y]$.

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Suppose $\dot{x}$ is a $\mathbb{S}$-name for x. As $(r, S)^{\#}$ exists, by the previous lemma applied to the model $L[r, S]$, the set

$$D = \{T \in \mathbb{S} : \text{for some } S \in \mathbb{S},\ T \leq S,\ T \Vdash_{\mathbb{S}} \dot{x} \text{ is } \mathbb{C}\text{-generic over } L[r, S]\}$$

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Therefore, $V[G] \models x$ is $\mathbb{C}$-generic over $L[r, S]$ modulo π_S. □

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Note that j' is elementary and non-trivial. Then $\bar{j} := j' \restriction_{L[y]}$ witnesses the existence of $y^{\#}$ in $V[x]$. □

Other forcings

Pretty much the same ideas that we used before work by considering Silver, Mathias, Laver and Miller forcing. Basically, if for every $x \in {}^{\omega}\omega$, $x^{\#}$ exists and $r \in {}^{\omega}\omega$ then we can prove:

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These allow us to show that all the aforementioned forcing notions preserve sharps for reals.

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where κ is the critical point of the top measure of $z^\#$.

Therefore, under the presence of sharps for reals, E is a Δ^1_3 equivalence relation.

Claim: E is thin.

Suppose that there is a perfect set $P \subset {}^{\omega}\omega$ such that $[P]^2 \subset \mathbb{R}^2 \setminus E$. Since E is Δ^1_3, the formula

$$\forall x, y \in P(x \neq y \implies (x, y) \in \mathbb{R}^2 \setminus E)$$

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As V is closed under sharps for reals we have Σ^1_3 absoluteness for any provably Σ^1_2 c.c.c. forcing notion. Then if c is Cohen generic over V it follows that

$$V[c] \models [P]^2 \subset \mathbb{R}^2 \setminus E$$

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As V is closed under sharps for reals we have Σ^1_3 absoluteness for any provably Σ^1_2 c.c.c. forcing notion. Then if c is Cohen generic over V it follows that

$$V[c] \models [P]^2 \subset \mathbb{R}^2 \setminus E$$

Notice that P induces a Δ^1_3 well-ordering of the reals by taking

$$x \prec y \text{ iff } \omega_1^{+L[\varphi(x)]} < \omega_1^{+L[\varphi(y)]}$$

where $\varphi : {}^{\omega}\omega \to P$ is a recursive bijection with parameters in the ground model.

Therefore, there exists $a \in {}^{\omega}\omega \cap V$ and a $\Delta^1_3(a)$ formula $\phi(x, y)$ such that

$$V[c] \models \{(u, v) : \phi(u, v, a)\} \text{ is a well-ordering of } \mathbb{R} \quad (*)$$

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This means that the Cohen generic real c is definable with a formula using parameters from the ground model which is impossible. Thus, E is thin.

Question:

Let $\mathcal{T} = \{\mathbb{S}, \mathbb{V}, \mathbb{M}, \mathbb{L}, \mathbb{ML}\}$. Under the existence of sharps for reals, does any of the tree forcings in $\mathcal{T}$ add new equivalence classes to E?

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Let E be the equivalence relation defined by $xEy \iff \omega_1^{+L[x]} = \omega_1^{+L[y]}$ and let $\mathbb{P}$ be a forcing notion in $\mathcal{T}$. Then, for every $x \in V^{\mathbb{P}}$ there exists $x' \in V$ such that xEx'.

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Let $x \in V^{\mathbb{P}}$. Then, there exists some $z \in {}^{\omega}\omega \cap V$ such that $x \in L[z][g]$ where g is $\mathbb{Q}$ generic over $L[z]$, $\mathbb{Q}$ being either Cohen or Mathias forcing.

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Suppose that $z^{\#} = (J_\alpha(z), \in, U)$ and let $M = M_{\omega_1}$ be the ω_1-th iterate of $z^{\#}$ by U. Let $j : z^{\#} \to M$ be the induced elementary embedding. Observe that if $\kappa = \mathrm{crit}(U) = \mathrm{crit}(j)$ then $j(\kappa) = \omega_1^V$.

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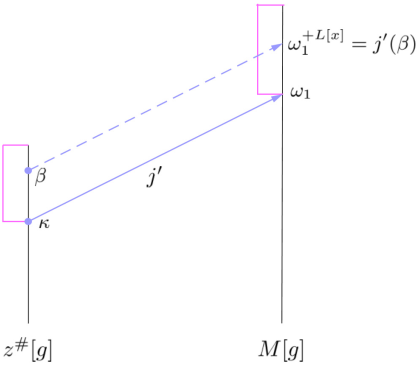

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Therefore $\gamma = j'(\beta) = \omega_1^{+L[x']}$, i.e. xEx'.

Uniform indiscernibles

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$$u^x_\gamma = \begin{cases} \mathrm{Next}(x, 0) & \text{if } \gamma = 1 \\ \mathrm{Next}(x, u^x_\alpha) & \text{if } \gamma = \alpha + 1 \\ \sup_{\alpha \in \lambda} u^x_\alpha & \text{if } \gamma \text{ is limit} \end{cases}$$

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Since all the cardinals in V are indiscernibles for every real, we have that $u_1 = \omega_1$. For the same reason, $u_2 \leq \omega_2$.

Theorem (Kunen-Martin)

If for every $x^{\in \omega}\omega$, $x^{\#}$ exists the following are all equal:

- u_2;
- $\sup\{(\omega_1)^{+L[x]} : x \in {}^{\omega}\omega\}$ where $\omega_1 = \omega_1^V$;
- $\sup\{\alpha : \alpha$ is the rank of a Π^1_1 well-founded relation$\}$;
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Corollary

Suppose that $x^{\#}$ exists for every real x and let $\mathbb{P}$ be a forcing notion in $\mathcal{T}$. Then $\mathbb{P}$ does not change the value of u_2, i.e. $u_2^V = u_2^{V^{\mathbb{P}}}$.

Open questions and further work

- Our results about preservation of sharps can be extended to any Σ^1_2 provably strongly proper forcing. Also, every such a forcing does not change the value of u_2.

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- Can exist a projective proper forcing $\mathbb{P}$ adding a new class to E such that in $V^{\mathbb{P}}$, ${}^{\omega}\omega$ is still closed under sharps?
- In which scenario can a projective proper forcing $\mathbb{P}$ increase δ^1_2?

Many thanks!