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supercompact cardinals + the mitchell order

abstract comparison processes

<u>defn.</u> supp. U_0, U_1 are ctbly. complete ultrafilters. a comparison of (U_0, U_1) by internal ultrafilters is a pair (W_0, W_1) s.t.

1. W_0 is an M_{U_0} - internal ultrafilter
$$M_{U_0} = \mathrm{ult}(V; U_0)$$
2. W_1 is an M_{U_1} - internal ultrafilter
3. $(M_{W_0})^{M_{U_0}} = (M_{W_1})^{M_{U_1}}$
4. $j_{W_0}^{M_{U_0}} \circ j_{U_0} = j_{W_1}^{M_{U_1}} \circ j_{U_1}$

$$\begin{tikzcd} & M_{U_0} \arrow[dr, "W_0"] & \\ V \arrow[ur, "U_0"] \arrow[dr, "U_1"'] & & N \\ & M_{U_1} \arrow[ur, "W_1"'] & \end{tikzcd}$$

<u>example</u>: suppose U_0, U_1 are κ-complete ultrafilters on κ. supp. $U_0 <_M U_1$

mitchell order

$$\begin{tikzcd}
 & M_{U_0} \arrow[dr, "j_{U_0}(U_1)"] & \\
V \arrow[ur, "U_0"] \arrow[dr, "U_1"'] & & N \\
 & M_{U_1} \arrow[ur, "U_0"'] &
\end{tikzcd}$$

<u>def.</u> (ultrapower axiom) any pair of ctbly. complete ultrafilters admits a comparison by internal ultrafilters.

<u>def.</u> Supp. M, N are transitive models of ZFC. an elementary embedding $k : M \longrightarrow N$, k cof., is close to M if any ultrafilter derived from k is in M.

<u>def.</u> (woodin) Supp. κ is inaccessible. say V_κ satisfies <u>weak comparison</u> iff there is a w.o. $<$ of V_κ s.t. f.a. M_0, M_1 finitely gen. and elementarily embeddable in $(V_\kappa, <)$, either $\mathcal{P}(\omega) \cap M_0 \neq \mathcal{P}(\omega) \cap M_1$, or there are close $k_0 : M_0 \longrightarrow N$ and $k_1 : M_1 \longrightarrow N$.

<u>th.</u> Supp. V_κ sat. weak comparison. then $V_\kappa \models$ ultrapower axiom.

<u>lem.</u> Supp. M is fin. generated and el. embeddable into $(V_\kappa, <)$. if U is a ctbly. complete ultrafilter of M then $\mathrm{ult}(M; U)$ is fin. generated and el. embeddable into $(V_\kappa, <)$.

<u>lem.</u> supp. ~~[illegible]~~ M is a transitive model of ZFC and $\mathcal{U}_0$, $\mathcal{U}_1$ are internal ultrafilters of M. supp.

$$\begin{tikzcd}
 & M_{\mathcal{U}_0} \arrow[dr, "k_0"] & \\
M \arrow[ur, "\hat{j}_{\mathcal{U}_0}"] \arrow[rr, "j"] \arrow[dr, "\hat{j}_{\mathcal{U}_1}"'] & & N \\
 & M_{\mathcal{U}_1} \arrow[ur, "k_1"'] &
\end{tikzcd}$$

have k_0, k_1 elem s.t. $k_0 \circ \hat{j}_{\mathcal{U}_0} = j = k_1 \circ \hat{j}_{\mathcal{U}_1}$.

then in M, $(\mathcal{U}_0, \mathcal{U}_1)$ admits comparison.

the lem easily gives the th.

~~of~~ on p. 3.

proof: $H = \mathrm{Hull}^{H}\left(j''M \cup \{k_0([\mathrm{id}]_{\mathcal{U}_0}), k_1([\mathrm{id}]_{\mathcal{U}_1})\}\right)$.

hen

$$\begin{tikzcd}
 & M_{U_0} \arrow[dr, "W_0"] \arrow[drr, bend left, "k_0"] & & \\
M \arrow[ur, "j_{U_0}"] \arrow[dr, "j_{U_1}"'] & & H \arrow[r] & N \\
 & M_{U_1} \arrow[ur, "W_1"'] \arrow[urr, bend right, "k_1"'] & &
\end{tikzcd}$$

let W_0 be derived from k_0 using $k_1([id]_{U_1})$, and let W_1 be derived from k_1 using $k_0([id]_{U_0})$.

$$H = M_{W_0}^{M_{U_0}} = M_{W_1}^{M_{U_1}} . \qquad \dashv$$

__Thm.__ (ultrapower axiom) The Mitchell order is linear on normal ultrafilters.

proof: let U_0, U_1 be normal ultrafilters, on the same κ.

let (W_0, W_1) be a comparison of (U_0, U_1).

<u>case 1.</u> $j_{W_0}^{M_{U_0}}(\kappa) = j_{W_1}^{M_{U_1}}(\kappa)$.

say $X \subset \kappa$.

$$X \in U_0 \iff \kappa \in j_{U_0}(X)$$

$$\iff j_{W_0}^{M_{U_0}}(\kappa) \in j_{W_0}^{M_{U_0}} \circ j_{U_0}(X)$$

$$\qquad\quad \| \qquad\qquad\qquad \|$$

$$j_{W_1}^{M_{U_1}}(\kappa) \in j_{W_1}^{M_{U_1}} \circ j_{U_1}(X)$$

$$\iff \kappa \in j_{U_1}(X)$$

$$\iff X \in U_1 .$$

<u>case 2.</u> $j_{W_0}^{M_{U_0}}(\kappa) < j_{W_1}^{M_{U_1}}(\kappa)$.

say $X \subset \kappa$.

then

$$X \in U_0 \iff \hat{j}_{W_0}^{M_{U_0}}(\kappa) \in \hat{j}_{W_1}^{M_{U_1}} \circ \hat{j}_{U_1}(X)$$

$$\iff \hat{j}_{W_0}^{M_{U_0}}(\kappa) \in \underbrace{\hat{j}_{W_1}^{M_{U_1}} \circ \hat{j}_{U_1}(X) \cap \hat{j}_{W_1}^{M_{U_1}}(\kappa)}_{= \, \hat{j}_{W_1}^{M_{U_1}}\big(\underbrace{\hat{j}_{U_1}(X) \cap \kappa}_{= \, X}\big)}$$

$$\iff \hat{j}_{W_0}^{M_{U_0}}(\kappa) \in \hat{j}_{W_1}^{M_{U_1}}(X) .$$

so $U_0 \in M_{U_1}$.

<u>Case 3</u>. Symmetrically.

<u>def.</u> Supp. $\mathcal{U}$ is an ultrafilter.

$\mathcal{U}$ is uniform iff $\mathcal{U}$ is a ultrafilter on an ordinal α and if $\beta < \alpha$, $\beta \notin \mathcal{U}$. $SP(\mathcal{U}) = \alpha$.

if $\mathcal{U}$ is a ctbly. complete uniform ultrafilter, $\mathcal{U}$ is an <u>u.f.</u>

$\mathcal{U}$ is <u>dodd-solid</u> iff the fcn. $E_{\mathcal{U}}$,

$$E_{\mathcal{U}}(X) = j_{\mathcal{U}}(X) \cap [id]_{\mathcal{U}} \; ; \; \mathrm{dom}(E_{\mathcal{U}}) = \mathcal{P}(SP(\mathcal{U})).$$

is in $M_{\mathcal{U}}$.

<u>thm.</u> (ultrafilter axiom) the mitchell order is linear on dodd-solid ultrafilters.

proof: let $\mathcal{U}_0, \mathcal{U}_1$ be dodd-solid ultrafilters on δ_0, δ_1.

etc. same pf. as before. ⊣

<u>th.</u> supp. $2^{<\lambda} = \lambda$. then every normal fine ultrafilter on $\mathcal{P}(\lambda)$ is RK-equivalent to a unique dodd-solid ultrafilter. (RK = rudin keisler)

<u>defn.</u> the internal relation $<_I$ is defined on ctbly. complete ultrafilters by

$$U_0 <_I U_1 \iff j_{U_0} \restriction M_{U_1} \text{ is an amenable class of } M_{U_1}.$$

<u>thm.</u> (ultrapower axiom) if $2^{<\lambda} = \lambda$. then $<_I$ well-orders the normal fine ultrafilters on $\mathcal{P}(\lambda)$.

remark. supp. U_0 is κ_0-complete on κ_0, U_1 is κ_1-complete on κ_1. then

$$U_0 <_I U_1 \text{ and } U_1 <_I U_0.$$

by kunen, $j_{U_0}(j_{U_1}) = j_{U_1} \restriction M_{U_0}$, $j_{U_1}(j_{U_0}) = j_{U_0} \restriction M_{U_1}$.