

Case I

strongly compact cardinals + the Mitchell order

abstract comparison processes

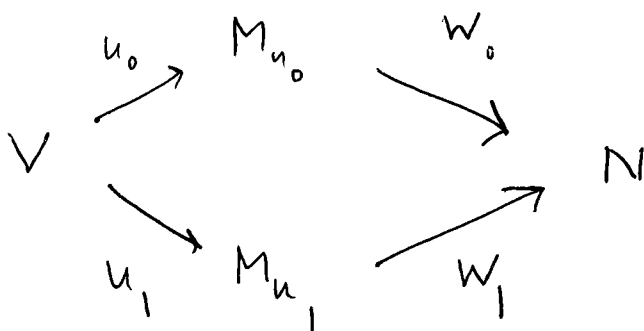
defn. supp. u_0, u_1 are cthy. extn. ultrafilters. a comparison of (u_0, u_1) by internal ultrafilters is a pair (W_0, W_1) s.t.

① W_0 is an M_{u_0} -internal ultrafilter
" " $int(V; u_0)$

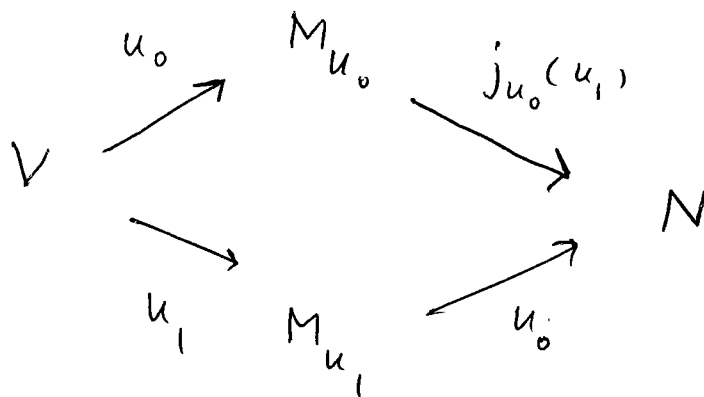
② W_1 is an M_{u_1} -internal ultrafilter

③ $(M_{W_0})^{M_{u_0}} = (M_{W_1})^{M_{u_1}}$

④ $j_{W_0}^{M_{u_0}} \circ j_{u_0} = j_{W_1}^{M_{u_1}} \circ j_{u_1}$



example: suppose U_0, U_1 are κ -cyclic
ultrafilters on κ . supp. $U_0 <_M U_1$
↑
 Mitchell order



def. (ultrapower axiom) any pair
of c.m.g. cyclic ultrafilters admits
a comparison by initial ultrafilters.

def. sup. M, N are transitive models of ZFC.
an elementary embedding $k: M \rightarrow N$, k cof.,
is close to M if any ultrafilter
derived from k is in M .

def. (Woodin) $\text{supp. } \kappa$ is inaccessible. say V_κ satisfies weak comparison iff there is a w.o. $<$ of V_κ s.t. f.a.

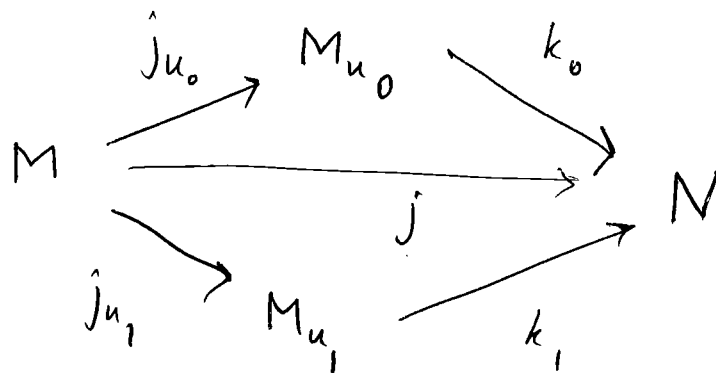
M_0, M_1 finitely gen. and elementarily embedded in $(V_\kappa, <)$, with $\mathcal{P}(w) \cap M_0 \neq \mathcal{P}(w) \cap M_1$, or then we close

$$k_0 : M_0 \rightarrow N \text{ and } k_1 : M_1 \rightarrow N.$$

thm. $\text{supp. } V_\kappa$ sat. weak comparison. then $V_\kappa \models$ ultrapow axiom.

lea. $\text{supp. } M$ is fin. generated and el. embedded into $(V_\kappa, <)$. if U is a cthy. cofinal ultrapow of M then $\text{ult}(M; U)$ is fin. generated and el. embedded into $(V_\kappa, <)$.

Let \mathcal{A} be a transition model of ZFC and u_0, u_1 are ordinal ultrafilters of \mathcal{A} . Suppose



Let k_0, k_1 be chosen s.t. $k_0 \circ \hat{j}_{u_0} = \hat{j} = k_1 \circ \hat{j}_{u_1}$.

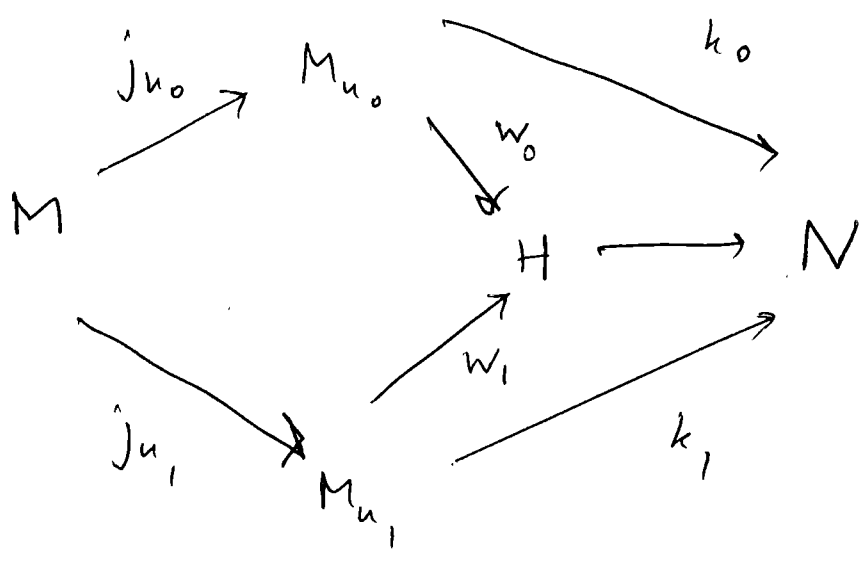
then in \mathcal{A} , (u_0, u_1) admits comparison.

the lemma easily gives the result.

see on p.3.

proof: $H = \text{Hull}^H (j'' M \cup \{k_0([id]_{u_0}), k_1([id]_{u_1})\})$

then



Let w_0 be derived for k_0 using $k_1([id]_{u_1})$, and let w_1 be derived for k_1 using $k_0([id]_{u_0})$.

$$H = M_{w_0}^{M_{u_0}} = M_{w_1}^{M_{u_1}} \quad \square$$

thm. (ultrafilter axiom) the mitchell order is linear on normal ultrafilters.

proof: let u_0, u_1 be normal ultrafilters, on the same κ .

Let (W_0, W_1) be a comparison of
 (u_0, u_1) .

case 1. $j_{W_0}^{M_{u_0}}(\kappa) = j_{W_1}^{M_{u_1}}(\kappa)$.

say $X \subset \kappa$.

$$X \in u_0 \Leftrightarrow \kappa \in j_{u_0}(X)$$

$$\Leftrightarrow j_{W_0}^{M_{u_0}}(\kappa) \in j_{W_0}^{M_{u_0}} \circ j_{u_0}(X)$$

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$$j_{W_1}^{M_{u_1}}(\kappa) \in j_{W_1}^{M_{u_1}} \circ j_{u_1}(X)$$

$$\Leftrightarrow \kappa \in j_{u_1}(X)$$

$$\Leftrightarrow X \in u_1.$$

case 2. $j_{W_0}^{M_{u_0}}(\kappa) < j_{W_1}^{M_{u_1}}(\kappa)$.

say $X \subset \kappa$.

then

$$X \in u_0 \implies j_{W_0}^{M_{u_0}}(x) \in j_{W_1}^{M_{u_1}} \circ j_{u_1}(X)$$

$$\implies j_{W_0}^{M_{u_0}}(x) \in \underbrace{j_{W_1}^{M_{u_1}} \circ j_{u_1}(X) \cap j_{W_1}^{M_{u_1}}(x)}_{\parallel}$$

$$j_{W_1}^{M_{u_1}} \left(\underbrace{j_{u_1}(X) \cap x}_{\parallel} \right)$$

X

$$\implies j_{W_0}^{M_{u_0}}(x) \in j_{W_1}^{M_{u_1}}(X)$$

so $u_0 \in M_{u_1}$.

can 3. symbolically.

def. supp. u is an ultrapr.

u is uniform iff u is a ultrapr on an ordinal α and if $\beta < \alpha$, $\beta \notin u$. $SP(u) = \alpha$.

if u is a cthy. complete uniform ultrapr, u is an u.f.

u is dodd-solid iff the fcn. E_u ,

$$E_u(X) = \int_u (X) \cap [id]_u ; \text{ dom}(E_u) =$$

$$\mathcal{P}(SP(u)).$$

is in M_u .

thm. (ultrapr axiom) the mitchell order is linear on dodd-solid ultrapr.

proof: let u_0, u_1 be dodd-solid

ultrapr on δ_0, δ_1 .

etc. see pg. as before. \rightarrow

thm. supp. $2^{<\lambda} = \lambda$, then every
 normal fine ultrafilter on $\mathcal{P}(\lambda)$ is
 RK-equivalent to a unique closed-solid
 ultrafilter. (RK = Rudin Keisler)

def. the initial order $<_I$ is defined
 on cty. eqv. ultrafilters by

$$u_0 <_I u_1 \iff j_{u_0} \upharpoonright M_{u_1} \text{ is an} \\ \text{another class of} \\ M_{u_1}.$$

thm. (ultrafilter axiom) if $2^{<\lambda} = \lambda$.

then $<_I$ well-orders the
 normal fine ultrafilters on $\mathcal{P}(\lambda)$.

remark. supp. u_0 is κ_0 -complete on κ_0 ,
 u_1 is κ_1 -complete on κ_1 . then

$$u_0 <_I u_1 \text{ and } u_1 <_I u_0.$$

by known, $j_{u_0}(j_{u_1}) = j_{u_1} \upharpoonright M_{u_0}$, $j_{u_1}(j_{u_0}) = j_{u_0} \upharpoonright M_{u_1}$.