

gate II

the seed order

def. In U_f denote the class of all ufs. The seed order \leq_s is the binary relation on U_f defined by

$$u_0 \leq_s u_1 \text{ iff } \exists (w_0, w_1) \text{ fin. comparison}$$

$$j_{w_0}^{M_{u_0}}([id]_{u_0}) \leq j_{w_1}^{M_{u_1}}([id]_{u_1}).$$

th. (ultrafilter axiom) the seed order is a well-order.

def. $U_p =$ the collection of all ultrapowers of V by ultrafilters, with

$$\text{hom}(M_0, M_1) = \left\{ j_u^{M_0} : M_u^{M_0} = M_1, \right.$$

u an ultrafilter
u.f. of M_0 $\left. \right\}$

propositional (UA). U_p is a directed partial order.

If \therefore supp. u_0, u_1 are ufs and

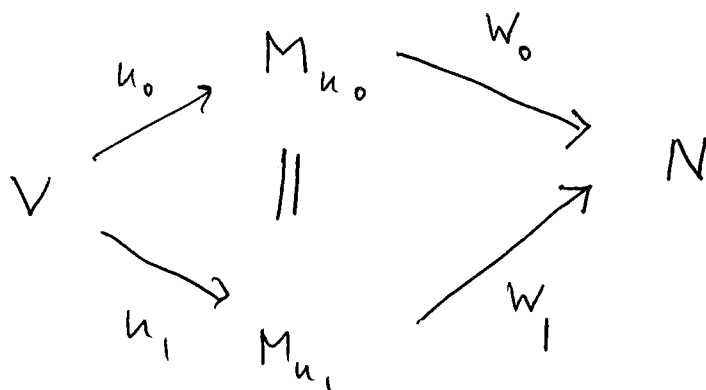
$M_{u_0} = M_{u_1}$. we must show $\hat{j}_{u_0} = \hat{j}_{u_1}$.

fact. supp. $j_i: V \rightarrow M$ are dy. tr. el. embeddings, $i=0,1$. (same M). then $j_0 = j_1$ on the ordinals.

proof: let α least s.t. for two Σ_n fdy el. embedding into the same target,

$j_0(\alpha) \neq j_1(\alpha)$. but then α is definite. $\exists \dashv$

fix, (w_0, w_1) a comparison of (u_0, u_1)



may arise w.l.o.g. that W_0 is
 derived fr j_{W_0} via $\hat{j}_{W_1}([\text{id}]_{u_1})$
 (cf. 1st part).

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$$\hat{j}_{W_0}([\text{id}]_{u_1})$$

$$j_{W_0}^{M_{u_0}} \circ j_{u_0} = j_{W_1}^{M_u} \circ j_{u_1}$$

$$\Rightarrow j_{u_0} = j_{u_1}$$

def. (UA) $\text{ln } N_\infty =$

$\text{dir lim } U_p$.

not nec. set-like.

for $M_0, M_1 \in U_p$, let \hat{j}_{M_0, M_1} be

the unique morphism if it ex.

$$\text{ln } \hat{j}_{M_0, \infty} : M_0 \longrightarrow N_\infty.$$

proof of "seed order is a w.o.",

define $\Phi : U_f \rightarrow OR^{N_\infty}$ be

$$\Phi(u) = \int_{M_u^\infty} ([id]_u).$$

① Φ is order preserving for

$$(U_f, \leq_s) \rightarrow (OR^{N_\infty}, \leq)$$

Let $u_0, u_1 \in U_f$.

$u_0 \leq_s u_1 \iff \exists (w_0, w_1)$ compatible

$$\int_{w_0}^{M_{u_0}} ([id]_{u_0}) \leq \int_{w_1}^{M_{u_1}} ([id]_{u_1})$$

$\underbrace{\qquad\qquad\qquad}_{= \int_{M_{u_0}} ([id]_{u_0})}$

$$\Leftrightarrow \int_{M_{w_0}^{M_{u_0}^\infty}} \left(\int_{w_0}^{M_{u_0}} ([id]_{u_0}) \right) \leq$$

$$\int_{M_{w_1}^{M_{u_1}^\infty}} \left(\int_{w_1}^{M_{u_1}} ([id]_{u_1}) \right)$$

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 $\int_{M_{w_1}} ([id]_{u_1})$ /

$$s_0 \Leftrightarrow \Phi(u_0) \leq \Phi(u_1).$$

(2) Φ is injective

Φ is invertible & since u is the
uf. derived from $\hat{J}_{\infty} \hat{V}_{\infty} u_j$
 $\Phi(u)$.

lea. ~~suppose~~ $u_0 \leq_s u_1$, ~~then~~ provided then
 $sp(u_0) \ll sp(u_1)$.

proof: note that $sp(u) =$ the least α
s.t. $u \leq_s P_\alpha =$ principal ultra filter w
 $\{\alpha\}$ in it.

$u_0 \leq_s P_{sp(u_0)} <_s u_1$. so $u_0 <_s u_1$.
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th. (UA) any uf is OD.

th. (UA) κ is supercompact.

then f.a. $A \subset \kappa$ s.t. $V_\kappa \subset HOD_A$.

then $V = HOD_A$.

M_f := the \mathcal{U} $f: \kappa \rightarrow V_\kappa$ s.t.

$$f \in \text{cod}_A$$

$$\forall x \in V \exists u \in \mathcal{U}_f \quad x = j_u(f)(\kappa)$$

$$X \in \text{OD}_A \quad \text{so} \quad V = \text{cod}_A \quad \rightarrow$$

Canonical comparisons.

def. syp. $u_0, u_1 \in \mathcal{U}_f$

(W_0, W_1) is a canonical comparison if

$$[id]_{W_0}^{M_{u_0}} = j_{W_1}^{M_{u_1}}([id]_{u_1}) \quad \text{and}$$

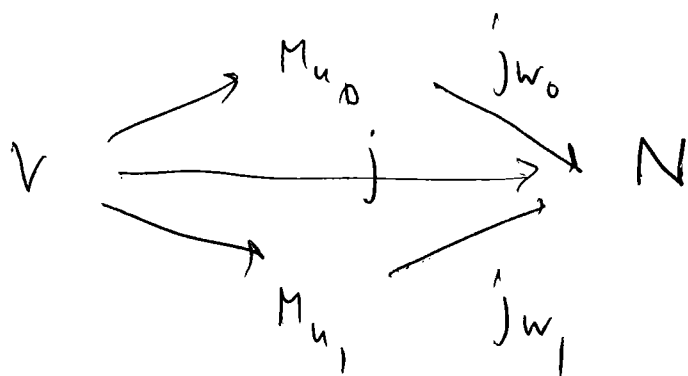
$$[id]_{W_1}^{M_{u_1}} = j_{W_0}^{M_{u_0}}([id]_{u_0})$$

prop. syp. (W_0, W_1) is a can.

comparison of (u_0, u_1)

$$\Phi(u_0) = \Phi^{M_{u_1}}(W_1)$$

$$\begin{aligned}
 \Phi(u_0) &= \hat{j}_{M_{u_0}, \infty}([\text{id}]_{u_0}) \\
 &= \hat{j}_{M_{w_0}, \infty}^{M_{u_0}}(\hat{j}_{w_0}^{M_{u_0}}([\text{id}]_{u_0})) \\
 &= \hat{j}_{N, \infty}([\text{id}]_{w_1}^{M_{u_1}}) =
 \end{aligned}$$



$$\begin{aligned}
 &= \left(\hat{j}_{M_{w_1}, \infty} \right)^{M_{u_1}}([\text{id}]_{w_1}^{M_{u_1}}) \\
 &= \Phi^{M_{u_1}}(w_1).
 \end{aligned}$$

proposition (UA) canonical comparison is unique,

def. sup. u_0 is an u.f. then

$$t_{u_0} : u_f \longrightarrow (u_f)^{M_{u_0}} \text{ by}$$

$t_{u_0}(u_1) = w_0$, wh (w_0, w_1) is the
can. compari of u_0, u_1 .

proposition (UA) $t_{u_0} : (u_f, \leq_s) \longrightarrow (u_f, \leq_s)^{M_{u_0}}$
is order preserving.

$$\text{pf.} \text{ : sup. } u_1 \leq u_2 \iff \Phi(u_1) \leq \Phi(u_2)$$

$$\iff \Phi^{M_{u_0}}(t_{u_0}(u_1)) \leq \Phi^{M_{u_1}}(t_{u_0}(u_2))$$

$$\iff t_{u_0}(u_1) \leq_s^{M_{u_0}} t_{u_0}(u_2).$$

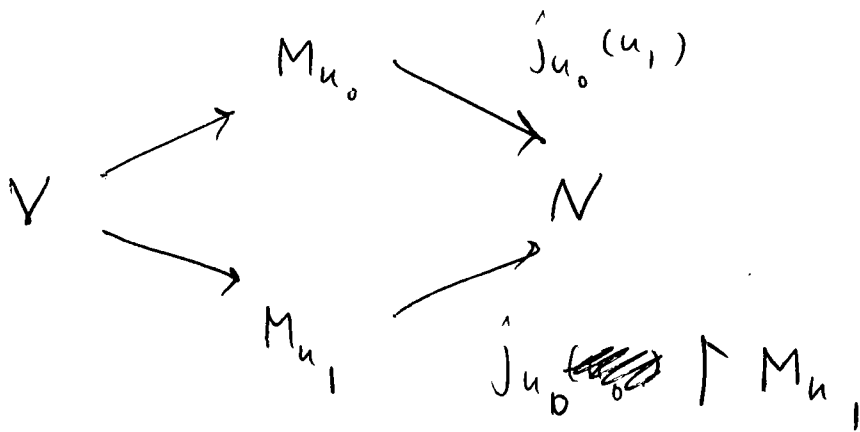
proposition (UA) $t_{u_0}(u_1) \leq_s^{M_{u_0}} j_{u_0}(u_1)$ and

equally holds iff $u_0 <_I u_1$.

proof: $\Phi^{M_{u_0}}(t_{u_0}(u_1)) = \Phi(u_1)$

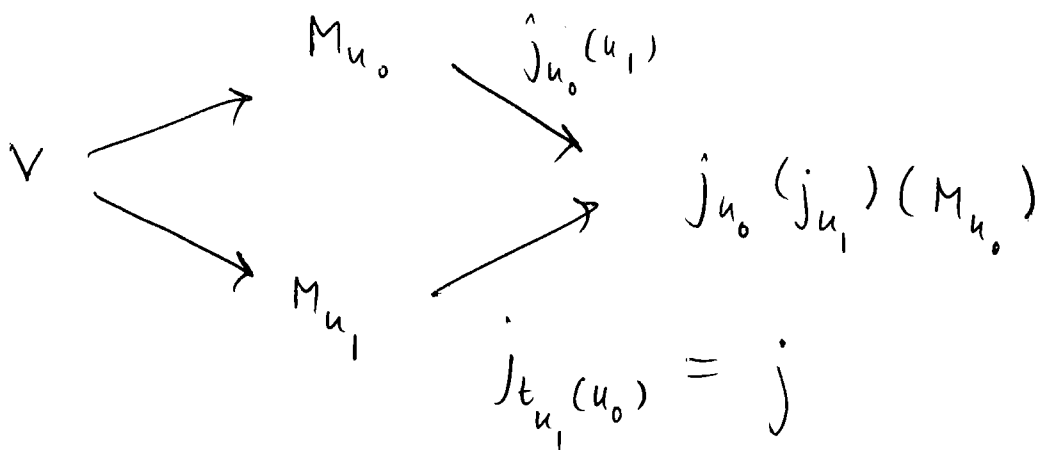
$\Phi^{M_{u_0}}(\hat{j}_{u_0}(u_1)) = \hat{j}_{u_0}(\Phi(u_1))$.

$\phi(u_1) \leq \hat{j}_{u_0}(\Phi(u_1))$.



ask; $u_0 <_{\mathbb{I}} u_1$, then in the picture.

suppon $\hat{j}_{u_0}(u_1) = t_{u_0}(u_1)$



def: $j_{t_{u_1}(u_0)}^{M_{u_1}} = \hat{j}_{u_0} \uparrow M_{u_1}$

pr:

① $\hat{j} \uparrow \hat{j}_{u_1} [u] = \hat{j}_{u_0} / \hat{j}_{u_1} [v]$

$$\begin{aligned} \hat{j} \circ \hat{j}_{u_1} &= \hat{j}_{u_0} (\hat{j}_{u_1}) \circ \hat{j}_{u_0} \\ &= \hat{j}_{u_0} \circ \hat{j}_{u_1} \end{aligned}$$

② $\hat{j} ([id]_{u_1}) = \hat{j}_{u_0} ([id]_{u_0})$

$$\hat{j} ([id]_{u_1}) = [id]_{\hat{j}_{u_0}(u_1)}^{M_{u_0}} =$$

$$[id]_{\hat{j}_{u_0}(u_1)}^{M_{u_0}} = \hat{j}_{u_0} ([id]_{u_0})$$