Gate II

the seed order

def. Let $\mathcal{U}f$ denote the class of all ufs. The seed order $\leq_s$ is the binary relation on $\mathcal{U}f$ defined by

$$U_0 \leq_s U_1 \quad \text{iff} \quad \exists\, (W_0, W_1) \text{ fin. comparison}$$

$$j^{M_{U_0}}_{W_0}([\mathrm{id}]_{U_0}) \leq j^{M_{U_1}}_{W_1}([\mathrm{id}]_{U_1}).$$

th. (ultrafilter axiom "UA") the seed order is a well-order.

def. $\mathcal{U}p$ = the collection of all ultrapowers of V by ultrafilters, with

$$\hom(M_0, M_1) = \{ j^{M_0}_u : M^{M_0}_u = M_1, \; u \text{ an internal u.f. of } M_0 \}$$

<u>proposition</u> (UA). U_p is a directed partial order.

pf.: supp. U_0, U_1 are ufs and $M_{U_0} = M_{U_1}$. we must show $j_{U_0} = j_{U_1}$.

<u>fact.</u> Supp. $j_i : V \longrightarrow M$ are def. th. el. embeddings, $i = 0, 1$. (same M). then $j_0 = j_1$ on the ordinals.

proof: let α least s.t. for two Σ_n fully el. embedding with the same target, $j_0(\alpha) \neq j_1(\alpha)$. but then α is definable. ↯ $\dashv$

fix, (W_0, W_1) a comparison of (U_0, U_1)

$$\begin{tikzcd}
 & M_{U_0} \arrow[dr, "W_0"] \arrow[d, equal] & \\
V \arrow[ur, "U_0"] \arrow[dr, "U_1"'] & {} & N \\
 & M_{U_1} \arrow[ur, "W_1"'] &
\end{tikzcd}$$

may assume w.l.o.g. that W_0 is derived from $\hat{j}_{W_0}$ using $\hat{j}_{W_1}([id]_{U_1})$ (cf. 1st part).

$$\hat{j}_{W_1}([id]_{U_1}) = \hat{j}_{W_0}([id]_{U_1})$$

$$\hat{j}^{M_{U_0}}_{W_0} \circ \hat{j}_{U_0} = \hat{j}^{M_U}_{W_1} \circ \hat{j}_{U_1}$$

$$\Rightarrow \quad \hat{j}_{U_0} = \hat{j}_{U_1} .$$

<u>def.</u> (UA) then $N_\infty =$ dir lim $\mathcal{U}_p$.

not nec. set-like.

for $M_0, M_1 \in \mathcal{U}_p$, let $\hat{j}_{M_0, M_1}$ be the unique morphism if it ex.

then $\hat{j}_{M_0 \infty} : M_0 \longrightarrow N_\infty$.

proof of "seed order is a w.o.":

define $\Phi : \mathcal{U}_f \longrightarrow OR^{N_\infty}$ be

$$\Phi(u) = j_{M_u \infty}([\mathrm{id}]_u).$$

① Φ is order preserving for

$$(\mathcal{U}_f, \leq_s) \longrightarrow (OR^{N_\infty}, \leq)$$

pf: let $u_0, u_1 \in \mathcal{U}_f$.

$u_0 \leq_s u_1$ iff $\exists (W_0, W_1)$ comparison

$$\cancel{\text{st}}\ j^{M_{u_0}}_{W_0}([\mathrm{id}]_{u_0}) \leq j^{M_{u_1}}_{W_1}([\mathrm{id}]_{u_1})$$

$$\Longleftrightarrow \overbrace{j_{M^{M_{u_0}}_{W_0} \infty}\left(j^{M_{u_0}}_{W_0}([\mathrm{id}]_{u_0}) \right)}^{= j_{M_{u_0}}([\mathrm{id}]_{u_0})} \leq$$

$$\underbrace{j_{M^{M_{u_1}}_{W_1} \infty}\left(j^{M_{u_1}}_{W_1}([\mathrm{id}]_{u_1}) \right)}_{= j_{M_1}([\mathrm{id}]_{u_1})},$$

so $\Longleftrightarrow \Phi(u_0) \leq \Phi(u_1)$.

(2) Φ is injective

Φ is invertible ∵ since U is the uf. derived from ~~$j_{A\infty}$~~ $j_{V\infty}$ using $\Phi(U)$.

<u>lem</u>: ~~suppose~~ $U_0 \leq_s U_1$, provided that ~~then~~ $sp(U_0) < sp(U_1)$.

proof: note that $sp(U) =$ the least α s.t. $U \leq_s P_\alpha$ = principal ultrafilter w/ $\{\alpha\}$ in it.

$U_0 \leq_s P_{sp(U_0)} <_s U_1$. so $U_0 <_s U_1$. ⊣

<u>th</u>. (UA) any uf is OD.

<u>th</u>. (UA) assume κ is supercompact. then f.a. $A \subset \kappa$ s.t. $V_\kappa \subset HOD_A$. then $V = HOD_A$.

pf. : then is $f : \kappa \longrightarrow V_\kappa$ s.t.

$f \in \mathrm{HOD}_A$.

$\forall x \in V \ \exists U \in \mathcal{U}_f \quad x = j_U(f)(\kappa)$.

$X \in \mathrm{OD}_A$. so $V = \mathrm{HOD}_A$. ⊣

<u>Canonical comparisons.</u>

<u>defn.</u> supp. $U_0, U_1 \in \mathcal{U}_f$

(W_0, W_1) is a canonical comparison iff

$$[\mathrm{id}]^{M_{U_0}}_{W_0} = j^{M_{U_1}}_{W_1}([\mathrm{id}]_{U_1}) \quad \text{and}$$

$$[\mathrm{id}]^{M_{U_1}}_{W_1} = j^{M_{U_0}}_{W_0}([\mathrm{id}]_{U_0}).$$

<u>proposition.</u> supp. (W_0, W_1) is a can. comparison of (U_0, U_1)

$$\Phi(U_0) = \Phi^{M_{U_1}}(W_1).$$

$$\Phi(u_0) = j_{M_{u_0}\infty}([id]_{u_0})$$

$$= j^{M_{u_0}}_{M_{W_0}\infty}\left(j^{M_{u_0}}_{W_0}([id]_{u_0})\right)$$

$$= j_{N\infty}\left([id]^{M_{u_1}}_{W_1}\right) =$$

$$\begin{tikzcd}
 & M_{u_0} \arrow[dr, "j_{W_0}"] & \\
V \arrow[ur] \arrow[rr, "j"] \arrow[dr] & & N \\
 & M_{u_1} \arrow[ur, "j_{W_1}"'] &
\end{tikzcd}$$

$$= \left(j_{M_{W_1}\infty}\right)^{M_{u_1}}\left([id]^{M_{u_1}}_{W_1}\right)$$

$$= \Phi^{M_{u_1}}(W_1).$$

<u>proposition</u> (UA) Canonical comparisons are unique.

<u>defn.</u> Supp. u_0 is an u.f. then

$$t_{u_0} : U_f \longrightarrow (U_f)^{M_{u_0}} \quad \text{by}$$

$$t_{u_0}(u_1) = W_0, \quad \text{where } (W_0, W_1) \text{ is the}$$

can. comparison of u_0, u_1.

<u>proposition</u> (UA) $t_{u_0} : (U_f, \leq_s) \longrightarrow (U_f, \leq_s)^{M_{u_0}}$

is order preserving.

pf.: ~~Supp.~~ $u_1 \leq u_2 \iff \Phi(u_1) \leq \Phi(u_2)$

$$\iff \Phi^{M_{u_0}}(t_{u_0}(u_1)) \leq \Phi^{M_{u_1}}(t_{u_0}(u_2))$$

$$\iff t_{u_0}(u_1) \leq_s^{M_{u_0}} t_{u_0}(u_2).$$

<u>proposition</u> (UA) $t_{u_0}(u_1) \leq_s^{M_{u_0}} j_{u_0}(u_1)$ and

equality holds iff $u_0 <_I u_1$.

proof: $\Phi^{M_{u_0}}(t_{u_0}(u_1)) = \Phi(u_1)$

$\Phi^{M_{u_0}}(j_{u_0}(u_1)) = j_{u_0}(\Phi(u_1))$.

$\phi(u_1) \leq j_{u_0}(\Phi(u_1))$.

$$\begin{tikzcd}
 & M_{u_0} \arrow[dr, "j_{u_0}(u_1)"] & \\
V \arrow[ur] \arrow[dr] & & N \\
 & M_{u_1} \arrow[ur, "j_{u_0} \restriction M_{u_1}"'] &
\end{tikzcd}$$

assuming $u_0 <_I u_1$, this is the picture.

suppose $j_{u_0}(u_1) = t_{u_0}(u_1)$

$$\begin{tikzcd}
 & M_{u_0} \arrow[dr, "j_{u_0}(u_1)"] & \\
V \arrow[ur] \arrow[dr] & & j_{u_0}(j_{u_1})(M_{u_0}) \\
 & M_{u_1} \arrow[ur, "j_{t_{u_1}(u_0)} = j"'] &
\end{tikzcd}$$

claim: $\quad j^{M_{u_1}}_{t_{u_1}(u_0)} = j_{u_0} \restriction M_{u_1}$.

pf.:

① $j \restriction j_{u_1}[u] = j_{u_0} / j_{u_1}[V]$.

$$j \circ j_{u_1} = j_{u_0}(j_{u_1}) \circ j_{u_0}$$

$$= j_{u_0} \circ j_{u_1}$$

② $\quad j([id]_{u_1}) = j_{u_0}([id]_{u_1})$

$$j([id]_{u_1}) = [id]^{M_{u_0}}_{t_{u_0}(u_1)} =$$

$$[id]^{M_{u_0}}_{j_{u_0}(u_1)} = j_{u_0}([id]_{u_1}).$$