group IV

proof of generation of pointclasses below non-domestic.

K is <u>non-domestic</u> if κ is a measurable lim of woodins and $<\kappa$ strongs

<u>thm.</u> $(AD_{\mathbb{R}})$ AD^+ + no non-domestic hod mice, then every set $A \subset \mathbb{R}$ is reducible to a hod pair.

pf.: Supp. not. Let $\Gamma = \{ A :$
$A \leq_W$ a hod pair $\}$.

$L(\Gamma, \mathbb{R}) \models AD_{\mathbb{R}} + \Theta$ is regular.

$H = Lp^{\Omega}(H^-)$, $H^- =$ hod limit of $L(\Gamma, \mathbb{R})$,

$\Omega =$ iteration strategy

<u>def.</u> for $g \subset \mathrm{Col}(\omega, <\kappa)$.

we say (Q, Λ) is a <u>generic generator</u> iff

① $Q \cap OR = \omega_1$

② $Q \models$ all strong cardinals are lim of woodin cardinals.

③ $Q \models \omega_1$ is not a lim woodin

④ for any V-hod pair (R, Ψ), $\exists \alpha < \omega_1$, $\Lambda_{Q|\alpha} \restriction V \in V$ and $\mathrm{Code}(\Psi) <_w \mathrm{code}(\Lambda_{Q|\alpha})$.

<u>lem</u>. there are generic generators.

pf.: fix a Γ^+-woodin, $\Gamma^+ >_w \Gamma$, and Γ^+ is good.

(M, Ψ). let $\mathcal{M}$ = hod pair constructed of M.

let κ = the least strong of M.

which is not a lim of woodins in M.

<u>exercise</u>. supp. $cf(\Theta^\Gamma) = \omega_1$.

let $(M_\alpha, \pi_{\alpha\beta} : \alpha < \beta \le \omega_1)$ be a lin. it. of M via a normal measure on κ. then $(\pi_{0,\omega_1}(\mathcal{M}), \Lambda)$ is a generic generator where Λ is the induced strategy for $\Psi_{M_{\omega_1}}$.

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back to comp conjecture:

κ is a lim of woodins + $<\kappa$ strongs

$\mathcal{M}_0 \lhd \mathcal{M}_1$,

a) $L(\mathcal{M}_0^\omega, \mathrm{Hom}^*) \models AD^+$

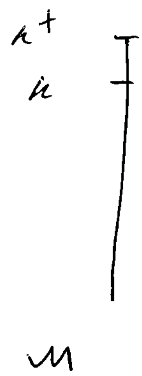
b) $o(\mathcal{M}_0) = \eta$ is the largest cardinal of $\mathcal{M}_1$

c) $\mathcal{M}_1$ has a $\square_\eta$-seq. that is not threadable.

<u>fact</u>. (ex.) Supp. $\mathcal{M}$ is a hod pm and κ is a reg. lim of $< \kappa$-strongs + woodins. Then $\mathcal{M}_\infty$ = dir lim of all iterates of $\mathcal{M}|\kappa$ via κ-iterates that fix κ. $i : \mathcal{M}|\kappa \longrightarrow \mathcal{M}_0$,

$\mathcal{M}_1 = i(\mathcal{M}|\kappa^+)$.

have a), b), c) as above.



what we want to show is that

$\mathcal{M}_0 \in V(\mathbb{R})$
$\uparrow$
$AD_{\mathbb{R}}$ – model.

the proof of the thm.

Suppose $g \subset Col(\omega_1, \mathbb{R})$ – gen.

we say that (P, Σ) ...

Last time: suppose not.

$$\Gamma = \{A : A \leq_W \mathrm{Code}(\Sigma)\}.$$

Lea.
$L(\Gamma, \mathbb{R}) \models AD_{\mathbb{R}} + \Theta$ is regular.

We say that (P, Σ) is a generic generator (g.g.) if

① $o(P) = \omega_1$, $\mathbb{P} \models$ no proper class of woodins,

② Σ has full normalization

③ $\mathcal{M}_\infty(P, \Sigma) \trianglelefteq H = Lp^{\Omega}(H^-)$

④ $\forall \alpha < \omega_1 \quad \Sigma_{P|\alpha} \restriction V \in V$

⑤ $\forall (Q, \Lambda) \in \Gamma \;\; \exists \alpha < \omega_1$
$\mathrm{Code}(\Lambda) \leq_W \mathrm{Code}(\Sigma_{P|\alpha})$

⑥ all strongs of P are lim of woodins.

Lem. there is a g.g.

pf.: let Γ_1 be any good pointclass beyond. let $x \mapsto (N_x^*, \delta_x, \Sigma_x)$ be as in "W_x^*-lem" for Γ_1.

fix x s.t. $(W_x^*, \delta_x, \Sigma_x)$ Suslin co-Suslin captures Γ.

let M = the hod pair construction of W_x^*. because of our hypothesis there is $\eta < \delta_x$ s.t. $M \models$ no woodins in $[\eta, \delta_x)$. let κ be the least strong in (η, δ_x) of W_x^*.

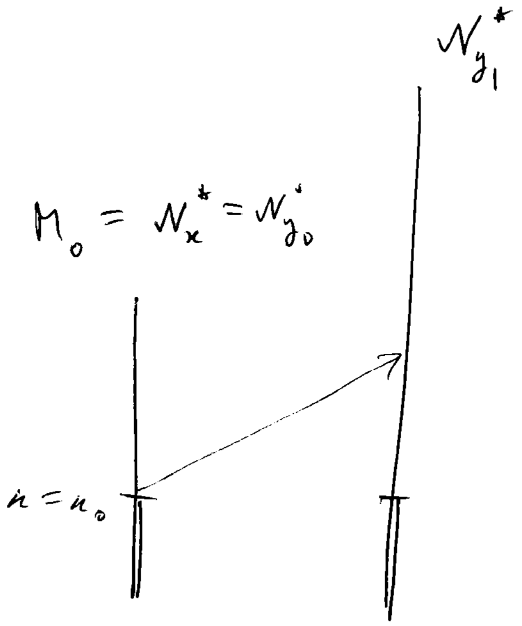
<u>observation 1.</u> Supp. $h \subset \mathrm{Col}(\omega, <\kappa)$ is generic. then $M|\kappa$ iterates past H^{Γ^*}, where Γ^* = the version of Γ in $W_x^*[h]$.

Let $(y_\alpha : \alpha<\omega_1)$ = a seq. of reals s.t.

① $\forall z \in \mathbb{R} \quad z \leq_T y_\alpha$, for α

② $(N^*_{y_\beta} : \beta<\alpha) \in HC^{N^*_{y_\alpha}}$

③ $y_0 = x$.



do $L[E](N^*_{y_0} | \kappa_0)$, but allowing overlapping extenders which come from $N^*_{y_0}$ + iterate thereof.

this produces $(M_\alpha, \pi_{\alpha\beta} : \alpha<\beta\leq\omega_1)$

$\pi_{\alpha\beta} : M_\alpha \longrightarrow M_\beta$, $\alpha<\beta\leq\omega_1$

$\text{crit}(\pi_{01}) = \kappa_0 = \kappa$

$\text{crit}(\pi_{\alpha\beta}) = \pi_{0\alpha}(\kappa_0)$

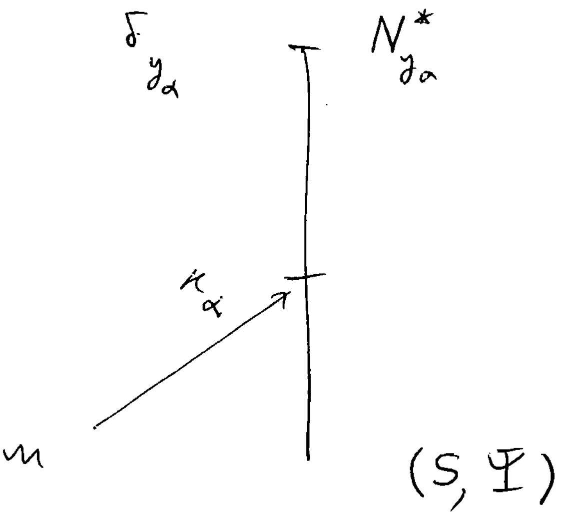
hod pair constr.

let $P = \pi_{0\omega_1}(M \mid \kappa_0)$

let Λ be the strategy of P induced by $\Sigma_{M_{\omega_1}}$

<u>Lem.</u> (P, Λ) is a g.g.

to verify ⑤ : fix a hod pair (S, Ψ). fix y_α s.t. (S, Ψ) is Suslin co Suslin, captured by $W^*_{y_\alpha}$.



(S, Ψ) gets absorbed by some $(L^\Psi[E])^{N^*_{y_\alpha}}$, but as κ_α is strg, it must be absorbed ~~between~~ by a mouse below κ_α.

so $\text{Code}(\Psi) \leq_w \text{Code}(\Sigma_{P|\gamma})$.

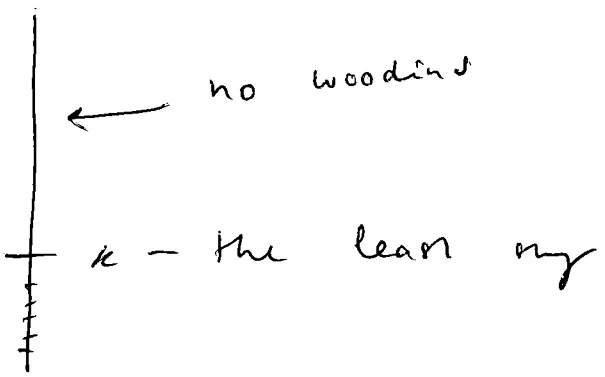
<u>next</u> we want to have a comparison theory for g.g.

<u>rmk:</u> "i am not counting woodins."

<u>simplification</u>: why is there a hod mouse with a strong past a woodin?

why does P have a woodin above its least strong?

towards a contradiction, assume no more woodins above the least strong of P.



fix a g.g. (P, Σ).

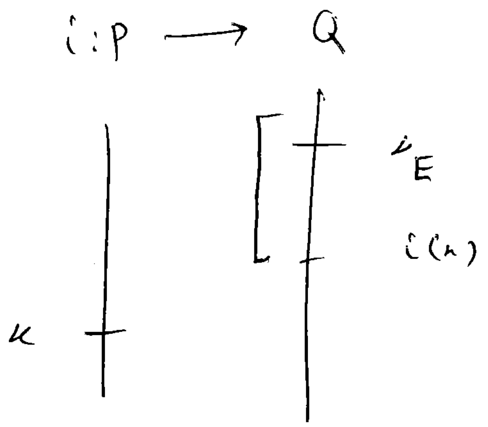
let $\pi^{\Sigma}_{P,\infty} : P \longrightarrow \mathcal{M}_\infty(P, \Sigma) \trianglerighteq H$.

let $X^{\Sigma}_{P} = \pi^{\Sigma}_{P,\infty}[\, P \mid \lambda^P \,]$,

λ^P = the succ. of the [illegible] of P.

observation 2. X^{Σ}_{P} defining extends with critical point κ.

i.e., suppose $i : P \longrightarrow Q$ it. via Σ.



$(a, A) \in E$ iff $\cancel{\pi_E(a)}$ $a \in \pi_E(A)$

$\iff \pi^{\Sigma_Q}_{Q,\infty}(a) \in \pi^{\Sigma}_{Q,\infty}(\pi_E(A))$

$\iff a_\infty \in \pi^{\Sigma}_{P,\infty}(B)(b_\infty)$

where $\pi_E(A) = \pi^{\Sigma}_{P,Q}(A)(b)$,

$b_\infty = \pi^{\Sigma}_{Q,\infty}(b)$.

<u>observation</u>. if (P,Σ), (Q,Λ) are two g.g., $X^{\Sigma}_{P} = X^{\Lambda}_{Q}$ and

$$P|\eta_{\cancel{+}} = Q|\eta, \quad \Sigma_{P|\eta} = \Lambda_{P|\eta}$$

$\Rightarrow$ the first overlapping extenders are the same.

given an $X \in \mathcal{P}_{\omega_1}(H)$, can talk about X-realization hod pairs (Q,Λ).

$$\begin{tikzcd} & H \\ Q \arrow[ur, "\tau^X"] \arrow[r, "\pi^X_Y"'] & R \arrow[u, "\tau^X_Y"'] \end{tikzcd}$$

<u>lem</u>.

supp. (S_0,Λ_0), (S_1,Λ_1) are two X-realization hod pairs, η ~~is the largest cardinal of~~

$$S_0|\eta = S_1|\eta, \quad \eta \geq \rho_w(S_0), \rho_w(S_1),$$

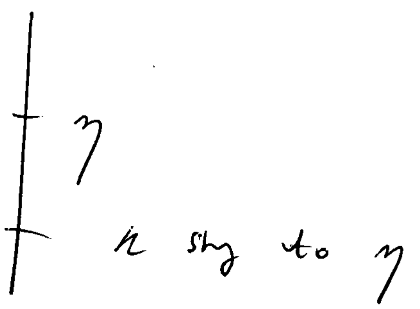
Some η.

they are both witness to the non-woodins of η, then

$$S_0 = S_1, \quad \Lambda_0 = \Lambda_1 .$$

making the lemma correct.

Supp.:



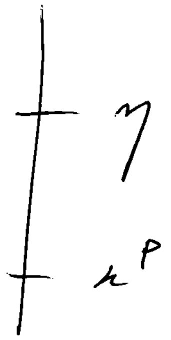
$$Act(\Lambda, S) = \{ \Lambda_Q : S \xrightarrow{\pi} Q \text{ by } \Lambda, \text{ generators are contained in } \pi(\kappa) \}$$

add to the statement of the lemma on p. 11:

$$Act(\Lambda_0, S_0|\eta) = Act(\Lambda_1, S_1|\eta).$$

Rem. one can make sense of action comparism.

$\Rightarrow$ P is a g.g. then



$$P \Vdash (\eta^+)^P \in OD_{X_P^\Sigma}(P|\eta).$$