trevor III

theorem ($ZF + DC_{\mathbb{R}}$) if $J_\kappa(\mathbb{R}) \models KP + AD$.

(1) $\det(\overline{OD^{<\kappa}})$

(2) $\overline{OD^{<\kappa}}$ is closed under $\exists^{\mathbb{R}}$ and $\forall^{\mathbb{R}}$

same for $\overline{OD_x^{<\kappa}}$, $x \in \mathbb{R}$.

the point: this "det. transfer" gives enough determinacy for Steel's construction of scales on $\Pi_1^{J_\kappa(\mathbb{R})}$ sets. the norm relation of those scales are then seen to exist in $\bigcup_{x \in \mathbb{R}} \overline{OD_x^{<\kappa}}$.

(the argument is similar to kechris-woodin: "the equivalence of partition properties + determinacy") proof of ① :

Supp. towards contradiction

$A \in \overline{OD^{<\kappa}}$ is non-determined.

by $DC_{\mathbb{R}}$, for each real $t \in \mathbb{R}$ we can find a cntbl. turing ideal $I \subset \mathbb{R}$ s.t. $t \in I$,

(closed under $\leq_T$ + $\oplus$)

A is not det. on I, meaning:

for every strategy $\sigma \in I$, there is a real in I that beats σ.

<u>note</u>: the nat. enumeration $(A_\alpha : \alpha < \kappa)$ of $OD^{<\kappa}$ pointsets is $\Delta_1^{J_\kappa(\mathbb{R})}$.

for every $t \in \mathbb{R}$ define $\alpha(t)$ = the least $\alpha < \kappa$ s.t. for some cntbl. turing ideal $I \ni t$, A_α is not det. on I.

note: this leads to a contradiction meaning that $\exists t \in \mathbb{R}\ \forall$ cntbl. turing ideals $I \ni t$, every $OD^{<\kappa}$ set is det. on I.

consider game

$$G \quad \begin{array}{ll} \mathrm{I} & x, \; z \\ \underline{\mathrm{II}} & y, \; s \end{array}$$

we say that I wins iff

$$x \oplus y \in A_{\alpha(z \oplus s)} .$$

this game is $\Delta_1^{J_\kappa(\mathbb{R})}$, so by $J_\kappa(\mathbb{R}) \models KP$, the payoff is in $J_\kappa(\mathbb{R})$, so by $J_\kappa(\mathbb{R}) \models AD$ it is det.

w.l.o.g., II has a w.s. σ in G.

take a real z coding σ.

take c.m. turing ideal $I \ni z$ s.t.

$A_{\alpha(z)}$ is not det. on I.

define σ', a strategy for II;

given x,

consider game in which I plays

x, z

I	x, z	
II	y, s	by σ.

describes σ': $\sigma'(x) = y$.

so $x \oplus y \notin A_{\alpha(z \oplus s)}$

$A_{\alpha(z)}$ is not det. on I, and

$\sigma' \leq_T z$. so $\sigma' \in I$. so there is

some $x \in I$ with $\underset{\substack{\| \\ x \oplus y}}{x * \sigma'} \in A_{\alpha(z)}$.

have $z \leq_T z \oplus s$

so $\alpha(z) \leq \alpha(z \oplus s)$.

also $z \oplus s \in I$, hence $A_{\alpha(z)}$ is

not det. on I. so $\alpha(z \oplus s) \leq \alpha(z)$.

so here $\alpha(z) = \alpha(z \oplus s)$. $\mathcal{S}$

proof of (2):

assuming $J_\kappa(\mathbb{R}) \models KP + AD$, want to show $\overline{OD^{<\kappa}}$ is closed under $\exists^{\mathbb{R}}$ (and $\forall^{\mathbb{R}}$).

<u>lemma</u>. $\forall A \subset \mathbb{R}$, TFAE.

(*) $A \in \overline{OD^{<\kappa}}$

(**) for any ctbl. $\sigma \subset \mathbb{R}$ there is a cone of turing degrees d s.t.

$$\exists A' \in OD_d^{<\kappa} \quad A \cap \sigma = A' \cap \sigma.$$

proof: similar to martin, "the largest ctbl. this, that, and the other."

(*) $\Longrightarrow$ (**) trivial.

assume (**). ~~[illegible]~~ $\cancel{d_0}$.

let $\sigma \subset \mathbb{R}$ be ctbl.

pick d_0 s.t. $\forall d \geq d_0 \ \exists A' \in OD_d^{<\kappa}$

$$A' \cap \sigma = \underbrace{A \cap \sigma}_{a}.$$

let $A_{a,\sigma,d}$ = the $OD_d^{<\kappa}$ least such A'.

by $J_\kappa(\mathbb{R}) \models KP$, have $\alpha < \kappa$ s.t.

$A_{a,\sigma,d} \in OD_d^{\alpha}$ f.a. $d \geq d_0$.

define $A_{a,\sigma} = \{x : x \in A_{a,\sigma,d}$ for a cone of $d \geq d_0\}$.

~~then~~

$\mathbb{R} \setminus A_{a,\sigma} = \{x : x \in \mathbb{R} \setminus A_{a,\sigma,d}$ for a cone of $d \geq d_0\}$.

want to show $A_{a,\sigma} \in OD^{<\kappa}$.

$(a_1, \sigma_1) \leq (a_2, \sigma_2) \iff$

$\forall^* d$ (the OD_d^α – least A_1 with

$A_1 \cap \sigma_1 = a_1$ is

$\leq_{OD_d^\alpha}$ the OD_d^α – least

A_2 with $A_2 \cap \sigma_2 = a_2$).

So $A_{a,\sigma} \in OD^{<\alpha}$ +

agrees with A on σ.