Square Principles in Fine Structural Models

~~Def. Let ϑ be regular.~~

<u>Def.</u> A sequence $(C_\alpha \mid \alpha < \vartheta)$, where ϑ is a cardinal or $\vartheta = \mathrm{OR}$, is a coherent sequence of clubs iff

a) each C_α is club in α,

b) if $\bar\alpha \in \lim(C_\alpha)$, then $C_{\bar\alpha} = C_\alpha \cap \bar\alpha$.

<u>Def.</u> Let ϑ be regular and $(C_\alpha \mid \alpha < \vartheta)$ be a coherent sequence of clubs.
A club $C \subseteq \vartheta$ is a thread for $(C_\alpha \mid \alpha < \vartheta)$ iff for every $\alpha \in \lim(C)$

$$C_\alpha = C \cap \alpha .$$

<u>Def.</u> A coherent sequence of clubs $(C_\alpha \mid \alpha < \vartheta)$ is threadable iff there is a thread for $(C_\alpha \mid \alpha < \vartheta)$.
Otherwise $(C_\alpha \mid \alpha < \vartheta)$ is nonthreadable; we

also say that $(C_\alpha \mid \alpha < \vartheta)$ is a $\square(\vartheta)$-sequence.

If there is a $\square(\vartheta)$-sequence then we say that $\square(\vartheta)$ holds.

<u>Fact (ZFC)</u>. $\square(\omega_1)$.

<u>Def.</u> A coherent sequence of clubs $(C_\alpha \mid \alpha < \kappa^+)$ is a $\square_\kappa$-sequence iff $\mathrm{otp}(C_\alpha) \leq \kappa$ for all $\alpha < \kappa^+$.

<u>Fact</u>. Every $\square_\kappa$-sequence is a $\square(\kappa^+)$-sequence.

<u>Fact</u>. Assume $V \subseteq W$ are transitive models s.t. κ, κ^+ are cardinals in both. Let $(C_\alpha \mid \alpha < \kappa^+)$ be a $\square_\kappa$-sequence in V. Then $(C_\alpha \mid \alpha < \kappa^+)$ is a $\square_\kappa$-sequence in W.

<u>Def.</u> Let ϑ be inaccessible or $\vartheta = OR$. A coherent sequence of clubs $(C_\alpha \mid \alpha < \vartheta)$ is a global $\square$-sequence iff $\mathrm{otp}(C_\alpha) < \alpha$ for

every singular $\alpha < \vartheta$.

<u>Theorem</u> (Jensen, late 60's)

$L \models$ There is a global $\square$-sequence

<u>Consistency strength</u>

Theorem (Jensen): Let κ be a singular cardinal. Then $(\kappa^+)^L = \kappa^+$ iff $0^\#$ does not exist.

<u>Fact</u> $\square \iff (\forall \kappa\ \square_\kappa)\ \&\ \square^{\text{sing. cards}}$

<u>Corollary</u>. If $\square_\kappa$ fails for a singular κ, then $0^\#$ exists.
In particular $\neg \square_{\aleph_\omega} \Rightarrow 0^\#$ exists.

<u>Proposition</u>. Assume κ^+ is not Mahlo in L. Then $V \models \square_\kappa$.

<u>Why</u>: In L there is a club $C \subseteq \kappa^+$ of singular cardinals. So we have a coherent sequence of clubs $(C_\alpha \mid \alpha \in C)$ s.t. $\forall \alpha \in C$

$\mathrm{otp}(C_\alpha) < \alpha$.

By filling in + some combinatorial argument this sequence can be turned into a $\square_\kappa$ - sequence.

<u>Theorem</u> (Solovay) If λ is Mahlo and $\kappa < \lambda$ is regular, then in $V^{\mathrm{Col}(\kappa, <\lambda)}$ we get $\lambda = \kappa^+$ & $\neg \square_\kappa$.

<u>Theorem</u>. ~~Let $\kappa > \omega_1$ be regular~~. The following are equiconsistent:

a) $\neg \square(\omega_2)$, (Baumgartner)

b) ω_2 is weakly compact in L. (Jensen)

<u>Theorem</u>. (Solovay)

κ is κ^+-supercompact $\Longrightarrow \neg \square_\kappa$.

<u>Def</u>. κ is subcompact iff the set

$$S^* := \{ X \in [H_{\kappa^+}]^{<\kappa} \mid (X; \in) \cong (H_{\mu^+}; \in) \text{ for some } \mu < \kappa \}$$

is stationary.

<u>Theorem</u> (Z.) If κ is a measurable ~~sup~~ subcompact cardinal then there is a forcing extension s.t.

$$\kappa = \aleph_{\omega+1} \text{ and } \neg \square_{\aleph_\omega}.$$

If GCH holds in V, then GCH holds in $V[G]$.

<u>Def.</u> (Schimmerling)
$\square_{\kappa,\lambda}$ is defined similarly to $\square_\kappa$ but this time $\mathcal{C}_\alpha \subseteq P(\alpha)$ is a family of clubs of size $\leq \lambda$.
Coherence: $C \in \mathcal{C}_\alpha$, $\bar{\alpha} \in \lim(C) \Rightarrow C \cap \bar{\alpha} \in \mathcal{C}_{\bar{\alpha}}$
order type: $\mathrm{otp}(C) \leq \kappa \quad \forall C \in \mathcal{C}_\alpha$.

In the theorem above we actually get

$$\neg \square_{\aleph_\omega, <\omega} \;\&\; \square_{\aleph_\omega, \omega}$$

Proving $\square_\kappa$ in fine structural models:

- $L[U]$ Solovay
- 0^{long} ("zero long") Koepke
- $o(\kappa) < 1$ D. Wylie (actually larger $o(\kappa)$)
- ~~1 strong cardinal~~
- below 1 Woodin: Schimmerling: various forms of $\square_{\kappa,\lambda}$
  - $\square_{\kappa,2}$ up to 1 strong
  - $\square_\kappa$ somewhere below

- 1 strong Jensen, also $\square_{\kappa,2} \not\Rightarrow \square_\kappa$
- $\square_{\kappa,<\omega}$ Schimmerling
- Measurable Woodin in Z.

  $L[E]$ with λ-indexing

- Full characterization Schimmerling-Z.

Theorem (Schimmerling - Z.)

If $L[E]$ is an extender model* with λ-indexing then TFAE

a) $\square_\kappa$

b) $\square_{\kappa_1 < \kappa}$

c) κ is not subcompact

d) $S = \{\nu \in (\kappa, \kappa^+) \mid \nu$ indexes an E-extender$\}$ is non-stationary.

* including: fine structural,
solidity,
soundness of levels,
condensation

i.e. some consequences of iterability

Proof of $\square_\kappa$ in L:

The heart of the construction is the construction of a $\square(\kappa^+)$-sequence $(B_\alpha \mid \alpha \in C^*)$, where $C^* \subseteq \kappa^+$ is the set

$$C^* = \{ \alpha \in (\kappa, \kappa^+) \mid J_\alpha \prec J_{\kappa^+} \}$$

To each $\alpha \in C^*$ we assign the following parameters:

- β_α; the collapsing level of α, i.e. α is a cardinal in J_{β_α} but not in $J_{\beta_\alpha + 1}$.

  Write $N_\alpha := J_{\beta_\alpha}$. Note $\alpha = (\kappa^+)^{N_\alpha}$

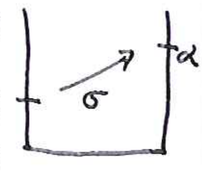
- n_α; the definability level of a collapsing function for α. In this case n_α is the unique n s.t. $\rho^{n+1}_{N_\alpha} \leq \kappa < \rho^n_{N_\alpha}$

- p_α; the standard parameter p_{N_α}

We let for $\alpha \in C^*$:

$\bar\alpha \in B_\alpha$:$\Longleftrightarrow$ a) $n_{\bar\alpha} = n_\alpha$ $\quad \sigma : N_{\bar\alpha} \to N_\alpha$

b) There is a map σ s.t.



i) $crit(\sigma) = \bar\alpha$, $\sigma(\bar\alpha) = \alpha$

ii) $\sigma(p_{\bar\alpha}) = p_\alpha$

iii) $\sigma : N_{\bar\alpha} \longrightarrow_{\Sigma_0^{(n_\alpha)}} N_\alpha$

Note: Think of the case $n_\alpha = 0$. Then $\Sigma_0^{(n)}$ is just Σ_0.

In general $\Sigma_0^{(n)}$ means Σ_0 as a map between the reducts $\sigma : N_{\bar{\alpha}}^{n_{\bar{\alpha}}} \longrightarrow N_\alpha^{n_\alpha}$ where $N_{\bar{\alpha}}^{n_{\bar{\alpha}}}$, $N_\alpha^{n_\alpha}$ have master codes as predicates.

Fact 1. σ is uniquely determined, can write $\sigma_{\bar{\alpha}\alpha}$.

Fact 2. $\sigma_{\bar{\alpha}\alpha}$ is not $\Sigma_1^{(n_\alpha)}$, hence it is not cofinal at the n_α-th projectum.

From now on assume $n = 0$ and write Σ_0 for $\Sigma_0^{(n)}$.

Recall. Important point: Σ_1-Skolem functions have a uniform Σ_1-definition. This means there is a Σ_0-formula $\psi(u, v, w, z)$ s.t. for every $\alpha \in OR$ and $x, y \in J_\alpha$, $i < \omega$

$$y = h_{J_\alpha}(i, x) \text{ iff } J_\alpha \models \exists z\, \psi(y, i, x, z)$$

We write

$$H_\alpha = \{ (y, i, x, z) \in J_\alpha \times \omega \times J_\alpha \times J_\alpha \mid J_\alpha \models \psi(y, i, x, z) \}$$

So $y = \underset{\underset{h_{J_\alpha}}{\shortparallel}}{h_\alpha}(i,x)$ iff $\exists z\, H_\alpha(y,i,x,z)$

Also notice: If $y, y', x, z, z' \in J_\alpha$, $i<\omega$ then

$$H_\alpha(y,i,x,z) \wedge H_\alpha(y',i,x,z') \Longrightarrow y=y'$$

<u>To see fact 1</u>: Assume $y = \cancel{H_{\bar\alpha}}\ h_{\bar\alpha}(i,x)$.

So, for some $z \in J_{\bar\alpha}$, $H_{\bar\alpha}(y,i,x,z)$.

Apply σ: $H_\alpha(\sigma(y), i, \sigma(x), \sigma(z))$.

Hence $\sigma(y) = h_\alpha(i,\sigma(x))$.

By soundness: There is $\xi<\kappa$ s.t. $y = h_{\bar\alpha}(i, \langle \xi, p_{\bar\alpha}\rangle)$.

Apply the above to $x = \langle \xi, p_{\bar\alpha}\rangle$, then

$$\sigma(y) = h_\alpha(i, \langle \underset{\underset{\sigma(\xi)}{\shortparallel}}{\xi}, \underset{\underset{\sigma(p_{\bar\alpha})}{\shortparallel}}{p_\alpha} \rangle)$$

Thus σ is unique.

<u>To see fact 2</u>: What if σ were Σ_1-preserving?

Then $\mathrm{rng}(\sigma) = N_\alpha$ because for $y \in N_\alpha$ then

$\cancel{N_\alpha \models \exists w}$

, say $y = h_\alpha(i, \langle \xi, p_\alpha \rangle)$, $\xi < \kappa$, so

$$\underbrace{N_\alpha \models \exists w : w = h_\alpha(\underset{\sigma(i)}{\overset{\shortparallel}{i}}, (\underset{\sigma(\xi)}{\overset{\shortparallel}{\xi}}, \underset{\sigma(p_{\bar\alpha})}{\overset{\shortparallel}{p_\alpha}}))}_{\Sigma_1}$$

If σ is Σ_1 - preserving then is some $y' \in rng(\sigma)$ for $\exists w \ldots$ but y is the unique witness and hence $y = y' \in rng(\sigma)$.

This holds for every $y \in N_\alpha$. $\lightning$

<u>Fact 3</u>. Assume $\alpha^*, \bar\alpha \in B_\alpha$ and $\alpha^* < \bar\alpha$. Then

$$rng(\sigma_{\alpha^*}) \subseteq rng(\sigma_{\bar\alpha})$$

and therefore we can define the factor map

$$\sigma_{\alpha^* \bar\alpha} : N_{\alpha^*} \longrightarrow N_{\bar\alpha}$$

by $\sigma_{\alpha^* \bar\alpha}(x) = \sigma_{\bar\alpha}^{-1} \circ \sigma_{\alpha^*}(x)$

<u>Proof.</u> Let $\gamma^* = \sup \sigma_{\alpha^*}[OR \cap N_{\alpha^*}] \in N_\alpha$

$$\bar{\gamma} = \sup \sigma_{\bar\alpha}[OR \cap N_{\bar\alpha}] \in N_\alpha .$$

We prove

(1) $\gamma^* < \gamma$.

With this we argue as follows:

Assume $y = h_{\alpha^*}(i, \langle \xi, p_{\alpha^*} \rangle)$, some $\xi < \kappa$,

i.e. for some $z \in N_{\alpha^*}$

$$H_{\alpha^*}(y, i, \langle \xi, p_{\alpha^*} \rangle, z)$$

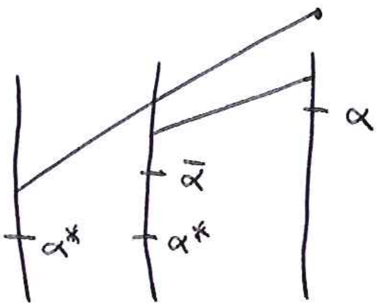
Apply σ_{α^*}:

$$H_\alpha(\sigma_{\alpha^*}(y), i, \langle \xi, p_\alpha \rangle, \sigma_{\alpha^*}(z))$$

$$N_\alpha \models \underbrace{\exists z' \in J_\delta \; H_\alpha(\sigma_{\alpha^*}(y), i, \langle \xi, p_\alpha \rangle, z')}_{\Delta_0}$$

can pull back to $N_{\bar\alpha}$

<u>Proof of (1).</u> $\gamma^* < \bar\gamma$. Suppose not.



Notice $\alpha^* = h_{\bar\alpha}(i, \langle \xi, p_{\bar\alpha}\rangle)$ for some $i<\omega$, $\xi<\kappa$.

As before - apply $\sigma_{\bar\alpha}$

$$\alpha^* = h_\alpha(i, \langle \xi, p_\alpha\rangle)$$

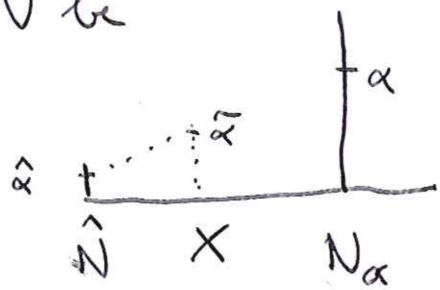
A witness for the existential quantifier for this statement is in $J_{\bar\gamma} \subseteq J_{\gamma^*}$.

As in the proof of (1) $\Rightarrow$ Fact 3 we can conclude $\alpha^* \in \mathrm{rng}(\sigma_{\alpha^*})$. $\lightning$ $\dashv$ Fact 3

<u>Fact 4</u> $\sigma_{\alpha^*\bar\alpha} : N_{\alpha^*} \longrightarrow N_{\bar\alpha}$ has properties a), b) from the definition of σ_α.

<u>Lemma A</u>. If $cf(\alpha) > \omega$ then B_α is unbounded in α.

Let $\alpha' < \alpha$. We find $\bar\alpha \in B_\alpha$ s.t. $\bar\alpha > \alpha'$.
Let $\alpha' \in X \prec N_\alpha$ be countable, $\hat N$ be its transitive collapse,



$\sigma : \hat N \longrightarrow N_\alpha$
be the inverse map.

$\tilde{\alpha} := \sup(X \cap \alpha) = \sup \sigma[\hat{\alpha}] < \alpha$

So $\sigma \upharpoonright (\hat{N} \upharpoonright \hat{\alpha}) : \hat{N} | \hat{\alpha} \longrightarrow_{\Sigma_0} N_\alpha | \tilde{\alpha}$ cofinal.

Let $\tilde{N}$ = ultrapower of $\hat{N}$ by (the long extender derived from) the map $\sigma \upharpoonright (\hat{N} | \hat{\alpha})$.

Elements of $\tilde{N}$ are of the form $[\gamma, f]$, $\gamma < \kappa$ and $f \in \hat{N}$, $f : \hat{\kappa} \longrightarrow \hat{N}$, $\hat{\kappa} = \sigma^{-1}(\kappa)$.

We get a map

$$\sigma' : \tilde{N} \longrightarrow N_\alpha$$
$$[\gamma, f] \longmapsto \sigma(f)(\gamma)$$

Also if $\tilde{\sigma} : \hat{N} \longrightarrow \tilde{N}$ is the canonical embedding then $[\gamma, f] = \tilde{\sigma}(f)(\gamma)$.

We get:
- σ' is Σ_0-preserving
- $\tilde{\sigma}$ is Σ_0-preserving & cofinal
- $\mathrm{crit}(\sigma') = \tilde{\alpha}$ and $\sigma'(\tilde{\alpha}) = \alpha$
- $\tilde{\sigma}(\hat{p}) = p_{\tilde{N}}$ and $\sigma'(p_{\tilde{N}}) = p_\alpha$

$$\hat{p} = \sigma^{-1}(p_\alpha)$$

- $S^1_{\tilde{N}} = \kappa$

All of this gives $\tilde{\alpha} \in B_\alpha$ and $\sigma' = \sigma_{\tilde{\alpha}}$.
Also $\tilde{\alpha} > \alpha'$ as $\alpha' \in X$.

<u>Lemma B</u> B_α is a closed subset of α.