Tutorial III, part 4

Want to define the set B_α.

First task: Tell what the collapsing structure is.

<u>Def.</u> Let N_α be the collapsing level for α in $L[E]$. A pair (q,μ) is a <u>divisor</u> for N_α iff

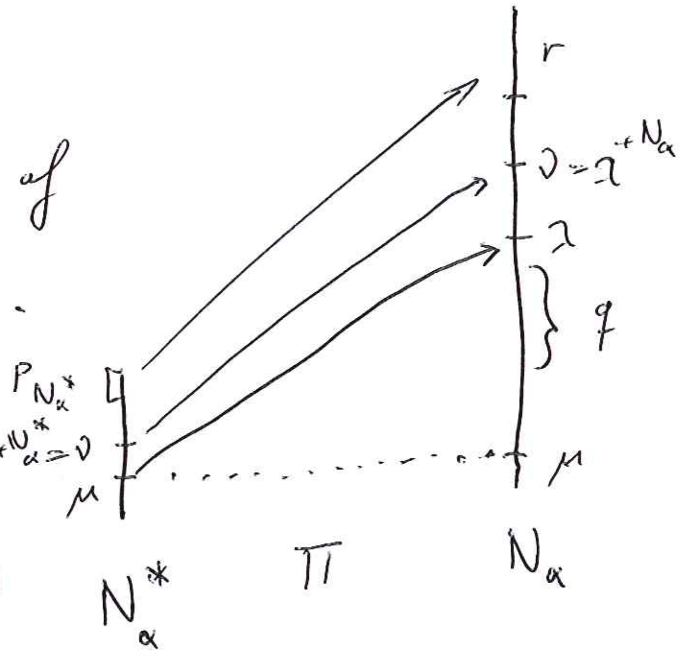
a) q is a bottom part of p_α, i.e. there is some r s.t. $p_\alpha = r \cup q$ and $\min(r) > \max(q)$.

b) $\tilde{h}^{n_\alpha+1}_{N_\alpha}(\mu \cup r)$ is cofinal in $\varrho^{n_\alpha}_{N_\alpha}$. ~~where~~

c) $\max(q) \cap \tilde{h}^{n_\alpha+1}_{N_\alpha}(\mu \cup r) \subseteq \mu$.

Again WLOG $n_\alpha = 0$

We let N^*_α = the collapse of $h_{N_\alpha}(\mu \cup r)$.



We allow $\mu = \kappa$ in which case we call (q,μ) an <u>impossible divisor</u>.

Claim 1. $N_\alpha^* \trianglelefteq N_\alpha$ (Unless $\kappa = \mu$ and $q = \cancel{p_\alpha}\ \emptyset$?

in which case $N_\alpha^* = N_\alpha$.)

1) N_α^* is sound.

With 1) we can apply the CL (condensation lemma).

Notice: $\rho^1_{N_\alpha^*} = \rho^\omega_{N_\alpha^*} = \mu$.

Why: $\leq$ easy by definition of N_α^*.

Equality follows from

2) μ is a limit cardinal in N_α.

Why: Because $\mu < \kappa^{+N_\alpha} \leq \lambda$.

Now discuss the options in CL:

a) $N_\alpha^* = \mathrm{core}(N_\alpha)$

~~Impossible~~ Only possible if $(q, \mu) = (\overset{\emptyset}{\cancel{p_\alpha}}, \kappa)$.

Otherwise $N_\alpha^* \in N_\alpha$.

c) $\mathrm{crit}(\pi)$ is a successor cardinal in N_α^*.
Impossible by 2).

d) ~~N^*_α~~ This would mean that $\mu = \mathrm{crit}(\pi)$ indexes an extender in N_α. Impossible by 2).

What remains is b): $N^*_\alpha \trianglelefteq N_\alpha$.

<u>Proof of 1).</u> Let $\bar{N} = \mathrm{core}(N^*_\alpha)$.

<u>Notice:</u> $P(\mu) \cap \bar{N} = P(\mu) \cap N^*_\alpha$.

Let F be the (μ, λ)-extender derived from π.

$$\begin{tikzcd}
\tilde{N} \arrow[r, "k"] & N_\alpha \\
\bar{N} \arrow[u, "\pi_F"] \arrow[r, "\sigma = \text{core map}"'] & N_{\alpha^*} \arrow[u, "\pi"']
\end{tikzcd}$$

<u>Have:</u> • k is Σ_0-preserving & cofinal

• $k \upharpoonright \nu = \mathrm{id}$

• $\tilde{N}$ is sound above λ.

Use CL to prove $\tilde{N} = N_\alpha$. Then $\bar{N}, N^*_\alpha$ are both the transitive collapse of $h_\alpha(\mu \cup r)$.

So $\bar{N} = N^*_\alpha$ is sound.

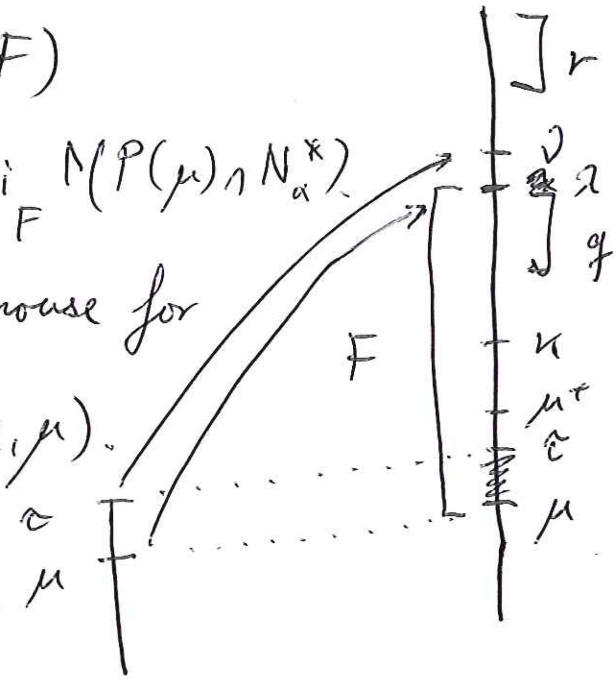
<u>Alternative:</u> Direct argument if $\sigma(p_{\bar{N}}) <^* p_{N^*_\alpha}$

we could calculate the Σ_1-theory of p_{N_α} inside N_α.

<u>Def.</u> $N_\alpha(q,\mu) = (J_\nu^E, F)$

Identify F with $\pi = i_F^{M}(P(\mu) \cap N_\alpha^*)$

$N_\alpha(q,\mu)$ is the protomouse for α associated with (q,μ).



<u>Task.</u> Find one canonical divisor for α.

<u>Def.</u> A divisor (q,μ) is <u>strong</u> iff for every $x \in N_\alpha \cap P(\lambda)$: if $x \in h_{N_\alpha}(\mu, p_\alpha)$ then there is $y \in h_{N_\alpha}(\mu, r)$ s.t. $x \cap \mu = y \cap \mu$.

<u>Lemma.</u> (q,μ) is strong iff any of the following holds:

a) $P(\mu) \cap N_\alpha^* = P(\mu) \cap N_\alpha'$ where

N_α' = the collapse of $h_{N_\alpha}(\mu \cup p_\alpha)$.

b) $|p_{N_\alpha^*}| = |p_{N'_\alpha}| = |r|$.

c) $N_\alpha^* = \operatorname{core}(N'_\alpha)$.

<u>Lemma</u> (crucial!)

~~Let (q,μ) be~~

Let q be a bottom part of p_α.

$$\mathcal{D}_q := \{\mu < \kappa \mid (q,\mu) \text{ is a strong divisor}\}$$

is closed and bounded in κ.

<u>Proof</u>. Let $\mu^* \in \lim(\mathcal{D}_q)$.

Easy: (q,μ^*) is a divisor.

To see (q,μ^*) is strong:

~~Let $x = h_{N_\alpha}(i,(\xi,p_\alpha))$, $\xi < \mu^*$.~~

~~As $\mu^* \in \lim(\mathcal{D}_q)$ can find $\mu \in \mathcal{D}_q$ s.t.~~

~~$\xi < \mu$. But (q,μ) is strong, so there~~

~~is some $\xi' < \mu < \mu^*$ s.t.~~

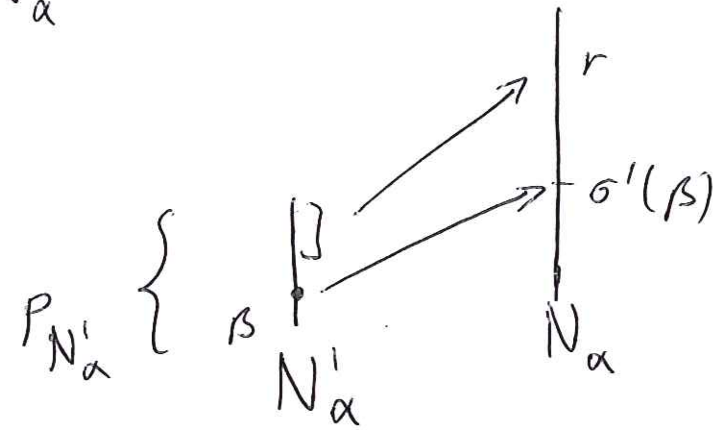
We use b) of the previous lemma:

Assume $p_{N'_\alpha}$ is longer than $|r|$.

We have $\sigma' : N'_\alpha \longrightarrow N_\alpha$.

Notice that $\sigma'^{-1}(r)$ is a top part of $p_{N'_\alpha}$.

Let $\beta = \max(p_{N'_\alpha} \setminus \sigma'^{-1}(r))$

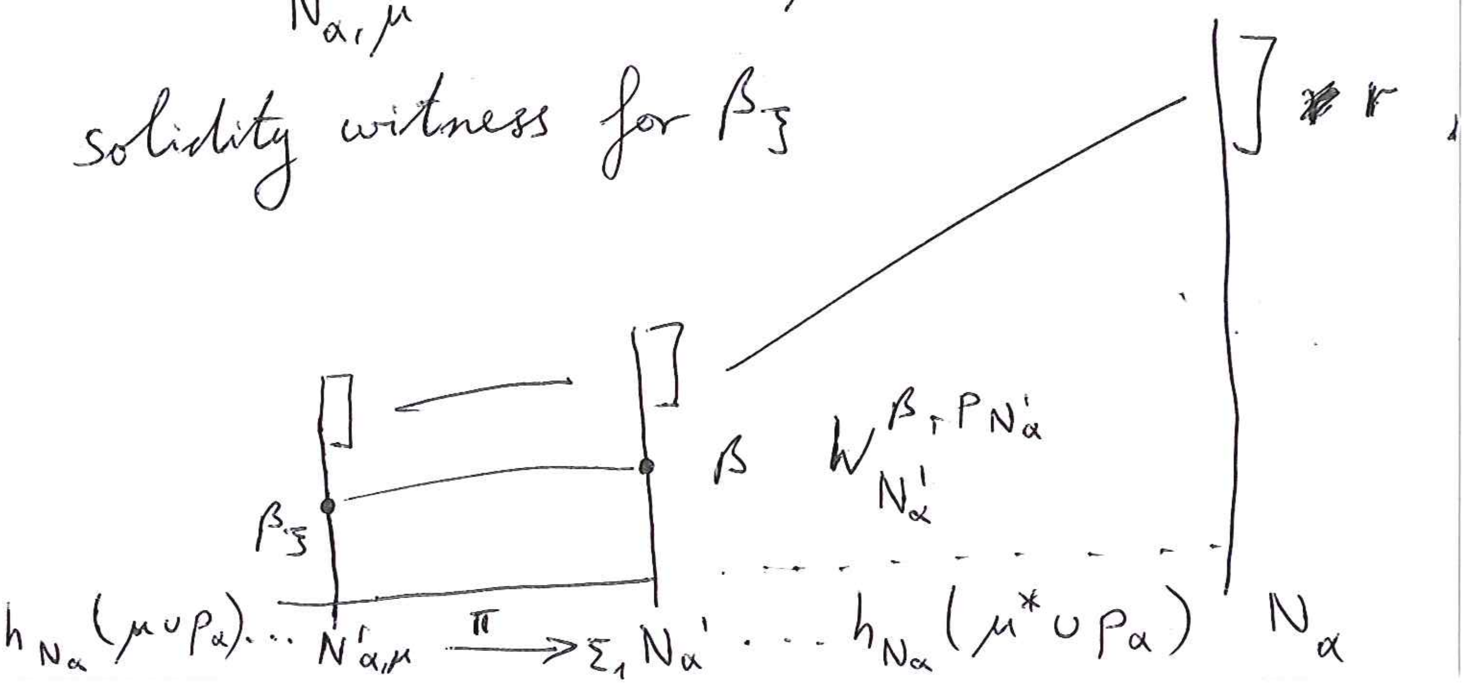


N'_α is solid, so $W^{\beta, p_{N'_\alpha}}_{N'_\alpha} \in N'_\alpha$.

Hence $W^{\beta, p_{N'_\alpha}}_{N'_\alpha} \in h_{N'_\alpha}(i, (\xi, p_{N'_\alpha}))$ some $\xi < \mu^*$.

Since $\mu^* \in \lim(\mathcal{D}_q)$, we can pick $\mu \in \mathcal{D}_q$ s.t. $\xi < \mu < \mu^*$.

Then $\overbrace{h_{N'_{\alpha,\mu}}(i, (\xi, p_{N'_{\alpha,\mu}}))}^{\pi^{-1}(W^{\beta, p_{N'_\alpha}}_{N'_\alpha})}$ is a generalized solidity witness for β_ξ



But $\mu \in \mathcal{D}_q$, so $\cancel{\pi}$ $\pi^{-1} \circ \sigma'(r)$ is the full standard parameter for $\cancel{N'_{\alpha,\bar\gamma}}$. Contradiction!

$$N'_{\alpha,\mu}$$

$\dashv$

This proves $\mathcal{D}_q$ is closed.

The proof also works for $\mu^* = \kappa$.

To see $\mathcal{D}_q$ is bounded, we prove

(q,κ) is not strong.

If $q = p_\alpha$ then (q,κ) is not a divisor.

If $q \neq p_\alpha$ then $\underbrace{N^*_\alpha(q,\kappa)}_{h_\alpha(\kappa \cup r)} \in N_\alpha$

But $N'_\alpha(q,\kappa) = h_\alpha(\kappa \cup p_\alpha) = N_\alpha$.

So $P(\kappa) \cap N^*_\alpha(q,\kappa) \neq P(\kappa) \cap N'_\alpha(q,\kappa)$.

It follows that $\sup(\mathcal{D}_q)$, being a strong divisor, is strictly less than κ.