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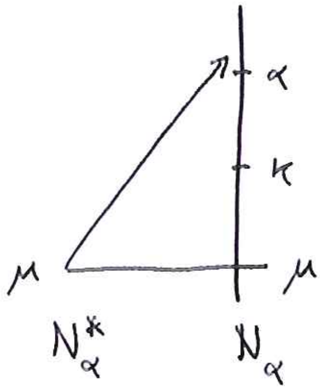
Tutorial III, part

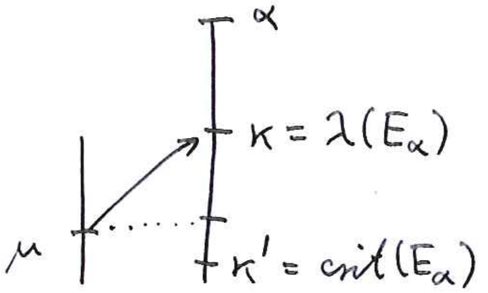
Define B_α on a club $C^* \subseteq \kappa^+$.

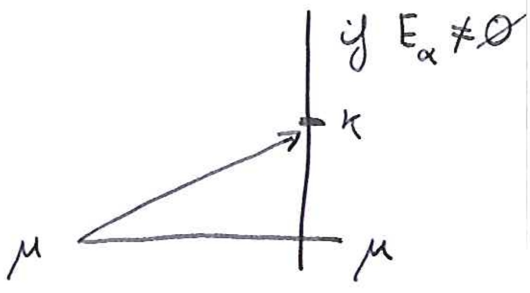
$C^* \subseteq$ the set of all $\alpha \in (\kappa, \kappa^+)$ s.t.

- $J^E_\alpha \prec J^E_{\kappa^+}$
- $E_\alpha = \emptyset$



If $E_\alpha \neq \emptyset$





C^* exists iff

$$S = \{\alpha < \kappa^+ \mid E_\alpha \neq \emptyset\}$$

is non-stationary.

<u>Prop.</u> (Jensen) If S is stationary then $\square_\kappa$ fails.

Jensen's argument shows:

<u>Prop.</u> If S is stationary then κ is subcompact.

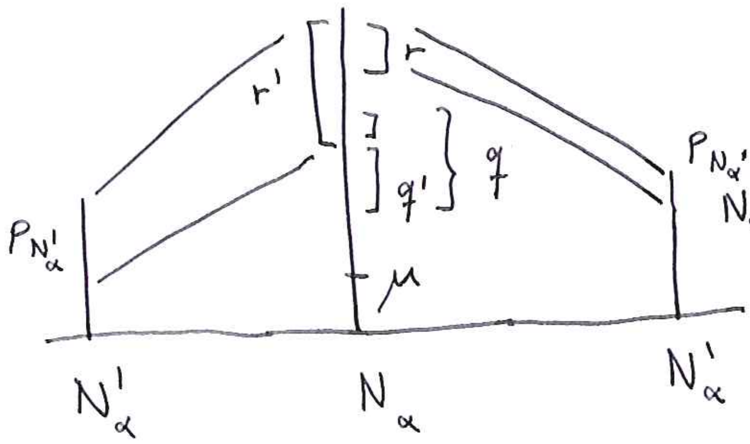
Note. We have equivalence in this last proposition.

$S^0 = C^* \setminus S^1$

$S^1 = \{\alpha \in C^* \mid N_\alpha \text{ has a strong divisor}\}$

Fact. If (q,μ) is a strong divisor for N_α and q' is a proper bottom part of q then (q',μ) is not a strong divisor.

Proof.



$N'_\alpha(q,\mu)$ = collapse of $h_{N_\alpha}(\mu \cup p_\alpha)$

⊣

Cor. If (q,μ) is a strong divisor for N_α with largest possible μ then q is shortest possible s.t. (q,μ) is a divisor for N_α.

Def. Assume N_α has a strong divisor. We define (q_α, μ_α) = the unique strong divisor with largest possible μ_α.

B_α for $\alpha \in S^0$: $\bar\alpha \in B_\alpha$ iff

i) $n_{\bar\alpha} = n_\alpha$

ii) there is a map $\sigma_{\bar\alpha\alpha} : N_{\bar\alpha} \longrightarrow N_\alpha$ s.t.

a) $\mathrm{crit}(\sigma_{\bar\alpha\alpha}) = \bar\alpha$ & $\sigma_{\bar\alpha\alpha}(\bar\alpha) = \alpha$,

~~b~~b) $\sigma_{\bar\alpha\alpha}(p_{\bar\alpha}) = p_\alpha$,

c ~~b)~~) σ is $\Sigma_0^{(n_\alpha)}$-preserving,

d) for every $\beta \in p_\alpha$ there is some generalized witness (Q, r) for β w.r.t. p_α in N_α s.t. $(Q, r) \in \mathrm{rng}(\sigma_{\bar\alpha\alpha})$.

<u>Lemma</u>. There is some $\hat\alpha < \alpha$ s.t. $B_\alpha \setminus \hat\alpha \subseteq S_0$.

If we have this define B'_α coherent sequence as before. Then let $\alpha' =$ the least $\bar\alpha \in$ ~~B_α~~ s.t. $B'_\alpha \setminus \bar\alpha \subseteq S^0$.

Let $B^*_\alpha = B'_\alpha - \alpha'$. Then B^*_α is coherent and we can thin them out as before to get the $\square_\kappa$-sequence C_α.

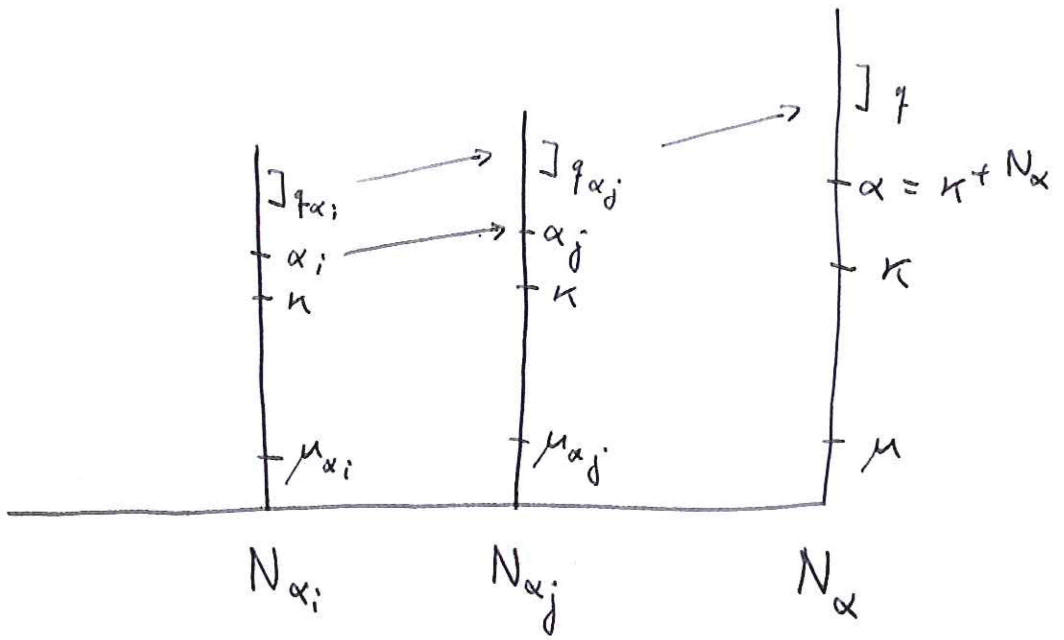
<u>Proof of Lemma</u>. We show: If $N_{\bar\alpha}$ admit a strong divisor $(q_{\bar\alpha}, \mu_{\bar\alpha})$ for cofinally many $\bar\alpha < \alpha$, then N_α admits a strong divisor – contradicting our

assumption on α.

Assume $\gamma = cf(\alpha)$ and $(\alpha_i \mid i < \gamma)$ is an increasing sequence of $\bar{\alpha}$'s as above.

WLOG 1) $|q_{\alpha_i}| = |q_{\alpha_j}|$ for all $i \le j < \gamma$

2) $\mu_{\alpha_i} \le \mu_{\alpha_j}$ for all $i \le j < \gamma$



Let $q = \sigma_{\alpha_i, \alpha}(q_{\alpha_i})$ (independent of i)

$$\mu = \sup_{i<\gamma} \mu_{\alpha_i}$$

3) (q, μ) is a divisor for N_α.

Criterion on (q,μ) being a divisor:

a) $h_\alpha(\mu \cup \overset{r}{\cancel{q}})$ is cofinal in the last projectum

b) if $h_\alpha(k, (\xi, r)) < \max(q)$, then
$h_\alpha(k(\xi, r)) < \mu$ for all $k < \omega$, $\xi < \mu$

a) Exercise.

b) Let $\zeta = h_\alpha(k, (\zeta, \cancel{p} r)) < \max(q)$.

Let i be large enough s.t. $h_{\alpha_i}(k, (\zeta, r_{\alpha_i}))$ is defined. ~~Because~~ $\cancel{(q_{\alpha_i}, \mu_{\alpha_i})}$

By Σ_0-preservation

$$h_{\alpha_i}(k, (\zeta, r_{\alpha_i})) < \max(q_{\alpha_i}).$$

As $(q_{\alpha_i}, \mu_{\alpha_i})$ is a divisor

$$h_{\alpha_i}(k, (\zeta, r_{\alpha_i})) < \mu_{\alpha_i}$$

By Σ_0-preservation

$$h_\alpha(k, (\zeta, r)) < \mu, \quad \text{i.e.}$$

(q, μ) is a divisor.

4) (q, μ) is strong.

Criterion for (q, μ) being strong:

$$|p_{N'_\alpha(\mu)}| = |r|$$

$$N'_\alpha(\mu) := ch_\alpha(\mu \cup p_\alpha).$$

collapse

$\sigma'_{\alpha,\mu} : N'_\alpha(\mu) \longrightarrow N_\alpha$ inverse of the collapse.

Suppose not. Let $\sigma^{1-1}_{\alpha,\mu}(r) \cup \{\beta\}$ be the top part of $p_{N'_\alpha}$ of length $|r|+1$.

Then there is some $\xi < \mu$ s.t.

$$(\underset{\sigma^{1-1}_{\alpha,\mu}(\beta)}{\underset{\wedge}{\cancel{\beta}}}, W^{\beta, p_{N'_\alpha}}_{N'_\alpha}) = h_{N'_\alpha}(k, (\xi, \sigma^{1-1}_{N'_\alpha}(p_\alpha)))$$

$\sigma^{1}_{\alpha,\mu}$ is $\Sigma_1^{(n_\alpha)}$-preserving

$$\sigma^{1}_{\alpha,\mu}(W^{\beta, p_{N'_\alpha}}_{N'_\alpha}) = h_\alpha(k, (\xi, p_\alpha)).$$

For $i < j$ large enough s.t. $h_{\alpha_i}(k, (\xi, p_{\alpha_i}))$ is defined and $\xi < \mu_{\alpha_i}$.

By Σ_0-preservation

$$(\sigma^{1-1}_{\alpha,\mu}(\beta), \sigma^{-1}_{\alpha_i\alpha} \circ \sigma^{1}_{\alpha\mu}(W^{\beta, p_{N'_\alpha}}_{N'_\alpha})) = h_{\alpha_i}(k, (\xi, p_{\alpha_i})).$$

By using $\sigma^{1-1}_{\alpha_i\mu_i}$

$$(\overbrace{\sigma^{1-1}_{\alpha_i\mu_i} \circ \sigma^{-1}_{\alpha_i\alpha}(\beta)}^{\bar\beta}, \overbrace{\sigma^{1-1}_{\alpha_i\mu_i} \circ \sigma^{-1}_{\alpha_i\alpha} \circ \sigma^{1}_{\alpha\mu}(W^{\beta, p_{N'_\alpha}}_{N'_\alpha})}^{\bar W})$$

$$= h_{N'_{\alpha_i}(\mu_i)}(k, (\xi, \sigma^{1-1}_{\alpha_i,\mu_i}(p_{\alpha_i}))).$$

The point is that $(\bar W, \sigma^{1-1}_{\alpha_i,\mu_i}(r_i))$ is a generalized witness for $\bar\beta$ w.r.t. $\sigma^{1-1}_{\alpha_i,\mu_i}(r_i)$ in $N'_{\alpha_i}(\mu_i)$.

Contradiction as $\sigma^{-1}_{\alpha_i,\mu_i}(r_1) = p_{N'_{\alpha_i}(\mu_i)}$ as we are assuming $(q_{\alpha_i}, \mu_{\alpha_i})$ is a strong divisor for N_{α_i}. $\dashv$

Def. A premouse N is <u>plurisupotent</u> iff

a) $\mathrm{crit}(E^N_{top}) < \kappa$,

b) $\rho^1_N = \kappa$

For $\alpha \in S^1_{M_\alpha}$ we define the canonical collapsing structure M_α as follows.

a) if N_α has a strong, then

$$M_\alpha = N_\alpha(q_\alpha, \mu_\alpha)$$

= the protomouse derived from N_α and its canonical divisor

b) Otherwise N_α is plurisupotent and in this case we let

$$M_\alpha = N_\alpha.$$

<u>Recall</u>: $N(q,\mu) = (J^{E^N}_\nu, F)$ where i_F is the inverse of the collapsing map associated with $h_N(\mu \cup r)$. $\lambda = i_F(\mu)$, $\mu = \mathrm{crit}(i_F)$, $\nu = \lambda^{+N}$, $F = i_F(P(\kappa) \cap N^*)$, $N^* = ch_N(\mu \cup r)$.

S^1 = set of all $\alpha \in C^*$ s.t. ~~either~~

a) N_α admits a strong divisor or

b) N_α is pluripotent.

In particular we demand, which we haven't mentioned, that N_α is <u>not</u> pluripotent for $\alpha \in S^0$.

Fine structure for protomice is done in language for coherent structures, i.e. the only symbols are for the extender sequence and the top extender.

We need to translate statements between N_α and $N_\alpha(q_\alpha, \mu_\alpha)$.

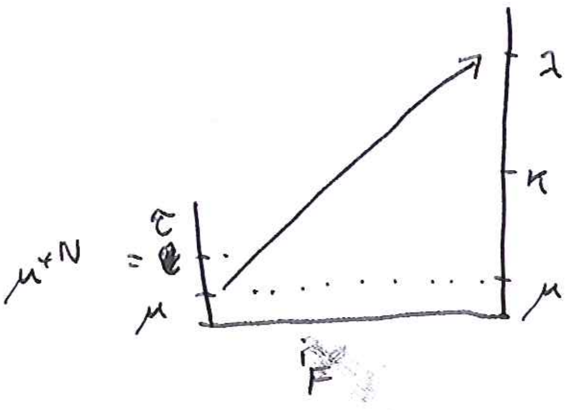
Consider general case N, $M = N(q, \mu)$.

Let $N^* = N^*(q, \mu)$, $F = F(q, \mu)$

$$N = \mathrm{Ult}^*(N^*, F).$$

If $\varphi(\vec{v})$ is a $\Sigma_1^{(n)}$-formula, $n = n(N)$ and $\vec{\gamma} = (\gamma_1, \dots, \gamma_e)$, $\gamma_i < \lambda$

$$N \models \varphi(\vec{\gamma})$$



$$(\exists J < \rho^n_{N^*})\, N \models \exists z^n \in i_F(S^E_J)\, \bar{\varphi}(\vec{\gamma}, z)$$

where $\varphi = \exists v^n \bar{\varphi}$, $\bar{\varphi}$ is $\Sigma^{(n)}_0$

$$(\exists J < \rho^n_{N^*})(\exists a)\ a = \{ \xi < \mu \mid N^* \models (\exists z^n \in S^E_J)\, \bar{\varphi}(\xi, z)\}$$

$$\wedge \langle \vec{\gamma} \rangle \in F(a)$$

$(\exists N^*)$, $(\exists a)$, $(\exists J))$ $[N^*$ is the collapsing level of M for τ (Σ_0) $\wedge\ J < \rho^n_{N^*}$ (Σ_0) $\wedge\ a = \{\dots\}$ (Σ_0) $\wedge \langle \vec{\gamma} \rangle \in F(a)$ (Σ_1) $]$.

So we have a Σ_1-formula $\varphi^*(u, v_1, \dots, v_\ell)$ s.t.

$$N \models \varphi(\vec{\gamma}) \iff M \models \varphi^*(\tau, \vec{\gamma}).$$

We can also go the other way around.
Given a Σ_1-formula $\psi(v_1, \dots, v_\ell)$ in the language for coherent structures, we can find a $\Sigma^{(n)}_1$-formula ${}^*\varphi$ in the language of premice s.t.

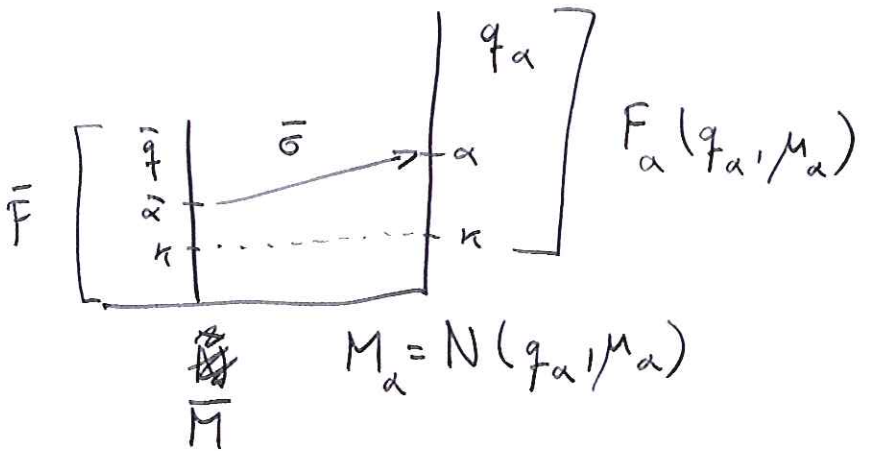
$$M \models \varphi(\vec{\gamma}) \iff N \models {}^*\varphi(\tau, \vec{\gamma}).$$

This can be used to prove:
Let $M = N(\varphi, \mu)$

1. $\varsigma^{1}_{M} = \varsigma^{n+1}_{N}$,
2. $p^{1}_{M} = p^{n+1}_{M}$,
3. M is sound iff N is sound,
4. M is solid iff N is solid.

In fact: Inside J^{E}_{ν} we can turn solidity witnesses for N into ones for M and vice versa.

Now we are given $\alpha \in S^1$ and obtain $\bar{\alpha}$ using the interpolation argument on $\bar{\alpha} \in \lim(B_\alpha)$



$\overline{M}$ is obtained as a liftup or as a direct limit.

<u>First we prove</u>: $\overline{M} = N_{\bar{\alpha}}(q_{\bar{\alpha}}, \mu_{\bar{\alpha}})$

Using the fact that M_{α^*} for $\alpha^* \in B_\alpha$ are sound and solid yield, by the same argument as for premise, that $\bar{M}$ is sound and solid $+\ \varrho^1_M = \kappa$.

We get CL for premise: If

$$\sigma : \bar{N} \to N_\alpha(q_\alpha, \mu_\alpha)$$

is as above then $\bar{N} = N_{\bar{\alpha}}(q_{\bar{\alpha}}, \mu_{\bar{\alpha}})$.

Enough to prove: $Ult^*(\bar{N}^*, \underset{\underset{E^{\bar{N}}_{top}}{\parallel}}{\bar{F}}) \lhd N_\alpha$

Why is this true:

$\bar{N}^* \lhd \bar{N}^*_\alpha$. So

$$\begin{tikzcd} & i_F(\bar{N}^*) \\ \bar{N}^* \arrow[ur, "i_F"] \arrow[dr, "i_{\bar{F}}"'] & \\ & Ult^*(\bar{N}^*, \bar{F}) \arrow[uu, "k"'] \end{tikzcd}$$

$k \restriction \lambda_{\bar{M}} = \bar{\sigma} \restriction \lambda_{\bar{M}}$

In particular

$k \restriction \underset{\underset{\varrho^1_{\bar{M}}}{\parallel}}{\kappa} = id$

Because $\bar{M}$ is sound & solid, the above translation formulae guarantee that $Ult^*(\bar{N}^*, \bar{F})$ is sound & solid. Now apply CL to show that $Ult^* = N_{\bar{\alpha}}$.

Next we need to check:

Ⓐ $(\bar{q}, \mu_\alpha)$ is a strong divisor for $N_{\bar{\alpha}}$.

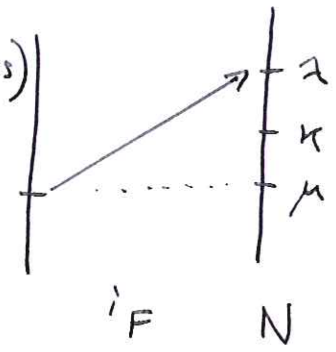
Ⓑ $(\bar{q}, \mu_\alpha)$ is the canonical divisor.

As before, we can check this only for a tail-end.

For Ⓐ

~~If~~ $\cancel{\gamma < \lambda}$ General case: $M = N(q, \mu)$

If $\gamma < \lambda$ is s.t. $\gamma \in hull(\mu \cup r \cup s)$ where $s \in [\lambda]^{<\omega}$.



Then we have a function

$f : {}^{\mu}\cancel{\kappa} \to {}^{\mu}\cancel{\kappa}$, $f \in N^*$ s.t.

$$\gamma = i_F(f)(s, \xi), \quad \xi < \mu$$

Similarly for subsets of λ in place of γ.

Other characterization for strong divisors:

If $h_\alpha(i, (\xi, p_\alpha)) \subseteq \lambda$, $\xi < \mu$

$$h_\alpha(i, (\xi, p_\alpha)) \cap \mu = h_\alpha(i, (\xi', r_\alpha)) \cap \mu, \text{ some } \xi' < \mu.$$

$$\parallel$$

$$F_\alpha(f)(q, \xi) \cap \mu \in J^E_\tau$$

With this we prove that $(\bar{q}, \mu_\alpha)$ is a strong divisor for $N_{\bar{\alpha}}$.

Easier case: $\bar{M}$ is a direct limit.

Assume

$$\bar{F}(f)(\bar{q}, \xi) \leq \bar{\lambda} \quad \Sigma_1$$

So we can find $\alpha^* < \bar{\alpha}$ large enough s.t.

$$F_{\alpha^*}(f)(q_{\alpha^*}, \xi) \leq \lambda_{\alpha^*}$$

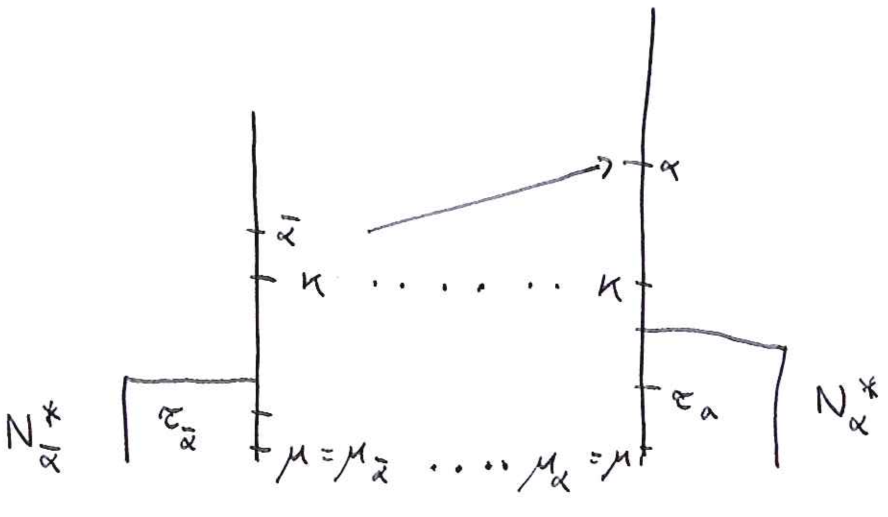
Point: f and ξ are not moved by $\bar{\sigma}_{\alpha^*} : M_{\alpha^*} \to \bar{M}$, because $\xi < \mu$ and $f : \mu \to \mu$.

But $(q_{\alpha^*}, \mu_{\alpha^*})$ is strong by assumption, so

$$F_{\alpha^*}(f)(q_{\alpha^*}, \xi) \cap \mu \in J^E_{\tau_{\alpha^*}} \subseteq J^E_{\bar{\tau}}$$

$$\shortparallel$$

$$\bar{F}(f)(\bar{q}, \xi) \cap \mu$$

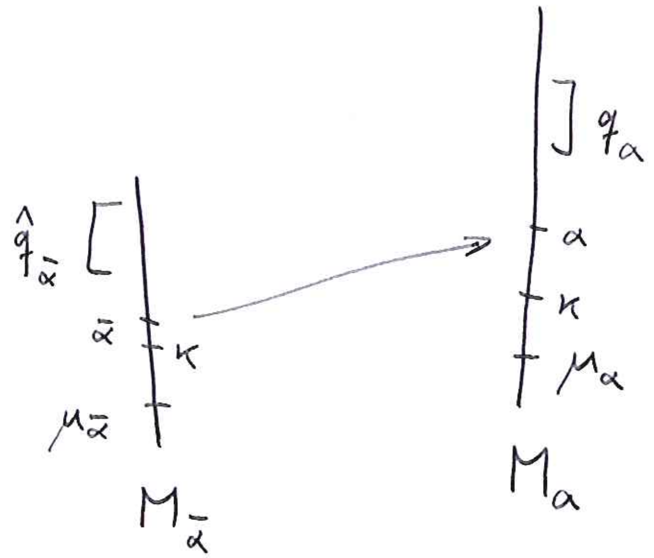


Finally, as before, we need to check $\cancel{B_\alpha X \alpha}$ that for a tail-end of $\bar\alpha \in B_\alpha$: the divisor dictated by the construction is the canonical one.

~~For $\bar\alpha \in B_\alpha$ let $\hat q_{\bar\alpha} = \sigma^{-1}_{\bar\alpha\alpha}(q_\alpha)$~~

For $\bar\alpha < \alpha$ let $\hat q_{\bar\alpha}$ be the parameter for $M_{\bar\alpha}$ dictated by the construction.

<u>WTS</u>. $(\hat q_{\bar\alpha}, \mu_\alpha) = (q_{\bar\alpha}, \mu_{\bar\alpha})$ for a tail end of $\bar\alpha$.



For a contradiction assume $(\hat q_{\bar\alpha}, \mu_\alpha) \neq (q_{\bar\alpha}, \mu_{\bar\alpha})$ for unboundedly many $\bar\alpha < \alpha$. This means $\mu_{\bar\alpha} > \mu_\alpha$ and $q_{\bar\alpha}$ is a ~~proper~~ bottom part of $\hat q_{\bar\alpha}$.

Then run the argument as for $\alpha \in S^0$ but with the difference that all statements are translated into the language of protomice.

e_M is the $<^*$-least finite set of ordinals e s.t.

$$\gamma_M \in h_M(\kappa \cup p_M \cup e)$$

$d_M = p_M \cup e_M$ is the ~~Dodd~~ Dodd parameter.

This is important to deal with type B premice and will replace the standard parameter in this case.