Martin's Maximum^{++} implies Woodin's Axiom $(*)$

David Asperó[*] and Ralf Schindler[†]

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Abstract

We show that Martin's Maximum^{++} implies Woodin's $\mathbb{P}_{\mathrm{max}}$ axiom $(*)$. This answers a question from the 1990's and amalgamates two prominent axioms of set theory which were both known to imply that there are $\aleph_2$ many real numbers.

1 Introduction.

Cantor's Continuum Problem, which later became Hilbert's first Problem (see [18]), asks how many real numbers there are. After having proved his celebrated theorem according to which $\mathbb{R}$ is uncountable, i.e., $2^{\aleph_0} > \aleph_0$, see [4], Cantor conjectured that every uncountable set of reals has the same size as $\mathbb{R}$, i.e., $2^{\aleph_0} = \aleph_1$. This statement is known as Cantor's Continuum Hypothesis (CH). Gödel [14] proved in the 1930's that CH is consistent with the standard axiom system for set theory, ZFC, by showing that CH holds in his constructible universe L, the minimal transitive model of ZFC containing all the ordinals. The axiom $V = L$, saying that the universe V of all sets is simply identical with L, has often been rejected, however, as an undesirable minimalistic assumption about V. For instance, L cannot have measurable cardinals by a result of Scott [43]. Gödel himself believed that CH would be shown not to

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[*] School of Mathematics, University of East Anglia, Norwich Research Park, Norwich, Norfolk, NR4 7TJ, UK. Funded by EPSRC Grant EP/N032160/1.

[†] Institut für Mathematische Logik und Grundlagenforschung, Universität Münster, Einsteinstr. 62, 48149 Münster, FRG. Funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany's Excellence Strategy EXC 2044 390685587, Mathematics Münster: Dynamics - Geometry - Structure.

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follow from ZFC, and at least for part of his life he held the view that CH is indeed false and that actually

$$2^{\aleph_0} = \aleph_2 \tag{1}$$

(see [17] and [16, pp. 173ff.]). In 1947, Gödel [15] wrote:

> [...] one may on good reason suspect that the role of the continuum problem in set theory will be this, that it will finally lead to the discovery of new axioms which will make it possible to disprove Cantor's conjecture.

As we shall now try to explain, in the light of our unifying result, Theorem 1.2, one could make the case that with the two axioms MM^{++} and $(*)$, natural and strong such axioms have already been found.

Luzin [32] proposed a related hypothesis which also refutes CH, namely

$$2^{\aleph_0} = 2^{\aleph_1}. \tag{2}$$

That CH does not follow from ZFC was confirmed by Cohen in 1963 through the discovery of the method of *forcing*: Every model of ZFC can be generically extended to a model of ZFC in which CH fails, see [7]. In fact, using forcing one can show that it is relatively consistent with ZFC that the cardinality of the continuum is $\aleph_1$, $\aleph_2$, $\aleph_{155}$, $\aleph_{\omega^2+17}$, or $\aleph_\alpha$ for many other values of α, see [45].

1.1 New axioms.

Ever since Cohen's work, set-theorists have been searching for natural new axioms which extend ZFC and which settle the Continuum Problem (see e.g. [56], [57], [25], and the discussion in [11]). One family of such axioms is the hierarchy of large cardinal axioms. It was realized early on, though, that these axioms cannot settle the Continuum Problem: one can always force CH to hold or be false by small forcing notions, and all large cardinals which exist in V will retain their large cardinal properties in the respective extensions, see [31].

One axiom which does settle the Continuum Problem is CH itself; after all, CH looks natural in that it gives the least possible value to $2^{\aleph_0}$ consistent with Cantor's theorem, $2^{\aleph_0} > \aleph_0$. CH allows the "diagonal" construction of objects of size $\aleph_1$ with specific combinatorial properties, e.g. Luzin or Sierpiński sets. In 1985, Woodin proved his Σ^2_1 absoluteness result conditioned on CH. Namely, if CH holds true, there is a proper class of measurable Woodin cardinals, and σ is a statement of the form "There is a set of reals X such that $\varphi(X, r)$," where r is a real and $\varphi(X)$ is a formula of set theory all of whose quantifiers are restricted to reals, such that σ can be forced over V, then σ actually holds true in V, see e.g. [9, Theorem 4.1]. Over the last decade, Woodin has developed a sophisticated scenario for set theory according to which CH is true, see e.g. [58] and [59].

Nevertheless, and despite the appeal of Σ^2_1 absoluteness, CH is often regarded as a minimalistic assumption on a par with its parent, $V = L$. To give an illustrative example, under CH one can easily find sets X and Y of reals without endpoints which are both $\aleph_1$-dense—in the sense that every interval of points contains exactly $\aleph_1$ many points—but such that X and Y are not order-isomorphic. On the other hand, by a theorem of Baumgartner [3], given any such X and Y, there is a nicely behaved forcing notion which adds an order-isomorphism between X and Y. Thus, adopting CH precludes the existence of sufficiently generic filters for such forcing notions—which may consistently exist.

A dual approach to CH is to formulate axioms stipulating the existence of objects which may possibly exist, i.e., to look for "maximality principles" expressing some form of saturation of the universe of all sets with respect to its generic extensions. Such principles are known as *forcing axioms*. Shortly after the discovery of forcing, it was realized that it is possible to iterate the process of forming generic extensions $V \subset V[g_0] \subset V[g_1] \subset \ldots \subset V[g_\alpha]$ of V in any length α in such a way that the final model is itself a generic extension of V. By "closing off" one may then get to final models which are in fact saturated with respect to the existence of certain (partial) generics in the way prescribed by forcing axioms.

1.2 Forcing axioms.

Forcing axioms are generalizations of the Baire Category Theorem. Formally, they assert the existence of sufficiently generic filters for all members of some reasonably large class of forcing notions. In a general form, given an infinite cardinal κ and a class $\mathcal{K}$ of forcing notions, the forcing axiom $\mathrm{FA}_\kappa(\mathcal{K})$ is the statement that for every $\mathbb{P} \in \mathcal{K}$ and for every collection $\mathcal{D}$ of dense subsets of $\mathbb{P}$ such that $|\mathcal{D}| = \kappa$ there is a filter g of $\mathbb{P}$ which is $\mathcal{D}$-generic (i.e., is such that $g \cap D \neq \emptyset$ for each $D \in \mathcal{D}$). $\mathrm{FA}_\kappa(\mathcal{K})$ is to be seen as a maximality principle with respect to forceability via forcing notions from $\mathcal{K}$: If $\mathrm{FA}_\kappa(\mathcal{K})$ holds, then all Σ_1 statements with parameters in H_{κ^+} that can be forced to hold by some forcing notion in $\mathcal{K}$ already hold in the universe.[1] The answers to questions about H_{κ^+} provided by forcing axioms $\mathrm{FA}_\kappa(\mathcal{K})$ are often

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[1] As a matter of fact, forcing axioms of the form $\mathrm{FA}_\kappa(\mathcal{K})$ can be fully characterized in terms of a suitable form of Σ_1-absoluteness with respect to generic extensions via members from $\mathcal{K}$ (see e.g. [55] or [6, Theorem 1.3]).

regarded as being natural in that $\mathrm{FA}_{\kappa^+}(\mathcal{K})$ offers a uniformly 'saturated' picture of H_{κ^+}, ruling out the type of pathological objects that one can construct when H_{κ^+} has an artificially constrained structure.

In what follows we will consider only forcing axioms $\mathrm{FA}_\kappa(\mathcal{K})$ for $\kappa = \omega_1$.[2] The first such forcing axiom shown to be consistent was Martin's Axiom at ω_1, MA_{ω_1}, see [46] and [34]. MA_{ω_1} is $\mathrm{FA}_{\omega_1}(\mathcal{K})$, where $\mathcal{K}$ is the class of partial orders $\mathbb{P}$ with the countable chain condition (i.e., such that there is no uncountable family of pairwise incompatible conditions in $\mathbb{P}$). Over the following years, a number of generalizations of MA_{ω_1} were isolated. PFA, the Proper Forcing Axiom, is $\mathrm{FA}_{\omega_1}(\mathcal{K})$, where $\mathcal{K}$ is the class of partial orders $\mathbb{P}$ which are proper.

The following is a list of examples of natural statement which are implied by forcing axioms.

- MA_{ω_1} implies that there are no Suslin lines ([46]).
- MA_{ω_1} implies that every union of $\aleph_1$-many Lebesgue null subsets of reals is Lebesgue null ([34]).
- MA_{ω_1} implies the existence of a non-free Whitehead group ([44]).
- PFA implies Baumgartner's Axiom that all $\aleph_1$-dense sets of reals are order-isomorphic (essentially [3]).[3]
- PFA implies Kaplansky's conjecture ([51]).
- PFA implies that there is a 5-element basis for the class of uncountable linear orders ([37]).
- PFA implies that every automorphism of the Calkin algebra of a separable Hilbert space is inner ([10]).

This line of research culminated in the proof by Foreman-Magidor-Shelah of the consistency of *Martin's Maximum*, MM, see [13].

Martin's Maximum is $\mathrm{FA}_{\omega_1}(\mathcal{K})$, where $\mathcal{K}$ is the class of partial orders $\mathbb{P}$ such that forcing with $\mathbb{P}$ preserves the stationarity of all stationary subsets of ω_1 in V. MM is provably maximal in the sense that the forcing axiom $\mathrm{FA}_{\omega_1}(\{\mathbb{P}\})$ fails for any forcing notion $\mathbb{P}$ destroying some stationary subset S of ω_1. At the same time, MM can be

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[2] This is the first κ for which $\mathrm{FA}_\kappa(\mathcal{K})$ does not follow outright from ZFC. Also, $\kappa = \omega_1$ is the only level for which we currently have a reasonably complete picture of the available forcing axioms.

[3] But it does not follow from MA_{ω_1} (s. [1]).

forced by means of a forcing iteration $\mathbb{P} \subset V_\kappa$, assuming that κ is a supercompact cardinal. The natural such forcing $\mathbb{P}$ actually produces a model of a strengthening of MM, called MM^{++}. This is the statement that if $\mathbb{P}$ is a forcing notion preserving stationary subsets of ω_1, $\mathcal{D}$ is a collection of size $\aleph_1$ consisting of dense subsets of $\mathbb{P}$, and $\{\tau_\alpha : \alpha < \omega_1\}$ is a collection of $\mathbb{P}$-names for stationary subsets of ω_1, then there is a filter $g \subset \mathbb{P}$ which is $\mathcal{D}$-generic and which, furthermore, interprets every τ_α, $\alpha < \omega_1$, as a truly stationary set in V (i.e., $\{\nu < \omega_1 : \exists p \in g, p \Vdash_{\mathbb{P}} \check{\nu} \in \tau_\alpha\}$ is stationary for every $\alpha < \omega_1$).

Already MA_{ω_1} contradicts CH, and it even proves Luzin's hypothesis (2), i.e., $2^{\aleph_0} = 2^{\aleph_1}$. More interestingly, MM (in contrast to MA_{ω_1}) decides the cardinality of the continuum, and in fact it confirms Gödel's conjecture (1), $2^{\aleph_0} = \aleph_2$. This is shown by producing an affirmative answer to Friedman's Problem under MM, see [13, Theorems 9 and 10].[4] MM^{++} is—by its very definition and the fact that no strictly stronger forcing axiom can be consistent—a prototype maximality principle for V. Remarkably, the empirical evidence seems to suggest that MM^{++} provides a complete theory of the initial segment H_{ω_2} of the universe of sets, at least with respect to natural questions. Here, H_{ω_2} is the collection of all sets which are hereditarily of size $< \aleph_2$.

1.3 The $\mathbb{P}_{\max}$ axiom $(*)$.

There is another maximality principle, though, which Magidor called a "competitor" of MM, see [33, p. 18], and which is denoted by $(*)$. Its formulation involves the notion of $\mathbb{P}_{\max}$, a forcing which was isolated by W.H. Woodin, see [55, Definition 4.33] and Definition 2.2 below. In much the same way as MM, $(*)$ is inspired by and formulated in the language of forcing, and they both have "the same intuitive motivation: Namely, the universe of sets is rich" ([33, p. 18]). $(*)$, introduced by Woodin in [55, Definition 5.1], is the conjunction of the following two statements.

(i) AD, the Axiom of Determinacy,[5] holds in $L(\mathbb{R})$, and

(ii) there is some g which is $\mathbb{P}_{\max}$-generic over $L(\mathbb{R})$ such that $\mathcal{P}(\omega_1) \subset L(\mathbb{R})[g]$.

Item (i), that AD holds in $L(\mathbb{R})$, follows from the existence of large cardinals, e.g. from the existence of infinitely many Woodin cardinals with a measurable cardinal above them all, see [35, p. 91]. Item (ii) is the part of $(*)$ which goes beyond assuming

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[4] It was later verified by Moore that already the much weaker forcing axiom BPFA implies (1), see [36].

[5] See e.g. [41, Chapter 12].

the existence of large cardinals. $\mathbb{P}_{\max}$ arose out of earlier work by Steel-Van Wesep [49] and by Woodin [54] on the size of $\boldsymbol{\delta}^1_2$ and the question if NS_{ω_1}, the nonstationary ideal on ω_1, can be saturated. Here, NS_{ω_1} is called saturated iff there is no collection $\mathcal{A}$ of stationary subsets of ω_1 of size $\aleph_2$ such that $S \cap T$ is nonstationary for all S, $T \in \mathcal{A}$, $S \neq T$.

$\mathbb{P}_{\max}$ consists of countable transitive structures, membership in $\mathbb{P}_{\max}$ is uniformly Π^1_2 in the codes, and the order $<_{\mathbb{P}_{\max}}$ is arithmetical. $\mathbb{P}_{\max}$ is ω-closed and homogeneous, see [55, Lemma 4.43]. The fact that a forcing $\mathbb{P}$ is homogeneous means that the validity in the forcing extension of a given statement is decided in the ground model by the trivial condition in $\mathbb{P}$. The homogeneity of $\mathbb{P}_{\max}$ then yields that under $(*)$, the theory of $L(\mathcal{P}(\omega_1))$ becomes part of the theory of $L(\mathbb{R})$ in the sense that if φ is any sentence, then

$$L(\mathcal{P}(\omega_1)) \vDash \varphi \text{ if and only if } \Vdash^{\mathbb{P}_{\max}}_{L(\mathbb{R})} \varphi. \tag{3}$$

If AD holds in $L(\mathbb{R})$, then there is no well-order of the reals in $L(\mathbb{R})$ (see e.g. [41, Lemma 12.2]), but if g is $\mathbb{P}_{\max}$-generic over $L(\mathbb{R})$, then ZFC is true in $L(\mathbb{R})[g]$ (see [55, Theorem 4.54]), and moreover NS_{ω_1} is saturated in $L(\mathbb{R})[g]$ (see [55, Theorem 4.50]) and $L(\mathbb{R})[g]$ provides an effective failure of CH in that $\boldsymbol{\delta}^1_2 = \omega_2$ is true in $L(\mathbb{R})[g]$ (see [55, Theorem 4.53]).

Like the "classical" forcing axioms culminating with MM^{++}, $(*)$ is also a maximality principle. While $(*)$ implies none of the stronger forcing axioms, see e.g. [40, Theorem 1.3], it does imply MA_{ω_1}. In particular, $(*)$ implies the first three implications of MA_{ω_1} which are listed on p. 4. As it turns out, $(*)$ also implies that every automorphism of the Calkin algebra of a separable Hilbert space is inner, see [27], [10]; and, at least in conjunction with the existence of a Woodin cardinal,[6] it also implies Baumgartner's Axiom on $\aleph_1$-dense sets of reals, as this can be expressed by a Π_2 sentence over H_{ω_2}, as well as the existence of a 5-element basis for the uncountable linear orders (since, in the presence of Baumgartner's Axiom and MA_{ω_1}, the existence of such a basis follows from every Aronszajn line containing a Countryman line, see [37], which again can be expressed by a Π_2 sentence over H_{ω_2}).

$(*)$ implies (and is in fact equivalent to) what is dubbed "Π_2 maximality." A sentence σ (in the language of set theory, possibly augmented with some additional predicates) is said to be Π_2 if it is of the form $\forall x \exists y \varphi(x, y)$, with $\varphi(x, y)$ being a formula with only restricted quantifiers. There is a whole family of interesting statements which are Π_2 in the language for the structure

$$(H_{\omega_2}; \in, \mathrm{NS}_{\omega_1}),$$

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[6] This is by the proof of Theorem 1.1.

see e.g. the discussion in [8]. The formulation of "Π_2 maximality" involves the concept of Ω-logic, see [55, Section 10.4]; for a sentence σ to be "Ω-consistent" is stronger than it just being consistent in that σ needs to be true in models which are closed under arbitrarily complicated universally Baire operations, see [55, Definition 10.144]. The Π_2 maximality theorem, see [55, Theorem 10.150], then runs as follows.

Theorem 1.1 (Woodin) *Suppose there is a proper class of Woodin cardinals. Then the following statements are equivalent.*

(1) $(*)$.

(2) Let σ *be a* Π_2 *sentence in the language for the structure*

$$(H_{\omega_2}; \in, \mathrm{NS}_{\omega_1}, A\colon A \in \mathcal{P}(\mathbb{R}) \cap L(\mathbb{R})).$$

If σ *is* Ω*-consistent, then* σ *is true.*

One specific instance of σ in (2) of Theorem 1.1 is called ψ_{AC}, see [55, Definition 5.12]. It is in spirit a local version of an affirmative solution to Friedman's Problem. Woodin showed, see [55, Theorem 5.14, Lemmata 5.15 and 5.18], that ψ_{AC} follows from both MM and $(*)$ and that ψ_{AC} implies Gödel's conjecture (1), i.e. $2^{\aleph_0} = \aleph_2$.

The homogeneity of $\mathbb{P}_{\max}$ gives that in the presence of large cardinals, $(*)$ yields a complete theory for $L(\mathcal{P}(\omega_1))$ modulo set-forcing: by (3), all set-generic extensions of V in which $(*)$ holds true agree on the theory of $L(\mathcal{P}(\omega_1))$.

Despite its nice properties, in order for $(*)$ to be a convincing candidate for a natural axiom, it would have to be compatible with all consistent large cardinal axioms. While $L(\mathbb{R})[g]$ is trivially a model of $(*)$, provided that g is $\mathbb{P}_{\max}$-generic over $L(\mathbb{R})$, Scott's result [43] carries over from L to $L(\mathbb{R})[g]$ and shows that $L(\mathbb{R})[g]$ cannot have measurable cardinals either. [55] and subsequent work left open the problem whether $(*)$ would be compatible with large cardinals beyond the level of Woodin cardinals.

1.4 Unifying forcing axioms and $(*)$.

Prior to the current paper, the relation between classical forcing axioms like MM, which could be forced by iterated forcing over models of ZFC with large cardinals, and the axiom $(*)$, whose models were obtained by forcing over models satisfying the Axiom of Determinacy, remained a complete mystery. It had been known by a result of P. Larson [26] that even $\mathsf{MM}^{+\omega}$, an axiom strictly between MM and MM^{++}, does not imply $(*)$.[7] One can build models of $\mathsf{MM}^{+\omega}$ with a well-order of H_{ω_2} which is

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[7] $\mathsf{MM}^{+\omega}$ is the strengthening of MM obtained by replacing, in the formulation of MM^{++}, collections of $\aleph_1$ many names for stationary sets with collections of only countably many such names.

definable over $(H_{\omega_2}; \in)$ by a formula without parameters, and the existence of such a well-order is incompatible with $(*)$ by the homogeneity of $\mathbb{P}_{\max}$. It remained even unclear whether classical strong forcing axioms would be compatible at all with $(*)$, see [55, p. 846]. See also [55, Question (18) a) on p. 924], [33, Conjecture 6.8 on p. 19], and [38, Problem 14.7].

This paper resolves the tension between MM and $(*)$. We prove:

Theorem 1.2 *Assume* Martin's Maximum^{++}. *Then Woodin's* $\mathbb{P}_{\max}$*-axiom* $(*)$ *holds true.*

In particular, MM and $(*)$ are compatible with one another, and $(*)$ is compatible with all consistent large cardinal axioms: If κ is a supercompact cardinal and $\mathbb{P} \subset V_\kappa$ is the partial order from [13] to force MM^{++}, then by Theorem 1.2 the axiom $(*)$ holds in $V^{\mathbb{P}}$, and all the large cardinals of V above κ are preserved by $\mathbb{P}$.

Theorem 1.2 renders MM^{++} a particularly appealing axiom. Not only is MM^{++} a provably maximal forcing axiom providing the 'right' answers to questions pertaining to H_{ω_2},[8] but it follows from Theorem 1.2 that MM^{++} implies the form of Π_2 maximality for arbitrary set-forcing given by (2) of Theorem 1.1 and, moreover, that MM^{++} completely decides the theory of $L(\mathcal{P}(\omega_1))$ via set-forcing.

It also follows from Theorem 1.2 that $(*)$ can be characterized, in the presence of large cardinals, by a statement which on the face of its formulation is weaker than (2) of Theorem 1.1. We will prove the following theorem at the end of the next section.

Theorem 1.3 *Suppose there is a supercompact cardinal. Then the following statements are equivalent.*

(1) $(*)$.

(2) Let σ *be a* Π_2 *sentence in the language for the structure*

$$(H_{\omega_2}; \in, \mathrm{NS}_{\omega_1}, A \colon A \in \mathcal{P}(\mathbb{R}) \cap L(\mathbb{R})).$$

If there is a stationary set preserving forcing $\mathbb{P}$ *such that* σ *holds in* $V^{\mathbb{P}}$*, then* σ *is true in* V*.*

This equivalence of $(*)$ is a variant of one which we are going to state below, see Theorem 2.17, and which characterizes $(*)$ as a strong version of a bounded forcing axiom.

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[8] The 'right' answers from a conception of the universe as being uniformly saturated with respect to forcing.

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The reader should have some acquaintance with forcing, determinacy, and universally Baire sets of reals. The relevant material is covered e.g. in [41, Chap. 6 and sections 7.1, 8.1, and 12.1]. Familiarity with stationary set preserving forcings and Martin's Maximum, see e.g. [13] or [19, Chap. 37], and with $\mathbb{P}_{\max}$ forcing to the extent of say [55, Chap. 4] or [30, Sections 1-6] would be desirable. Knowledge of forcings which are similar to the one which will be designed here and which were developed earlier e.g. in [20], [5], [8], or [9] is by no means required or presupposed.

2 Preliminaries.

Let us first state again *Martin's Maximum*$^{++}$, abbreviated by MM^{++} and isolated by Foreman-Magidor-Shelah [13] (cf. also [55, Definition 2.45 (2)]).

Definition 2.1 MM^{++} *is the statement that if* $\mathbb{P}$ *is a forcing which preserves stationary subsets of* ω_1*, if* $\{D_i\colon i < \omega_1\}$ *is a collection of dense subsets of* $\mathbb{P}$*, and if* $\{\tau_i\colon i < \omega_1\}$ *is a collection of* $\mathbb{P}$*-names for stationary subsets of* ω_1*, then there is a filter* $g \subset \mathbb{P}$ *such that for every* $i < \omega_1$*,*

(i) $g \cap D_i \neq \emptyset$ *and*

(ii) $(\tau_i)^g = \{\xi < \omega_1\colon \exists p \in g\, p \Vdash_{\mathbb{P}} \check{\xi} \in \tau_i\}$ *is stationary.*

The forcing $\mathbb{P}_{\max}$ was designed by W. Hugh Woodin, see [55, Chapter 4], specifically [55, Definition 4.33].

In order to define $\mathbb{P}_{\max}$, we need the notion of "generic iterability" of structues of the form $(M; \in, I, a)$, where M is a transitive model of a sufficiently large fragment of ZFC, $(M; I)$ is amenable,[9] $a \subset \omega_1^M$, and $(M; I) \models$ "I is a normal uniform ideal on ω_1." Given an ordinal $\gamma \leq \omega_1$, $\langle\langle(M_i; \in, I_i, a_i)\colon i \leq \gamma\rangle, \langle\pi_{i,j}\colon i \leq j \leq \gamma\rangle, \langle g_i\colon i < \gamma\rangle\rangle$ is a generic iteration of $(M; \in, I, a)$ if the following hold true.

- $(M_0; \in, I_0, a_0) = (M; \in, I, a)$,
- for $i < \gamma$, g_i is a $\mathcal{P}(\omega_1)^{M_\alpha} \setminus I_i$-generic filter over M_i, M_{i+1} is the ultrapower of M_i by g_i, and $\pi_{i,i+1}\colon (M_i; \in, I_i, a_i) \to (M_{i+1}; \in, I_{i+1}, a_{i+1})$ is the corresponding generic elementary embedding,

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[9] I.e., $x \cap I \in M$ for all $x \in M$.

- $\pi_{i,k} = \pi_{j,k} \circ \pi_{i,j}$ for all $i \leq j \leq k$, and
- if β is a nonzero limit ordinal $\leq \gamma$, then $(M_\beta, (\pi_{i,\beta} \colon i < \beta))$ is the direct limit of $(M_i, \pi_{i,j} \colon i \leq j < \beta)$.

$(M; \in, I, a)$ being generically iterable means that all models in any generic iteration of $(M; \in, I, a)$ are well-founded, irrespective of the filters g_α chosen at any stage α, see [55, Definition 4.1].

Let us stress that the current paper will only consider such generic iterations rather than iterations of mice as being studied in inner model theory.

Definition 2.2 *The conditions in* $\mathbb{P}_{\max}$ *are countable transitive models of a sufficiently large fragment of* ZFC *plus* MA_{ω_1} *of the form* $(M; \in, I, a)$, *where*

(i) $(M; I)$ *is amenable and* $(M; I) \models$ *"I is a normal uniform ideal on ω_1,"*

(ii) $a \in \mathcal{P}(\omega_1^M) \cap M$ *and* $M \models$ *"$\omega_1 = \omega_1^{L[a,x]}$ for some real x," and*

(iii) $(M; \in, I)$ *is generically iterable.*

We construe $\mathbb{P}_{\max}$ *as a partial order by declaring that* $(N; \in, J, b)$ *is stronger than* $(M; \in, I, a)$, *denoted by* $(N; \in, J, b) < (M; \in, I, a)$, *if and only if* $(M; \in, I, a) \in N$ *and inside* N *there is a generic iteration of* $(M; \in, I, a)$ *of length* $\omega_1^N + 1$ *with last model* $(M^*; \in, I^*, a^*)$ *such that* $I^* = J \cap M^*$ *and* $a^* = b$.[10]

Most of [55] studies the effect of forcing with $\mathbb{P}_{\max}$ or variants thereof over a model of the Axiom of Determinacy. Let us state again Woodin's $\mathbb{P}_{\max}$ axiom $(*)$, see [55, Definition 5.1].

Definition 2.3 $(*)$ *says that*

(i) AD *holds in* $L(\mathbb{R})$ *and*

(ii) there is some g which is $\mathbb{P}_{\max}$*-generic over* $L(\mathbb{R})$ *such that* $\mathcal{P}(\omega_1) \subset L(\mathbb{R})[g]$.

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[10] $\mathbb{P}_{\max}$ according to our Definition 2.2 is a slightly bigger poset than the one according to [55, Definition 4.33]. The difference is that we weakened the requirement $I \in M$ of [55, Definition 4.33] to "$(M; I)$ is amenable." This natural move will make $(H_{\omega_2}; \in, \mathsf{NS}_{\omega_1}, A)$ for any $A \subset \omega_1$ a $\mathbb{P}_{\max}$ condition in a generic extension where H_{ω_2} is countable, cf. (11). It is easy to see that $\mathbb{P}_{\max}$ according to [55, Definition 4.33] is dense in $\mathbb{P}_{\max}$ according to our Definition 2.2, so that both forcing notions are forcing-equivalent.

Already the *Proper Forcing Axiom*, PFA, which is much weaker than MM^{++}, implies $\mathsf{AD}^{L(\mathbb{R})}$ and much more, see [47], [21], and [39, Chapter 12].

The current paper produces a proof of Theorem 1.2. Our key new idea is (Σ.8) on page 28 below. We try to give an overview of the proof of Theorem 1.2 at the end of this section.

Theorem 1.2 is optimal in that P. Larson [26] and [29] has shown that $\mathsf{MM}^{+\omega}$ is consistent with $\neg(*)$ relative to a supercompact limit of supercompact cardinals. Our proof is also optimal in that the forcing which we will use to verify Theorem 1.2 has size $2^{\aleph_2}$, while Woodin has shown that MM^{++} for forcings of size $2^{\aleph_0} = \aleph_2$ does not imply $(*)$, see [55, Theorem 10.90], and it is consistent with MM^{++} that $2^{\aleph_2} = \aleph_3$.

Throughout our entire paper, "ω_1" will *always* denote ω_1^V, the ω_1 of V. We shall also make permanent use of the following.

Convention 2.4 *Let us fix throughout this paper some $A \subset \omega_1$ such that $\omega_1^{L[A]} = \omega_1$. Let us define g_A as the set of all $\mathbb{P}_{\max}$ conditions $p = (N; \in, I, a)$ such that there is a generic iteration*

$$(N_i, \sigma_{ij} \colon i \leq j \leq \omega_1)$$

of $p = N_0$ of length $\omega_1 + 1$ such that if we write $N_{\omega_1} = (N_{\omega_1}; \in, I^*, a^*)$,[11] *then $I^* = (\mathsf{NS}_{\omega_1})^V \cap N_{\omega_1}$ and $a^* = A$.*

In the following statement, $X^\#$ denotes the sharp of X. While the formal definition of a sharp (see e.g. [41, Section 10.2]) won't play any role in what follows, the reader may think of "$\mathcal{P}(\omega_1)^\#$ exists" as just some extra large cardinal structure which is assumed to be present in the universe.

We are going to use now the concept of elementary substructures. For any two models $\mathcal{M}$ and $\mathcal{N}$ with underlying universes M and N, respectively, and with the same first order language associated to them, $\mathcal{M} \prec \mathcal{N}$ means that $\mathcal{M}$ is an elementary substructure of $\mathcal{N}$, i.e., $M \subset N$ and for all formulae φ of that common language and all $x_1, \ldots, x_k \in M$,

$$\mathcal{M} \models \varphi(x_1, \ldots, x_k) \Longleftrightarrow \mathcal{N} \models \varphi(x_1, \ldots, x_k). \tag{4}$$

Lemma 2.5 (Woodin) *Assume MA_{ω_1}, that NS_{ω_1} is saturated, and that $\mathcal{P}(\omega_1)^\#$ exists. In the notation of Convention 2.4:*

(1) g_A is a filter.

---

[11] Here and elsewhere we often confuse a model with its underlying universe.

(2) If g_A is $\mathbb{P}_{\max}$-generic over $L(\mathbb{R})$, then $\mathcal{P}(\omega_1) \subset L(\mathbb{R})[g]$.

Proof. This routinely follows from the proof of [55, Lemma 3.12 and Corollary 3.13] and from [55, Lemmas 3.10 and 3.14]. Let us sketch the argument. Let us first state the following.

Claim 2.6 *Let $\theta \geq (2^{\aleph_1})^+$ be a cardinal, let $X \prec H_\theta$ be countable with $A \in X$, and let $\sigma \colon M \cong X$ be such that M is transitive. Then*

(a) $\sigma^{-1}((H_{\omega_2}; \in, \mathsf{NS}_{\omega_1}, A))$ is a $\mathbb{P}_{\max}$-condition.

(b) $\{X \in \mathcal{P}(\omega_1^M) \cap M \colon \omega_1 \in \sigma(X)\}$ is $(\mathcal{P}(\omega_1^M) \cap M) \setminus \sigma^{-1}(\mathsf{NS})$-generic over M.

(c) For $i \leq \omega_1$ let

$$X_i = \mathrm{Hull}^{H_\theta}(X \cup \sup\{X_j \colon j < i\}),$$

let $\sigma_i \colon M_i \cong X_i$ be such that M_i is transitive, and let, for $i \leq j \leq \omega_1$, $\pi_{ij} = \sigma_j^{-1} \circ \sigma_i$. Then $(M_i, \pi_{ij} \colon i \leq j \leq \omega_1)$ is a generic iteration of

$$(M; \in, \sigma^{-1}(\mathsf{NS}_{\omega_1}), A \cap \omega_1^M)$$

(d) $\sigma^{-1}((H_{\omega_2}; \in, \mathsf{NS}_{\omega_1}, A)) \in g_A$.

Proof of Claim 2.6. (a): This is by [55, Lemmas 3.10 and 3.14]. (b) and (c): This is by [55, Lemma 3.12 and Corollary 3.13]. (d): This follows immediately from (a) and (c). □ (Claim 2.6)

Let us now prove Lemma 2.5.

(1): Let $\mathcal{N}_0 = (N; \in, I, a) \in g_A$ as being witnessed by the generic iteration $(\mathcal{N}_i, \sigma_{ij} \colon i \leq j \leq \omega_1)$. Let $\mathcal{M}_0 > \mathcal{N}_0$ as being witnessed by the generic iteration $(\mathcal{M}_i, \pi_{ij} \colon i \leq j \leq \omega_1^{\mathcal{N}_0}) \in N$. Then $\sigma_{0\omega_1}((\mathcal{M}_i, \pi_{ij} \colon i \leq j \leq \omega_1^{\mathcal{N}_0}))$ is easily seen to be a generic iteration of $\mathcal{M}_0$ which witnesses that $\mathcal{M}_0 \in g_A$.

Now let $\mathcal{N}_0^0 = (N^0; \in, I^0, a^0) \in g_A$ and $\mathcal{N}_0^1 = (N^1; \in, I^1, a^1) \in g_A$ as witnessed by the generic iterations $\mathcal{I}^0 = (\mathcal{N}_i^0, \sigma_{ij}^0 \colon i \leq j \leq \omega_1)$ and $\mathcal{I}^1 = (\mathcal{N}_i^1, \sigma_{ij}^1 \colon i \leq j \leq \omega_1)$. Let $\sigma \colon M \cong X$ be as in Claim 2.6 with $\{\mathcal{I}^0, \mathcal{I}^1\} \subset X$. Then $\mathcal{N}^0$ and $\mathcal{N}^1$ are both weaker than $\sigma^{-1}((H_{\omega_2}; \in, \mathsf{NS}_{\omega_1}, A)) \in \mathbb{P}_{\max}$, cf. Claim 2.6 (a), as witnessed by the generic iterations $\sigma^{-1}(\mathcal{I}^0) = (\mathcal{N}_i^0, \sigma_{ij}^0 \colon i \leq j \leq \omega_1^M)$ and $\sigma^{-1}(\mathcal{I}^1) = (\mathcal{N}_i^1, \sigma_{ij}^1 \colon i \leq j \leq \omega_1^M)$.

(2): Let $Z \in \mathcal{P}(\omega_1)$. Let $\sigma \colon M \cong X$ be as in Claim 2.6 with $Z \in X$. Then $\sigma^{-1}((H_{\omega_2}; \in, \mathsf{NS}_{\omega_1}, A)) \in g_A$ by Claim 2.6 (d) and in fact if $(M_i, \pi_{ij} \colon i \leq j \leq \omega_1)$ is as in Claim 2.6 (c), then $Z \in M_{\omega_1}$, so that trivially Z is also in the last iterate of $\sigma^{-1}((H_{\omega_2}; \in, \mathsf{NS}_{\omega_1}, A))$ via the generic iteration which is the restriction of $(M_i, \pi_{ij} \colon i \leq j \leq \omega_1)$ to $\sigma^{-1}((H_{\omega_2}; \in, \mathsf{NS}_{\omega_1}, A))$ and its images. □ (Lemma 2.5)

Let $1 \leq k < \omega$, and let $D \in \mathcal{P}(\mathbb{R}^k)$. We say that T is a tree on ${}^k\omega \times \mathrm{OR}$ iff $T \subset \bigcup_{n<\omega} ({}^n\omega)^k \times {}^n\mathrm{OR}$ and if $(s_0, ..., s_{k-1}, t) \in T$ and $m < \omega$, then

$$(s_0 \restriction m, ..., s_{k-1} \restriction m, t \restriction m) \in T$$

We write

$$[T] = \{(x_0, ..., x_{k-1}, f) \colon \forall m < \omega \, (x_0 \restriction m, ..., x_{k-1} \restriction m, f \restriction m) \in T\}$$

and $p[T]$ for the projection of T, i.e.,

$$p[T] = \{(x_0, ..., x_{k-1}) \colon \exists f \, (x_0, ..., x_{k-1}, f) \in [T]\}$$

Definition 2.7 The trees T and U on ${}^k\omega \times \mathrm{OR}$ witness that D *is* universally Baire *iff $D = p[T]$ and for all posets $\mathbb{P}$,*

$$\Vdash_{\mathbb{P}} \; p[U] = \mathbb{R}^k \setminus p[T]. \tag{5}$$

D is called universally Baire *iff there are trees T and U witnessing that D is universally Baire.*

We denote by Γ^∞ the collection of all $D \in \bigcup_{1 \leq k < \omega} \mathcal{P}(\mathbb{R}^k)$ which are universally Baire.

The concept of universally Baire set was isolated by Feng-Magidor-Woodin in [12, Section 2]; see also [41, Definition 8.6].

If $D \in \Gamma^\infty$, then there is an unambiguous version of D in any forcing extension $V[g]$ of V, which as usual we denote by D^* and which is equal to $p[T] \cap V[g]$ for some/all trees T and U which witness that D is universally Baire. See [41, p. 149f.].

We will call a pointclass consisting of universally Baire sets productive iff it is closed under complements and projections in a strong sense and for all $k < \omega$ and $D \in \Gamma \cap \mathbb{R}^{k+2}$,

$$(\exists^{\mathbb{R}} D)^* = \{\vec{x} \in \mathbb{R}^{k+1} \colon \exists y \in \mathbb{R} \, (\vec{x}, y) \in D^*\} \tag{6}$$

will be true in every generic extension. The formal definition runs as follows.

Definition 2.8 *Let $\Gamma \subset \bigcup_{1 \leq k < \omega} \mathcal{P}(\mathbb{R}^k)$. We say that Γ is* productive *iff*

- *(a)* $\Gamma \subset \Gamma^\infty$,
- *(b) for all $k < \omega$ and all $D \in \Gamma \cap \mathcal{P}(\mathbb{R}^{k+1})$, $\mathbb{R}^{k+1} \setminus D \in \Gamma$ and if $k > 0$, then $\exists^{\mathbb{R}} D = \{(x_0, \ldots, x_{k-1}) \colon \exists x_k (x_0, \ldots, x_{k-1}, x_k) \in D\} \in \Gamma$, and*

(c) *for all* $k < \omega$ *and all* $D \in \Gamma \cap \mathcal{P}(\mathbb{R}^{k+2})$*, if the trees* T *and* U *on* ${}^{k+2}\omega \times \mathrm{OR}$ *witness that* D *is universally Baire and if*

$$\tilde{U} = \{(s \restriction (k+1), (s(k+1), t)) \colon (s,t) \in U\}, \tag{7}$$

then there is a tree $\tilde{T}$ *on* ${}^{k+1}\omega \times \mathrm{OR}$ *such that for all posets* $\mathbb{P}$*,*

$$\Vdash_{\mathbb{P}} p[\tilde{U}] = \mathbb{R}^{k+1} \setminus p[\tilde{T}]. \tag{8}$$

While (c) of Definition 2.8 canonically ensures that every productive pointclass is closed under projections, at least on the face of its definition, Γ being productive is stronger than having that $\Gamma \subset \Gamma^\infty$ and Γ is closed under complements and projections ([12, Question 3] exactly asks if the former is really stronger than the latter).

Lemma 2.9 *If* Γ *is productive and if* $D \in \Gamma$*, then any projective statement about* D *is absolute between* V *and any forcing extension of* V*, i.e., if* φ *is projective,* $x_1, \ldots, x_k \in \mathbb{R}$*, and* $\mathbb{P}$ *is any poset, then*

$$V \vDash \varphi(x_1, \ldots, x_k, D) \iff \Vdash_{\mathbb{P}} \varphi(\check{x}_1, \ldots, \check{x}_k, D^*).$$

Lemma 2.9 is shown by a trivial induction on the complexity of φ.

Let $e \colon \mathbb{R} \to \mathrm{HC}$ be a fixed simple coding of hereditarily countable sets by reals, see e.g. [42, p. 179]. A set $D \subset \mathrm{HC}$ is then called *universally Baire in the codes* iff the code set $\{x \in \mathbb{R} \colon e(x) \in D\}$ of D is universally Baire. If this is the case, then every forcing extension of V will have its unique new version of D, which we denote by D^*. If the code set of D is a member of a productive pointclass, then for every forcing $\mathbb{P}$,

$$(\mathrm{HC}; \in, D) \prec (\mathrm{HC}^{V^{\mathbb{P}}}; \in, D^*). \tag{9}$$

A classical variant of Lemma 2.9 is Shoenfield's absoluteness theorem, see e.g. [41, Corollary 7.21]. It states that if $M \subset N$ are both transitive models of a sufficiently rich fragment of ZFC such that $\omega_1^V \subset M$, then

$$(\mathrm{HC}^M; \in) \prec_{\underset{\sim}{\Sigma}_1} (\mathrm{HC}^N; \in), \tag{10}$$

where (10) means that (4) holds true with φ restricted to Σ_1 formulae (and HC^M, HC^N playing the roles of $\mathcal{M}$, $\mathcal{N}$, respectively).

[12, Question 3] is concerned with the question about the connection of, on the one hand, projective absoluteness with respect to forcing extensions and, on the other hand, having that every projective set is universally Baire (see [12, Questions 1 and 7]).

Theorem 2.10 (Woodin) *Assume that there is a proper class of Woodin cardinals. Then* Γ^∞ *is productive.*

PROOF. A theorem of Woodin says that in the presence of a proper class of Woodin cardinals, every set in Γ^∞ is weakly homogeneously Suslin, see e.g. [28, Theorem 3.3.8] and [48, Theorem 1.2]. Every tree $\tilde{U}$ witnessing that a given set D of reals is weakly homogeneously Suslin comes with a canonical tree $\tilde{T}$ for $\mathbb{R} \setminus D$ in such a way that $\tilde{U}$ and $\tilde{T}$ are connected as in (c) of Definition 2.8. For the construction of $\tilde{T}$ see e.g. [22, p. 455]. [22, Proposition 32.6] formulates how the two trees are connected. The main result of Martin and Steel from [35] is then that Woodin cardinals may be used to show that $\tilde{T}$ is homogeneous, cf. [22, Theorem 32.11]. That way, it follows that Γ^∞ is productive provided that there is a proper class of Woodin cardinals. □ (Theorem 2.10)

For any set X, $M_\omega^\#(X)$ denotes the least active X-mouse which has infinitely many Woodin cardinals. See [50].

Theorem 2.11 (Steel) *Assume* PFA. *Then the universe is closed under the operation* $X \mapsto M_\omega^\#(X)$. *In particular, every set of reals in* $L(\mathbb{R})$ *is universally Baire, and* $\bigcup_{k<\omega} \mathcal{P}(\mathbb{R}^k) \cap L(\mathbb{R})$ *is productive.*

PROOF. The proof from [47] produces the result that under PFA, the universe is closed under the operation $X \mapsto M_\omega^\#(X)$. The rest is given by standard inner model theoretic arguments, see e.g. [42, Section 3, pp. 187f.]. □ (Theorem 2.11)

By Lemma 2.5 and Theorem 2.11, Theorem 1.2 follows from the following more general statement.

Theorem 2.12 *Let* $\Gamma \subset \bigcup_{1 \le k<\omega} \mathcal{P}(\mathbb{R}^k)$. *Assume that*

(i) $\Gamma = \bigcup_{1 \le k<\omega} \mathcal{P}(\mathbb{R}^k) \cap L(\Gamma, \mathbb{R})$,

(ii) Γ *is productive, and*

(iii) Martin's Maximum^{++} *holds true.*

Then g_A *is* $\mathbb{P}_{\max}$*-generic over* $L(\Gamma, \mathbb{R})$.

The abbreviation $(*)_\Gamma$ was introduced in [42, Definition 4.1] to denote a straightforward generalization of $(*)$ to larger pointclasses. For a pointclass $\Gamma \supset \mathcal{P}(\mathbb{R}) \cap L(\mathbb{R})$, $(*)_\Gamma$ is the statement that every set in Γ is determined, and there is a filter $g \subset \mathbb{P}_{\max}$ which has nonempty intersection with every dense set (coded by a set) in Γ and is such that $\mathcal{P}(\omega_1) \subset L(\mathbb{R})[g]$.

Corollary 2.13 *Assume that there is a proper class of Woodin cardinals. Let* $\Gamma \subset \bigcup_{1\leq k<\omega} \mathcal{P}(\mathbb{R}^k) \cap \Gamma^\infty$. *Suppose that (i)–(iii) from the statement of Theorem 2.12 are satisfied. Then* $(*)_\Gamma$ *holds true.*

Theorem 2.12 readily follows from the following Lemma via a standard application of MM^{++}.

Lemma 2.14 *Let* $\Gamma \subset \bigcup_{1\leq k<\omega} \mathcal{P}(\mathbb{R}^k)$. *Assume that*

(i) $\Gamma = \bigcup_{1\leq k<\omega} \mathcal{P}(\mathbb{R}^k) \cap L(\Gamma, \mathbb{R})$,

(ii) Γ *is productive, and*

(iii) NS_{ω_1} *is saturated.*

Let $D \subset \mathbb{P}_{\max}$ *be open dense,* $D \in L(\Gamma, \mathbb{R})$. *With* A *being as in Convention 2.4, there is then a stationary set preserving forcing* $\mathbb{P}$ *of size* $2^{\aleph_2}$ *such that in* $V^{\mathbb{P}}$ *there is some* $p = \mathcal{N}_0 = (N; \in, I, a) \in D^*$ *and some generic iteration*

$$(\mathcal{N}_i, \sigma_{ij} : i \leq j \leq \omega_1)$$

of $p = \mathcal{N}_0$ *of length* $\omega_1 + 1$ *such that if we write* $\mathcal{N}_{\omega_1} = (N_{\omega_1}; \in, I^*, a^*)$, *then* $I^* = (\mathsf{NS}_{\omega_1})^{V^{\mathbb{P}}} \cap N_{\omega_1}$ *and* $a^* = A$.

Proof of Theorem 2.12 from Lemma 2.14. MM implies that NS_{ω_1} is saturated, see [13, Theorem 12]. By Lemma 2.5, it remains to show that $D \cap g_A \neq \emptyset$ for every open dense $D \subset \mathbb{P}_{\max}$, $D \in \Gamma$. Here, g_A is as in Convention 2.4.

Let us fix such D. The statement that there is a p as in the conclusion of Lemma 2.14, which is tantamount to saying that there is a $p \in D \cap g_A$, is easily seen to be Σ_1 expressible over the structure $(H_{\omega_2}; \in, \mathsf{NS}_{\omega_1}, A, D)$. By the conclusion of Lemma 2.14, the existence of such a p may be forced by a stationary set preserving forcing. Hence by MM^{++}, cf. [55, Theorem 10.124], there is such a p in V. $\square$ (Theorem 2.12)

As the proof of Theorem 2.12 from Lemma 2.14 shows, we don't need the full power of MM^{++} in order to derive Theorem 2.12 from Lemma 2.14. Instead, a bounded version of MM^{++} suffices; it may be defined as follows, see [55, Definition 10.123].

Definition 2.15 *For* $D \in \Gamma^\infty$, D-BMM^{++} *is the statement that for all* $\mathbb{P}$ *which are stationary set preserving,*

$$(H^V_{\omega_2}; \in, \mathsf{NS}^V_{\omega_1}, D) \prec_{\Sigma_1} (H^{V^{\mathbb{P}}}_{\omega_2}; \in, \mathsf{NS}^{V^{\mathbb{P}}}_{\omega_1}, D^*).$$

Modulo large cardinals, $(*)$ is then actually *equivalent* to D-BMM^{++} for all $D \in \mathcal{P}(\mathbb{R}) \cap L(\mathbb{R})$. Let us first state a more general fact, Theorem 2.16, which gives the characterization of $(*)$, i.e. Theorem 2.17, as a special case.

Theorem 2.16 *Assume that there is a proper class of Woodin cardinals. Let* $\Gamma \subset \bigcup_{1 \leq k < \omega} \mathcal{P}(\mathbb{R}^k)$*. Assume that*

(i) $\Gamma = \bigcup_{1 \leq k < \omega} \mathcal{P}(\mathbb{R}^k) \cap L(\Gamma, \mathbb{R})$,

(ii) Γ *is productive.*

The following statements are then equivalent, with g_A *being as in Convention 2.4.*

(1) D-BMM^{++} *holds true for all* $D \in \Gamma$.

(2) g_A *is* $\mathbb{P}_{\max}$*-generic over* $L(\Gamma, \mathbb{R})$.

Proof. (2) $\Longrightarrow$ (1): This is exactly by the proof of (A) $\Longrightarrow$ (B) of [2, Theorem 2.7].

(1) $\Longrightarrow$ (2): We may first force NS_{ω_1} to be saturated by a stationary set preserving forcing, see e.g. [55, Theorem 2.64]. The rest is then by the proof of Theorem 2.12 from Lemma 2.14 which was given above. □ (Theorem 2.16)

Theorem 2.17 *Assume that there is a proper class of Woodin cardinals. The following statements are then equivalent.*

(1) D-BMM^{++} *holds true for all* $D \in \mathcal{P}(\mathbb{R}) \cap L(\mathbb{R})$.

(2) $(*)$.

Let us now give a Proof of Theorem 1.3 from Lemma 2.14. (1) $\Longrightarrow$ (2) is weaker than (1) $\Longrightarrow$ (2) of Theorem 1.1. Let us now assume (2) and show (1). Fix $D \subset \mathbb{P}_{\max}$, any open dense set in $L(\mathbb{R})$. As the statement of the theorem assumes a supercompact cardinal to exist, there is a semi-proper (and hence stationary set preserving) forcing $\mathbb{P}$ such that MM^{++} holds true in $V^{\mathbb{P}}$. Inside $V^{\mathbb{P}}$, we will have that MM^{++} yields via Lemma 2.14 that for all $A' \subset \omega_1$ with $\omega_1^{L[A']} = \omega_1$ there will be some $p = (N; \in, I^*, a^*) \in D^*$ and some generic iteration

$$(\mathcal{N}_i, \sigma_{ij} \colon i \leq j \leq \omega_1)$$

of $p = \mathcal{N}_0$ of length $\omega_1 + 1$ such that if we write $\mathcal{N}_{\omega_1} = (N_{\omega_1}; \in, I^*, a^*)$, then $I^* = (\mathsf{NS}_{\omega_1})^{V^{\mathbb{P}}} \cap N_{\omega_1}$ and $a^* = A'$. This is a statement which is Π_2 over the structure mentioned in (2) of Theorem 1.3. This statement will therefore be true in V, which readily implies that g_A is $\mathbb{P}_{\max}$-generic over $L(\mathbb{R})$ and $\mathcal{P}(\omega_1) \subset L(\mathbb{R})[g_A]$, where g_A is as in Convention 2.4. $\square$ (Theorem 1.3)

The forcing which we designed in order to produce Lemma 2.14 is a souped up version of the forcings from [5] and [8], which are in turn variants of the $\mathcal{L}$-forcing of Jensen as being developed e.g. in [20].[12] All these forcings may be construed as building uncountable models as term models of a given language, $\mathcal{L}$, with the forcing conditions being finite fragments of a consistent and complete $\mathcal{L}$-theory which will give those term models, augmented by "side conditions" which will guarantee that the forcing only collapses cardinals in a controlled way. Our forcing will change the cofinalities of ω_2 and ω_3 to ω and ω_1, respectively, and it won't collapse any other cardinal outside of the (possibly empty) half-open interval $(\omega_3, 2^{\aleph_2}]$.

Let us give an outline of the proof of Lemma 2.14.

To prove Lemma 2.14, we aim to build a stationary set preserving forcing $\mathbb{P}$ which adds a generic iteration of some $\mathbb{P}_{\max}$-condition $(N; \in, I, a)$ coded by a real in the projection of a tree $\tilde{T}$ projecting to the set of codes for conditions in our given dense set D. Moreover, we want this iteration to send the distinguished set a of $(N; \in, I, a)$ to A, and we want every I^*-positive set in the final model $(N^*; \in, I^*, A)$ to be a stationary subset of ω_1 in $V^{\mathbb{P}}$. Our approach is to think of all the relevant objects – $(N; \in, I, a)$, a branch through $\tilde{T}$ projecting to a real coding $(N; \in, I, a)$, and the generic iteration of $(N; \in, I, a)$ of length $\omega_1 + 1$ – as being given by "term models" in a suitable language, $\mathcal{L}$, and add them via finite approximations. Thus, the working parts of our forcing will be finite sets p of sentences from $\mathcal{L}$ providing partial information about the above objects. We will require these finite bits of information p to be realized in some outer model.[13] The existence of such an outer model will be absolute to any generic extension of V via $\mathrm{Col}(\omega, \omega_2)$.

In $V^{\mathrm{Col}(\omega,\omega_2)}$,

$$(H^V_{\omega_2}; \in, \mathsf{NS}^V_{\omega_1}, A)$$

becomes a $\mathbb{P}_{\max}$-condition and $p[\tilde{T}] = D^*$ is still dense, so that in $V^{\mathrm{Col}(\omega,\omega_2)}$ there is a $\mathbb{P}_{\max}$-condition $(N; \in, I, a) \in D^*$ which is stronger than $(H^V_{\omega_2}; \in, \mathsf{NS}^V_{\omega_1}, A)$. We may now iterate $(N; \in, I, a)$ in length $\omega_1^{V^{\mathrm{Col}(\omega,\omega_2)}} + 1 = \omega_3^V + 1$ so as to produce

$$\sigma\colon (N; \in, I, a) \to (N^*; \in, I^*, a^*).$$

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[12] One of the referees informs us that J. Keisler in [23] and [24] developed forcings which work in a similar fashion.

[13] W is an outer model iff W is a transitive model of ZFC with $W \supset V$ and which has the same ordinals as V; in other words, W is an outer model iff V is an inner model of W.

If $(M_i, \pi_{ij} \colon i \leq j \leq \omega_1^N) \in N$ is the generic iteration of $(H_{\omega_2}^V; \in, \mathsf{NS}_{\omega_1}^V, A)$ witnessing that $(N; \in, I, a)$ is stronger than $(H_{\omega_2}^V; \in, \mathsf{NS}_{\omega_1}^V, A)$, then $\sigma((M_i, \pi_{ij} \colon i \leq j \leq \omega_1^N)) = (M_i, \pi_{ij} \colon i \leq j \leq \omega_3^V)$ is an extension of that iteration. We have that

$$M_0 = (H_{\omega_2}^V; \in, \mathsf{NS}_{\omega_1}^V, A),$$

and

$$\pi_{0,\omega_3^V} \colon M_0 \to M_{\omega_3^V}$$

may be lifted to a generic iteration

$$\tilde{\pi} \colon V \to M$$

of V, for a transitive M, such that $\tilde{\pi} \supset \pi_{0,\omega_3^V}$ and $\tilde{\pi}(M_0) = M_{\omega_3^V}$. See [55, Lemma 3.8].

$$\begin{tikzcd}
 & p[\tilde{T}] \subseteq p[\tilde{\pi}(\tilde{T})] & \\
 & \rotatebox{90}{$\in$} & \\
 & (N; \in, I, a) \arrow[r, "\sigma"] & (N^*; \in, I^*, a^*) \\
 & \rotatebox{90}{$\in$} & \rotatebox{90}{$\in$} \\
M_0 \arrow[r, "\pi_{0\omega_1^N}"] & M_{\omega_1^N} \arrow[r, "\pi_{\omega_1^N \omega_3^V}"] & M_{\omega_3^V} \\
\parallel & & \\
(H_{\omega_2}^V; \in, \mathsf{NS}_{\omega_1}^V, A) & & \rotatebox{-90}{$\in$} \\
\rotatebox{-90}{$\in$} & & \\
V \arrow[rr, "\tilde{\pi}"] & & M
\end{tikzcd}$$

We now see that $V^{\mathrm{Col}(\omega,\omega_2)}$ contains objects like the ones we intend to add by our forcing—namely $(N; \in, I, a)$, a branch through $\tilde{T}$ projecting to a real coding $(N; \in, I, a)$, and the generic iteration of $(N; \in, I, a)$ of length $\omega_1 + 1$—albeit not defined relative to the parameters $\tilde{T}$, $H_{\omega_2}^V$, and A, but relative to $\tilde{\pi}(\tilde{T})$, $H_{\omega_2}^M$, and $\tilde{\pi}(A)$. The statement that such objects exist is Σ_1 in the parameters $H_{\omega_2}^M$, $\tilde{\pi}(A)$, and a Skolem hull[14] of $\tilde{\pi}(\tilde{T})$ of size $\aleph_2^M$, which will both be elements of $\mathrm{HC}^{M^{\mathrm{Col}(\omega,\tilde{\pi}(\omega_2))}}$. By Shoenfield absoluteness (10), see e.g. [41, Corollary 7.21], such objects will also exist in $M^{\mathrm{Col}(\omega,\tilde{\pi}(\omega_2))}$.

The point that π_{0,ω_3^V} could be lifted to $\tilde{\pi}$ is then the following. The statement that objects like the ones we intend to add by our forcing exist in $M^{\mathrm{Col}(\omega,\tilde{\pi}(\omega_2))}$ may now be pulled back via $\tilde{\pi}$. This buys us that objects like the ones we intend to add by our forcing exist in $V^{\mathrm{Col}(\omega,\omega_2)}$ – and this time with the right parameters $\tilde{T}$,

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[14] See Claim 3.1.

$H^V_{\omega_2}$, and A. The argument that combined lifting π_{0,ω_3^V} to $\tilde{\pi}$, applying Shoenfield absoluteness, and pulling back the statement of interest was crucial to arrive at the desired conclusion, viz. that objects like the ones we intend to add by our forcing exist in $V^{\mathrm{Col}(\omega,\omega_2)}$. This will be our starting point for cooking up the forcing $\mathbb{P}$.

In order to prove that our forcing $\mathbb{P}$ preserves stationary subsets of ω_1 we will need an argument exploiting lifting, Shoenfield absoluteness, and pulling back. In order to be able to run this argument we will need our forcing to approximate, not only the objects we are ultimately interested in obtaining, but also the iteration $(M_i, \pi_{ij} \colon i \leq j \leq \omega_1^N) \in N$. (See footnote 29.) We will think of the objects themselves, which exist in $V^{\mathrm{Col}(\omega,\omega_2)}$, as "certificates" for some finite piece of information about them. The idea is then to have our forcing consist of finite sets of $\mathcal{L}$-sentences for which there is a "certificate" in $V^{\mathrm{Col}(\omega,\omega_2)}$.

The problem with the above strategy is that, although a forcing $\mathbb{P}$ like the one we have described would in fact add the desired objects, one would still need to show that it preserves stationary subsets of ω_1 and that every positive set in the final model of the iteration being added by $\mathbb{P}$ is in fact stationary in that extension. Our forcing $\mathbb{P}$ will be a subset of H_{ω_3}, and one tool for taking care of these issues is the use of a diamond sequence $\langle (Q_\lambda, A_\lambda) : \lambda < \omega_3 \rangle$ consisting of transitive structures[15] in H_{ω_3} in order to guess $(H_{\omega_3}, \dot{C})$, where $\dot{C}$ is a $\mathbb{P}$-name for a club in ω_1, $\dot{C} \subset H_{\omega_3}$. That $(H_{\omega_3}, \dot{C})$ be guessed means that there are stationarily many $\lambda < \omega_3$ such that (Q_λ, A_λ) is an elementary substructure of $(H_{\omega_3}, \dot{C})$. See $(\diamondsuit)$ on p. 23.

Imagine that $\mathbb{P}$ is a forcing which adds the desired objects, but which also preserves stationary subsets of ω_1. Let $\dot{C}$ be a $\mathbb{P}$-name for a club in ω_1, $\dot{C} \subset H_{\omega_3}$, and let $S \subset \omega_1$ be stationary in V. Let g be $\mathbb{P}$-generic over V. There will be some $\lambda < \omega_3$ such that $(H_{\omega_3}, \dot{C})$ is guessed by (Q_λ, A_λ) and in $V[g]$ there will be some countable elementary substructure X of (Q_λ, A_λ) such that, writing $X[g] = \{\tau^g \colon \tau \in V^{\mathbb{P}} \cap X\}$,

(a) $X \cap \omega_1 \in S$, and

(b) $X[g] \cap Q_\lambda = X \cap Q_\lambda$.

That (Q_λ, A_λ) is an elementary substructure of $(H_{\omega_3}, \dot{C})$ will then mean in practice that $X \cap \omega_1 \in \dot{C}^g$, so that $X \cap \omega_1$ witnesses that S is still stationary in $V[g]$. Calling some g with the property (b) "$(\mathbb{P}, X)$-generic" there is, however, no reason to expect an X such that g is $(\mathbb{P}, X)$-generic to exist in V (in fact, V won't have such X).

When defining $\mathbb{P}$, we will turn this around and have our conditions also approximate finite bits of information about such elementary substructures X.

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[15] I.e., the underlying universe Q_λ will be transitive and $A_\lambda \subset Q_\lambda$ will be a distinguished predicate of the structure.

Our key tool for taking care of the above issue is then to define $\mathbb{P}$ as the last forcing from a recursively defined $\subset$-increasing sequence $\vec{\mathbb{P}} = (\mathbb{P}_\lambda : \lambda \leq \omega_3)$. Each $\mathbb{P}_\lambda$ will be a subset of Q_λ. Hence, when defining $\mathbb{P}_\eta$, $\eta \leq \omega_3$, if $\lambda < \eta$, then we already know what it means for some g to be (partially) $\mathbb{P}_\lambda$-generic over (Q_λ, A_λ), and if X is a countable elementary substructure of (Q_λ, A_λ), then $X[g]$ may be assigned a meaningful interpretation as $X[g] = \{\tau^g \colon \tau \in V^{\mathbb{P}_\delta} \cap X\}$. We will maintain that at each stage η in the construction of $\vec{\mathbb{P}}$ we define $\mathbb{P}_\eta$ by saying that a finite set p of $\mathcal{L} \cap Q_\eta$-sentences is in $\mathbb{P}_\eta$ if and only if there is a certificate for p which extends p and when intersected with each of the side conditions $X_\lambda \prec (Q_\lambda, A_\lambda)$ (also given by the certificate), for $\lambda < \eta$, is generic over X_η for the already defined forcing $\mathbb{P}_\lambda$. The role of condition (b) above (with X_λ replacing X) will be that if A_λ codes the name of a club subset $\dot{C}$ of ω_1, then some $p \in g$ must force that $X_\lambda \cap \omega_1 \in \dot{C}$, so that in the light of (a) above, p also forces that S has non-empty intersection with $\dot{C}$.

Our next section is entirely devoted to a proof of Lemma 2.14.

3 The forcing.

Recall our Convention 2.4 which we are now going to make use of without further notice. Let us assume throughout the hypotheses of Lemma 2.14. We aim to verify its conclusion.

Let us fix $D \subset \mathbb{P}_{\max}$, an open dense set in $L(\Gamma, \mathbb{R})$. The fact that D is open dense may be written as

$$\forall p \in \mathbb{P}_{\max} \, \exists q \leq p, \, q \in D \wedge \forall p \in D \, \forall q \leq p, \, q \in D.$$

By hypothesis (ii) in the statement of Lemma 2.14 (i.e., $\Gamma \subset \Gamma^\infty$ is productive), we may apply Lemma 2.9 to conclude that (9) on p. 14 holds true with D and $\mathbb{P} = \mathrm{Col}(\omega, \omega_2)$, i.e.,

$$(\mathrm{HC}; \in, D) \prec (\mathrm{HC}^{V^{\mathrm{Col}(\omega,\omega_2)}}; \in, D^*).$$

This will ensure that

(D.1) $V^{\mathrm{Col}(\omega,\omega_2)} \models$ "D^* is an open dense subset of $\mathbb{P}_{\max}$."

Let us identify D with a canonical set of reals coding the elements of D,[16] and let $\tilde{T} \in V$ be a tree on $\omega \times \theta$, for some ordinal θ, such that

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[16] We will later have to spell out a bit more precisely in which way we aim to have the elements of $p[T]$ code the elements of D, see (C.2) and (Σ.5) below.

(D.2) $V^{\mathrm{Col}(\omega,\omega_2)} \models D^* = p[\tilde{T}]$.[17]

Let h be $\mathrm{Col}(\omega,\omega_2)$-generic over V. Inside $V[h]$,

$$((H_{\omega_2})^V; \in, (\mathsf{NS}_{\omega_1})^V, A) \tag{11}$$

is a $\mathbb{P}_{\max}$ condition, call it p. Let $q^* \in (\mathbb{P}_{\max})^{V[h]}$, $q^* < p$, $q^* \in D^*$, cf. (D.1). Let $q^* = (N^*; \in, I^*, a^*)$. Identifying q^* with some real coding it, we have that $q^* \in p[\tilde{T}]$, cf. (D.2).

Claim 3.1 *There is a tree $T \in V$ on $\omega \times \omega_2$ such that*

$$q^* \in p[T] \subset p[\tilde{T}]. \tag{12}$$

Proof of Claim 3.1. Let $q^* = \sigma^h$, where $\sigma \in V^{\mathrm{Col}(\omega,\omega_2)}$. We may assume that $\sigma \in H_{\omega_3}$. Recall that $\tilde{T}$ is on $\omega \times \theta$. Let $X \in V$, $X \prec H^V_{\theta^+}$ be such that $\omega_2 + 1 \cup \{\sigma, \tilde{T}\} \subset X$ and $\mathrm{Card}(X) = \aleph_2$. Let $\pi\colon P \cong X \prec H^V_{\theta^+}$ be such that P is transitive, and write $T = \pi^{-1}(\tilde{T})$. We have that $\pi(\sigma) = \sigma$, and π lifts to $\tilde{\pi}\colon P[h] \to H^{V[h]}_{\theta^+}$ with $\tilde{\pi}(q^*) = \tilde{\pi}(\sigma^h) = \pi(\sigma)^h = \sigma^h = q^*$. As $q^* \in p[\tilde{T}]$, the elementarity of $\tilde{\pi}$ then yields that $q^* \in p[T]$. The tree T is on $\omega \times P \cap \mathrm{OR}$, but using a bijection of $P \cap \mathrm{OR}$ with ω_2, we may construe it as a tree on $\omega \times \omega_2$. $\square$ (Claim 3.1)

Let us fix T as in Claim 3.1. Let us write

$$\kappa = \aleph_3, \tag{13}$$

so that $T \in H_\kappa$. Let d be $\mathrm{Col}(\kappa,\kappa)$-generic over V. In $V[d]$, let $(\bar{A}_\lambda\colon \lambda < \kappa)$ be a $\diamondsuit_\kappa$-sequence, i.e., for all $\bar{A} \subset \kappa$, $\{\lambda < \kappa\colon \bar{A} \cap \lambda = \bar{A}_\lambda\}$ is stationary. Also, let $c\colon \kappa \to H^V_\kappa = H^{V[d]}_\kappa$, $c \in V[d]$, be bijective. For $\lambda < \kappa$, let

$$Q_\lambda = c``\lambda \ \text{ and } \ A_\lambda = c``\bar{A}_\lambda. \tag{14}$$

An easy closure argument will give us some club $C \subset \kappa$ such that for all $\lambda \in C$,

(i) Q_λ is transitive,

(ii) $\{T, ((H_{\omega_2})^V; \in, (\mathsf{NS}_{\omega_1})^V, A)\} \cup (\omega_2 + 1) \subset Q_\lambda$,

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[17] An easy Skolem hull argument may be used to show that we might actually pick $\tilde{T} \in V$ as a tree on $\omega \times 2^{\aleph_2}$. We won't need that, though, but we shall prove and make use of a related fact below, see (12).

(iii) $Q_\lambda \cap \mathrm{OR} = \lambda$ (so that $c \restriction \lambda \colon \lambda \to Q_\lambda$ is bijective), and

(iv) $(Q_\lambda; \in) \prec (H_\kappa; \in)$.

(ii) is true for all sufficiently large $\lambda < \kappa$, and (iv) is true for all λ such that $Q_\lambda = c``\lambda$ is closed under some fixed set of Skolem functions for H_κ. As the set of $\lambda < \kappa$ with (i) and (iii) is each easily seen to be club, a club of λ with the above properties certainly exists. We will fix from now on some club $C \subset \kappa$ with (i) through (iv) for all $\lambda \in C$.

In $V[d]$, for all P, $B \subset H_\kappa$, the set of all $\lambda \in C$ such that

$$(Q_\lambda; \in, P \cap Q_\lambda, B \cap Q_\lambda) \prec (H_\kappa; \in, P, B)$$

is club, and the set of all $\lambda \in C$ such that $B \cap Q_\lambda = A_\lambda$ is stationary, so that

($\diamond$) For all P, $B \subset H_\kappa$ the set

$$\{\lambda \in C \colon (Q_\lambda; \in, P \cap Q_\lambda, A_\lambda) \prec (H_\kappa; \in, P, B)\}$$

is stationary.

We shall sometimes also write $Q_\kappa = H_\kappa$. Readers who are familiar with Jensen's diamond will easily see that the principle which we refer to as ($\diamond$) is actually equivalent to $\diamond_\kappa$; see e.g. [41, Definition 5.34]. We shall use ($\diamond$) to guess information about names for club subsets of ω_1; this will play a crucial role in the verification that our forcing preserves stationary subsets of ω_1.

3.1 The definition of the forcing.

We shall now go ahead and produce a stationary set preserving forcing $\mathbb{P} \in V[d]$ of size κ which adds some $p \in D^*$ and some generic iteration

$$(N_i, \sigma_{ij} \colon i \leq j \leq \omega_1)$$

of $p = N_0$ such that if we write $N_{\omega_1} = (N_{\omega_1}; \in, I^*, a^*)$, then $I^* = (\mathsf{NS}_{\omega_1})^{V[d]^{\mathbb{P}}} \cap N_{\omega_1}$ and $a^* = A$. As the forcing $\mathrm{Col}(\kappa, \kappa)$ which added d is certainly stationary set preserving, this will verify Lemma 2.14.

NS_{ω_1} is still saturated in $V[d]$. This is true simply because forcing with $\mathrm{Col}(\kappa, \kappa)$ doesn't add any sequences of elements of V of length $\aleph_2$. Moreover, (D.1) and (D.2) are true with V being replaced by $V[d]$, as no reals are added and $\mathbb{P}_{\max}$ remains unchanged. Hence, in order to simplify our notation, we shall in what follows write

V for $V[d]$, i.e., assume that, in addition to "NS_{ω_1} is saturated" plus (D.1) and (D.2), $(\diamondsuit)$ is also true in V.

Working under these hypotheses, we shall now recursively define a $\subset$-increasing and continuous chain of forcings $\mathbb{P}_\lambda$ for all $\lambda \in C \cup \{\kappa\}$. The forcing $\mathbb{P}$ will be $\mathbb{P}_\kappa$. The conditions in each $\mathbb{P}_\lambda$ will be finite sets of formulae of an associated first order language, $\mathcal{L}^\lambda$, which will be defined below. The order of each $\mathbb{P}_\lambda$ will be just reverse inclusion, i.e., $q \leq_{\mathbb{P}_\lambda} p$ iff $q \supset p$ for p, $q \in \mathbb{P}_\lambda$.[18]

Assume that $\lambda \in C \cup \{\kappa\}$ and $\mathbb{P}_\mu$ has already been defined in such a way that $\mathbb{P}_\mu \subset Q_\mu$ for all $\mu \in C \cap \lambda$. We aim to define $\mathbb{P}_\lambda$.

Convention 3.2 *We say that* $x \subset \omega$ *is a real code for* $N_0 = (N_0; \in, I, a)$ *if there is some surjection* $f \colon \omega \to N_0$ *such that* x *is the monotone enumeration of the Gödel numbers of all expressions of the form* $\ulcorner \dot{N}_0 \models \varphi(\dot{n}_1, \ldots, \dot{n}_\ell, \dot{a}, \dot{I}) \urcorner$ *such that* φ *is a first order formula of the language associated to* $(N_0; \in, I, a)$[19] *and* $N_0 \models \varphi(f(n_1), \ldots, f(n_\ell), a, I)$ *holds true.*

We shall be interested in objects $\mathfrak{C}$ which exist in some outer model and which have the following properties. Any $\mathfrak{C}$ will be a triple of sets which are indexed by ω_1, ω, and K, respectively, with K being a subset of ω_1:

$$\mathfrak{C} = \langle \langle M_i, \pi_{ij}, N_i, \sigma_{ij} \colon i \leq j \leq \omega_1 \rangle, \langle (k_n, \alpha_n) \colon n < \omega \rangle, \langle \lambda_\delta, X_\delta \colon \delta \in K \rangle \rangle, \tag{15}$$

where

(C.1) M_0, $N_0 \in \mathbb{P}_{\max}$,

(C.2) $x = \langle k_n \colon n < \omega \rangle$ is a real code for $N_0 = (N_0; \in, I, a)$ in the sense of Convention 3.2 and x is $\langle (k_n, \alpha_n) \colon n < \omega \rangle \in [T]$,

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[18] Every $\mathbb{P}_\lambda$ will be designed to add certain objects by means of finite sets of formulae giving partial information on these objects. As indicated by the description at the end of Section 2, our intended objects are mentioned by the first forcing $\mathbb{P}_{\min(C)}$; the additional objects mentioned by the latter forcings are introduced in order to ensure that the final member of the sequence, i.e., $\mathbb{P}_\kappa$, has the desired property of preserving stationary sets and adding a correct generic iteration. (So it is natural to think of $\mathbb{P}_{\min(C)}$ as supporting the "working part" of the conditions in $\mathbb{P}_\kappa$, and all latter forcings as supporting also "side conditions.") It might therefore be useful, on a first reading of the remainder of this section, to only focus on the definition and analysis of $\mathbb{P}_{\min(C)}$—in other words, to ignore everything involving clauses (C.6) to (C.8) in the definition below of potential certificate, to also ignore clauses (Σ.6) to (Σ.8) in the definition below of potential certificates being pre-certified, and to skip Subsection 3.3—and to pay attention to the entirety of the present section only on a second reading.

[19] I.e., the language of set theory augmented by predicates for I and a

(C.3) $\langle M_i, \pi_{ij} \colon i \leq j \leq \omega_1^{N_0} \rangle \in N_0$ is a generic iteration of M_0 which witnesses that $N_0 < M_0$ in $\mathbb{P}_{\max}$,

(C.4) $\langle N_i, \sigma_{ij} \colon i \leq j \leq \omega_1 \rangle$ is a generic iteration of N_0 such that if

$$N_{\omega_1} = (N_{\omega_1}; \in, I^*, A^*),$$

then $A^* = A$,[20]

(C.5) $\langle M_i, \pi_{ij} \colon i \leq j \leq \omega_1 \rangle = \sigma_{0\omega_1}(\langle M_i, \pi_{ij} \colon i \leq j \leq \omega_1^{N_0} \rangle)$ and

$$M_{\omega_1} = ((H_{\omega_2})^V; \in, (\mathsf{NS}_{\omega_1})^V, A),^{21} \tag{16}$$

(C.6) $K \subset \omega_1$,

and for all $\delta \in K$,

(C.7) $\lambda_\delta \in \lambda \cap C$,[22] and if $\gamma < \delta$ is in K, then $\lambda_\gamma < \lambda_\delta$ and $X_\gamma \cup \{\lambda_\gamma\} \subset X_\delta$, and

(C.8) $X_\delta \prec (Q_{\lambda_\delta}; \in, \mathbb{P}_{\lambda_\delta}, A_{\lambda_\delta})$ and $X_\delta \cap \omega_1 = \delta$.

For future purposes, let us refer to any object $\mathfrak{C}$ as in (15) which satisfies the above properties (C.1) through (C.8) as a *potential certificate*. As this definition of potential certificates has A, T, $\langle A_\mu : \mu \in C \cap \lambda \rangle$, and $\langle \mathbb{P}_\mu : \mu \in C \cap \lambda \rangle$ as hidden parameters, we shall also refer to them as *potential certificates relative to* A, T, $H_{\omega_2}^V$, $\langle A_\mu : \mu \in C \cap \lambda \rangle$, and $\langle \mathbb{P}_\mu : \mu \in C \cap \lambda \rangle$.

We need to define a first order language $\mathcal{L}$ (independently from λ) whose formulae will be able to describe some $\mathfrak{C}$ with the above properties by producing the models M_i and N_i, $i < \omega_1$, as term models out of equivalence classes of terms of the form $\dot{n}$, $n < \omega$. The language $\mathcal{L}$ will have the following constants. There will be one constant for each set in H_κ; these constants will be underlined. In addition, there will be constants for all those objects outside of V which our forcing will add; those constants will be dotted.

$\underline{T}$	intended to denote T
$\underline{x}$ for every $x \in H_\kappa$	intended to denote x itself

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[20] There is no requirement on I^* matching the non-stationary ideal of some model in which $\mathfrak{C}$ exists.

[21] In particular, the distingushed ideal of M_{ω_1} is the true nonstationary ideal of V.

[22] Recall that λ is the index of the forcing $\mathbb{P}_\lambda$ which we are about to define.

$\dot{n}$ for every $n < \omega$	as terms for elements of M_i and $N_i, i < \omega_1$
$\dot{M}_i$ for $i < \omega_1$	intended to denote M_i
$\dot{\pi}_{ij}$ for $i \leq j \leq \omega_1$	intended to denote π_{ij}
$\dot{\vec{M}}$	intended to denote $(M_j, \pi_{jj'} \colon j \leq j' \leq \omega_1^{N_i})$ for $i < \omega_1$
$\dot{N}_i$ for $i < \omega_1$	intended to denote N_i
$\dot{\sigma}_{ij}$ for $i \leq j < \omega_1$	intended to denote σ_{ij}
$\dot{a}$	intended to denote the distinguished a-predicate of $M_i, N_i, i < \omega_1$
$\dot{I}$	intended to denote the distinguished ideal of $N_i, i < \omega_1$
$\dot{X}_\delta$ for $\delta < \omega_1$	intended to denote X_δ.

As $T \in H_\kappa$, the first line above is redundant. The constants $\dot{n}$, $n < \omega$, will produce the term models N_i for $i < \omega_1$; it is of course not important to use (dotted) natural numbers as these constants, the elements of any other fixed countable set (in H_κ) would be equally good.

The formulae of $\mathcal{L}$ will be exactly the expressions of the following form.[23]

$$\ulcorner \dot{N}_i \vDash \varphi(\underline{\xi}_1, \ldots, \underline{\xi}_k, \dot{n}_1, \ldots, \dot{n}_\ell, \dot{a}, \dot{I}, \dot{M}_{j_1}, \ldots, \dot{M}_{j_m}, \dot{\pi}_{q_1 r_1}, \ldots, \dot{\pi}_{q_s r_s}, \dot{\vec{M}}) \urcorner$$

for $i < \omega_1, \xi_1, \ldots, \xi_k < \omega_1, n_1, \ldots, n_\ell < \omega, j_1, \ldots, j_m < \omega_1, q_1 \leq r_1 < \omega_1, \ldots, q_s \leq r_s < \omega_1$ and for φ being a formula of the first order language associated with $\mathbb{P}_{\max}$-structures,

as well as:

$$\begin{array}{rl}
\ulcorner \dot{\pi}_{i\omega_1}(\dot{n}) = \underline{x} \urcorner & \text{for } i < \omega_1 \text{ and } x \in H_{\omega_2} \\
\ulcorner \dot{\pi}_{\omega_1\omega_1}(\underline{x}) = \underline{x} \urcorner & \text{for } x \in H_{\omega_2} \\
\ulcorner \dot{\sigma}_{ij}(\dot{n}) = \dot{m} \urcorner & \text{for } i \leq j < \omega_1, n, m < \omega \\
\ulcorner (\underline{\vec{u}}, \underline{\vec{\alpha}}) \in \underline{T} \urcorner & \text{for } \vec{u} \in {}^{<\omega}\omega \text{ and } \vec{\alpha} \in {}^{<\omega}\omega_2 \\
\ulcorner \underline{\delta} \mapsto \underline{\mu} \urcorner & \text{for } \delta < \omega_1, \mu < \kappa \\
\ulcorner \underline{x} \in \dot{X}_\delta \urcorner & \text{for } \delta < \omega_1, x \in H_\kappa
\end{array}$$

Let us write $\mathcal{L}^\lambda$ for the collection of all $\mathcal{L}$-formulae except for the formulae which mention elements outside of Q_λ, i.e., except for the formulae of the form $\ulcorner \underline{\delta} \mapsto \underline{\mu} \urcorner$ for $\delta < \omega_1$ and $\lambda \leq \mu < \kappa$ as well as $\ulcorner \underline{x} \in \dot{X}_\delta \urcorner$ for $\delta < \omega_1$ and $x \in H_\kappa \setminus Q_\lambda$. We

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[23] Again, the first order language associated with $\mathbb{P}_{\max}$-structures $(M; \in, I, a)$ is the language of set theory augmented by predicates for I and a.

may and shall assume that $\mathcal{L}$ is built in a canonical way so that $\mathcal{L}^\lambda \subset Q_\lambda$ and in fact $\mathcal{L}^\lambda = \mathcal{L} \cap Q_\lambda$.

We say that a potential certificate

$$\mathfrak{C} = \langle \langle M_i, \pi_{ij}, N_i, \sigma_{ij} \colon i \leq j \leq \omega_1 \rangle, \langle (k_n, \alpha_n) \colon n < \omega \rangle, \langle \lambda_\delta, X_\delta \colon \delta \in K \rangle \rangle$$

as in (15) is *pre-certified* by a collection Σ of $\mathcal{L}^\lambda$-formulae if and only if (C.1) through (C.8) are satisfied by $\mathfrak{C}$ and there are surjections $e_i \colon \omega \to N_i$ for $i < \omega_1$ such that the following hold true.

(Σ.1) $\ulcorner \dot{N}_i \models \varphi(\underline{\xi}_1, \ldots, \underline{\xi}_k, \dot{n}_1, \ldots, \dot{n}_\ell, \dot{a}, \dot{I}, \dot{M}_{j_1}, \ldots, \dot{M}_{j_m}, \dot{\pi}_{q_1 r_1}, \ldots, \dot{\pi}_{q_s r_s}, \dot{\vec{M}}) \urcorner \in \Sigma$ iff

- (a) $i < \omega_1$,
- (b) $\xi_1, \ldots, \xi_k \leq \omega_1^{N_i}$,
- (c) $n_1, \ldots, n_\ell < \omega$,
- (d) $j_1, \ldots, j_m \leq \omega_1^{N_i}$,
- (e) $q_1 \leq r_1 \leq \omega_1^{N_i}, \ldots, q_s \leq r_s \leq \omega_1^{N_i}$, and

$N_i \models \varphi(\xi_1, \ldots, \xi_k, e_i(n_1), \ldots, e_i(n_\ell), A \cap \omega_1^{N_i}, I^{N_i}, M_{j_1}, \ldots, M_{j_m}, \pi_{q_1 r_1}, \ldots, \pi_{q_s r_s}, \vec{M})$, where I^{N_i} is the distinguished ideal of N_i and $\vec{M} = \langle M_j, \pi_{jj'} \colon j \leq j' \leq \omega_1^{N_i} \rangle$,

(Σ.2) $\ulcorner \dot{\pi}_{i\omega_1}(\dot{n}) = \underline{x} \urcorner \in \Sigma$ iff $i < \omega_1$, $n < \omega$, and $\pi_{i\omega_1}(e_i(n)) = x$,

(Σ.3) $\ulcorner \dot{\pi}_{\omega_1\omega_1}(\underline{x}) = \underline{x} \urcorner \in \Sigma$ iff $x \in H_{\omega_2}$,

(Σ.4) $\ulcorner \dot{\sigma}_{ij}(\dot{n}) = \dot{m} \urcorner \in \Sigma$ iff $i \leq j < \omega_1$, n, $m < \omega$, and $\sigma_{ij}(e_i(n)) = e_j(m)$,

(Σ.5) letting F with $\mathrm{dom}(F) = \omega$ be the monotone enumeration of the Gödel numbers of all formulae of the form

$$\ulcorner \dot{N}_0 \models \varphi(\dot{n}_1, \ldots, \dot{n}_\ell, \dot{a}, \dot{I}) \urcorner$$

with $\ulcorner \dot{N}_0 \models \varphi(\dot{n}_1, \ldots, \dot{n}_\ell, \dot{a}, \dot{I}) \urcorner \in \Sigma$, we have that $\ulcorner (\underline{\vec{u}}, \underline{\vec{\alpha}}) \in \underline{T} \urcorner \in \Sigma$ iff there is some $n < \omega$ such that $\langle \vec{u}, \vec{\alpha} \rangle = \langle (F(m), \alpha_m) \colon m < n \rangle$ and $F(m) = k_m$ for all $m < n$,[24]

(Σ.6) $\ulcorner \underline{\delta} \mapsto \underline{\mu} \urcorner \in \Sigma$ iff $\delta \in K$ and $\mu = \lambda_\delta$, and

(Σ.7) $\ulcorner \underline{x} \in \dot{X}_\delta \urcorner \in \Sigma$ iff $\delta \in K$ and $x \in X_\delta$.

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[24] Here, $\langle (k_n, \alpha_n) \colon n < \omega \rangle$ is a component of $\mathfrak{C}$.

We say that a potential certificate $\mathfrak{C}$ as in (15) is *certified* by a collection Σ of formulae if and only if $\mathfrak{C}$ is pre-certified by Σ and, in addition,

(Σ.8) if $\delta \in K$, then $[\Sigma]^{<\omega} \cap X_\delta \cap E \neq \emptyset$ for every $E \subset \mathbb{P}_{\lambda_\delta}$ which is dense in $\mathbb{P}_{\lambda_\delta}$ and definable over the structure

$$(Q_{\lambda_\delta}; \in, \mathbb{P}_{\lambda_\delta}, A_{\lambda_\delta})$$

from parameters in X_δ.[25]

Item (Σ.8) is to play a crucial role in the proof that our forcing preserves stationary sets, see the proof of Lemma 3.13. If p is a condition with $\ulcorner \underline{\delta} \mapsto \underline{\lambda} \urcorner \in p$, and if the predicate A_λ guesses – via ($\Diamond$) – a name $\dot{C}$ for a club subset of ω_1, then (Σ.8) will guarantee that $p \Vdash \check{\delta} \in \dot{C}$, see the proof of Claim 3.17.

Definition 3.3 *Let $\mathfrak{C}$ as in (15) be a potential certificate relative to A, T, $H^V_{\omega_2}$, $\langle A_\mu : \mu \in C \cap \lambda \rangle$, and $\langle \mathbb{P}_\mu : \mu \in C \cap \lambda \rangle$. We call $\mathfrak{C}$ a* semantic certificate relative to *A, T, $H^V_{\omega_2}$, $\langle A_\mu : \mu \in C \cap \lambda \rangle$, and $\langle \mathbb{P}_\mu : \mu \in C \cap \lambda \rangle$, or just a* semantic certificate, *iff there is a collection Σ of formulae such that $\mathfrak{C}$ is certified by Σ. We call Σ a* syntactic certificate *iff there is a semantic certificate $\mathfrak{C}$ such that $\mathfrak{C}$ is certified by Σ.*

In the proofs of Lemma 3.6 and of Claim 3.15 we will run Definition 3.3 inside a generic iterate M of V. The corresponding iteration map π will move the parameters $A, T, H^V_{\omega_2}$, $\langle A_\mu : \mu \in C \cap \lambda \rangle$, and $\langle \mathbb{P}_\mu : \mu \in C \cap \lambda \rangle$, so that we will talk about semantic certificates relative to the parameters $\pi(A)$, $\pi(T)$, $\pi(H^V_{\omega_2})$, $\pi(\langle A_\mu : \mu \in C \cap \lambda \rangle)$, and $\pi(\langle \mathbb{P}_\mu : \mu \in C \cap \lambda \rangle)$ with the understanding that this will be an object obtained by running Definition 3.3 over M with these shifted parameters.

Given a syntactic certificate Σ, there is a *unique* semantic certificate $\mathfrak{C}$ such that $\mathfrak{C}$ is certified by Σ. Even though it is obvious how to construct $\mathfrak{C}$ from Σ, in the proof of Lemma 3.7 below we will provide details on how to derive a semantic certificate from a given Σ.

It is worth stressing that not every collection of $\mathcal{L}^\lambda$-formulae which is merely consistent is already a syntactic certificate. The requirement that the constant $x \in H_\kappa$ is to be interpreted by itself (cf. (Σ.2), (Σ.3), and (Σ.7)) may be restated as saying that for a consistent $\mathcal{L}^\lambda$-theory to be a syntactic certificate it is to be true that certain types are omitted.

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[25] Equivalently, $[\Sigma]^{<\omega} \cap E \neq \emptyset$ for every $E \subset \mathbb{P}_{\lambda_\delta} \cap X_\delta$ which is dense in $\mathbb{P}_{\lambda_\delta} \cap X_\delta$ and definable over the structure

$$(X_\delta; \in, \mathbb{P}_{\lambda_\delta} \cap X_\delta, A_{\lambda_\delta} \cap X_\delta)$$

from parameters in X_δ.

Let Σ and p be sets of formulae, where p is finite. We say that p is *certified by* Σ if and only if there is some (unique) $\mathfrak{C}$ as in (15) such that $\mathfrak{C}$ is certified by Σ and

(Σ.9) $p \in [\Sigma]^{<\omega}$.

We may also say that p is *certified by* $\mathfrak{C}$ as in (15) iff there is some Σ such that $\mathfrak{C}$ and p are both certified by Σ—and we will then also refer to Σ as a syntactical certificate for p and to $\mathfrak{C}$ as the associated semantic certificate.

We are then ready to define the forcing $\mathbb{P}_\lambda$. We say that $p \in \mathbb{P}_\lambda$ if and only if

$$V^{\mathrm{Col}(\omega,\lambda)} \models \text{“There is a set } \Sigma \text{ of } \mathcal{L}^\lambda\text{-formulae such that } p \text{ is certified by } \Sigma.\text{”} \tag{17}$$

Let p be a finite set of formulae of $\mathcal{L}^\lambda$. By the homogeneity of $\mathrm{Col}(\omega,\lambda)$, if there is some h which is $\mathrm{Col}(\omega,\lambda)$-generic over V and there is some $\Sigma \in V[h]$ such that p is certified by Σ, then for all h which are $\mathrm{Col}(\omega,\lambda)$-generic over V there is some $\Sigma \in V[h]$ such that p is certified by Σ. It is then easy to see that $\langle \mathbb{P}_\lambda \colon \lambda \in C \cup \{\kappa\}\rangle$ is definable over V from $\langle A_\lambda \colon \lambda < \kappa\rangle$ and C, and is hence an element of V.[26]

The following absoluteness fact will be relevant in the proofs of Lemma 3.6 and of Claim 3.15.

Lemma 3.4 *Let $\lambda \in C \cup \{\kappa\}$, and let p be a finite set of formulae of $\mathcal{L}^\lambda$. If there is any outer model in which there is some Σ which certifies p, then there is some $\Sigma \in V^{\mathrm{Col}(\omega,\lambda)}$ which certifies p.*

Proof. The statement that there is a Σ which certifies p is Σ_1 in the parameters A, T, $H^V_{\omega_2}$, $\langle A_\mu \colon \mu \in C \cap \lambda\rangle$, and $\langle \mathbb{P}_\mu \colon \mu \in C \cap \lambda\rangle$, all of which are elements of $\mathrm{HC}^{V^{\mathrm{Col}(\omega,\lambda)}}$. Hence by Shoenfield absoluteness (10), see [41, Corollary 7.21], if there is any outer model in which there is some Σ which certifies p, then there is some $\Sigma \in V^{\mathrm{Col}(\omega,\lambda)}$ which certifies p.[27] $\square$ (Lemma 3.4)

3.2 Some properties of the forcing.

It is easy to see that

(i) $\mathbb{P} = \mathbb{P}_\kappa \subset H_\kappa$,

(ii) if $\bar\lambda < \lambda$ are both in $C \cup \{\kappa\}$, then $\mathbb{P}_{\bar\lambda} \subset \mathbb{P}_\lambda$, and

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[26] To remind the reader, C is the club from p. 22.

[27] In fact, if P is a transitive model of KP plus the axiom *Beta* with A, T, $H^V_{\omega_2}$, $\langle A_\mu \colon \mu \in C \cap \lambda\rangle$, $\langle \mathbb{P}_\mu \colon \mu \in C \cap \lambda\rangle \in P$ and if $p \in \mathbb{P}_\lambda$, then there is some $\Sigma \in P^{\mathrm{Col}(\omega,\lambda)}$ which certifies p.

(iii) if $\lambda \in C \cup \{\kappa\}$ is a limit point of $C \cup \{\kappa\}$, then $\mathbb{P}_\lambda = \bigcup_{\bar\lambda \in C \cap \lambda} \mathbb{P}_{\bar\lambda}$,

so that there is some club $D \subset C$ such that for all $\lambda \in D$,

$$\mathbb{P}_\lambda = \mathbb{P} \cap Q_\lambda.$$

Hence $(\diamondsuit)$ gives us the following.

$(\diamondsuit(\mathbb{P}))$ For all $B \subset H_\kappa$ the set

$$\{\lambda \in C \colon (Q_\lambda; \in, \mathbb{P}_\lambda, A_\lambda) \prec (H_\kappa; \in, \mathbb{P}, B)\}$$

is stationary.

The first one of the following lemmas is entirely trivial.

Lemma 3.5 *Let Σ be a syntactic certificate, and let p, $q \in [\Sigma]^{<\omega}$. Then p and q are compatible conditions in $\mathbb{P}$.*

Lemma 3.6 $\emptyset \in \mathbb{P}_{\min(C)}$.

Proof. This is a simple variant of the proofs of [2, Theorem 2.8] and of [42, Theorem 4.2]. What needs to be done is to construct a semantic/syntactic certificate (for $\emptyset$) in some outer model.

Let h be $\mathrm{Col}(\omega, \omega_2)$-generic over V. Let $q^* = (N^*; \in, I^*, a^*) \in (\mathbb{P}_{\max})^{V[h]}$ be as in the paragraph preceeding (12), i.e., $q^* \in (\mathbb{P}_{\max})^{V[h]}$, $q^* < (H^V_{\omega_2}; \in, \mathsf{NS}^V_{\omega_1}, A)$, $q^* \in D^*$, and such that (12) is true, i.e., $q^* \in p[T] \subset p[\tilde{T}]$, cf. Lemma 3.1. Let $(M_i, \pi_{ij} \colon i \leq j \leq \omega_1^{N^*}) \in N^*$ be the unique generic iteration of the $(\mathbb{P}_{\max})^{V[h]}$-condition $(H^V_{\omega_2}; \in, \mathsf{NS}^V_{\omega_1}, A)$ which witnesses that q^* is stronger than this condition.

Let $(N_i, \sigma_{ij} \colon i \leq j \leq \kappa) \in V[h]$ be a generic iteration of $N_0 = N^*$ such that $\kappa = \omega_1^{N_\kappa}$.[28] Let

$$(M_i, \pi_{ij} \colon i \leq j \leq \kappa) = \sigma_{0\kappa}((M_i, \pi_{ij} \colon i \leq j \leq \omega_1^{N_0})) \tag{18}$$

Since $M_0 = ((H_{\omega_2})^V; \in, (\mathsf{NS}_{\omega_1})^V, A)$ and $(\mathsf{NS}_{\omega_1})^V$ is assumed to be saturated in V, every maximal antichain in V consisting of stationary subsets of ω_1 is an element of M_0. By [55, Lemma 3.8], we may hence lift the generic ultrapower map $\pi_{01} \colon M_0 \to$

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[28] If we wished, we could even arrange that writing $N_\kappa = (N_\kappa; \in, I', a')$, we have that $I' = (\mathsf{NS}_\kappa)^{V[h]} \cap N_\kappa$, but this is not relevant here; cf. footnote 20.

M_1 to act on all of V, and inductively we may lift the entire generic iteration (18) to a generic iteration

$$(M_i^+, \pi_{ij}^+ \colon i \leq j \leq \kappa) \tag{19}$$

of V in such a way that all M_i^+, $i \leq \kappa$, are transitive. Let us write $M = M_\kappa^+$ and $\pi = \pi_{0\kappa}^+$.

Let $\langle k_n, \alpha_n \colon n < \omega \rangle$ be such that $x = \langle k_n \colon n < \omega \rangle$ is a real code for N_0 in the sense of Convention 3.2 and $\langle (k_n, \alpha_n) \colon n < \omega \rangle \in [T]$. We then clearly have that $\langle (k_n, \pi(\alpha_n)) \colon n < \omega \rangle \in [\pi(T)]$.

$$\begin{tikzcd}
 & p[T] \subseteq p[\pi(T)] & \\
 & (N^*; \in, I^*, a^*) \arrow[u, phantom, sloped, "\subseteq"] \arrow[r, "\sigma_{0\kappa}"] & N_\kappa \\
M_0 \arrow[r, "\pi_{0\omega_1^{N^*}}"] \arrow[d, phantom, sloped, "="] & M_{\omega_1^{N^*}} \arrow[u, phantom, sloped, "\subseteq"] \arrow[r, "\pi_{\omega_1^{N^*}\kappa}"] & M_\kappa \arrow[u, phantom, sloped, "\subseteq"] \arrow[dd, phantom, sloped, "\subseteq"] \\
((H_{\omega_2})^V; \in, (\mathsf{NS}_{\omega_1})^V, A) \arrow[d, phantom, sloped, "\subseteq"] & & \\
V \arrow[rr, "\pi"] & & M_\kappa^+ = M
\end{tikzcd}$$

It is now easy to see that

$$\mathfrak{C} = \langle \langle M_i, \pi_{ij}, N_i, \sigma_{ij} \colon i \leq j \leq \kappa \rangle, \langle (k_n, \pi(\alpha_n)) \colon n < \omega \rangle, \langle \rangle \rangle \tag{20}$$

certifies $\emptyset$ relative to the parameters $\pi(A)$, $\pi(T)$, $\pi(H_{\omega_2}^V)$, $\langle \rangle$, and $\langle \rangle$, with $\emptyset$ being construed as the empty set of $\pi(\mathcal{L}^\kappa)$ formulae: as the third component $\langle \rangle$ of $\mathfrak{C}$ in (20) is empty, *any* set of surjections $e_i \colon \omega \to N_i$, $i < \omega_1$, will induce a syntactic certificate for $\emptyset$ relative to $\pi(A)$, $\pi(T)$, $\pi(H_{\omega_2}^V)$, $\langle \rangle$, and $\langle \rangle$, whose associated semantic certificate is $\mathfrak{C}$. The statement that there is a syntactic certificate for $\emptyset$ is Σ_1 in the parameters $H_{\omega_2}^M$, $\pi(A)$, and $\pi(T)$, which will all be in $\mathrm{HC}^{M^{\mathrm{Col}(\omega, \pi(\omega_2))}}$. Hence by Shoenfield absoluteness (10), see [41, Corollary 7.21], there is then some $\mathfrak{C} \in M^{\mathrm{Col}(\omega, \pi(\omega_2))}$ as in (20) which certifies $\emptyset$ relative to $\pi(A)$, $\pi(T)$, $\pi(H_{\omega_2}^V)$, $\langle \rangle$, and $\langle \rangle$, so that $\emptyset \in \pi(\mathbb{P}_{\min(C)})$, cf. Lemma 3.4.[29] By the elementarity of π, then, $\emptyset \in \mathbb{P}_{\min(C)}$. $\square$ (Lemma 3.5)

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[29] Exactly in order to be able to do this we let the forcing also search for $\langle M_i, \pi_{ij} \colon i \leq j \leq \omega_1 \rangle$ rather than just $\langle N_i, \sigma_{ij} \colon i \leq j \leq \omega_1 \rangle$. The presence of $\langle M_i, \pi_{ij} \colon i \leq j \leq \omega_1 \rangle$ allows us to lift $\pi_{0\kappa}$ to a map acting on all of V, so that we may then apply Shoenfield absoluteness and pull back the statement of interest—namely $\emptyset \in \pi(\mathbb{P}_{\min(C)})$. Cf. the discussion on p. 20.

Lemma 3.7 *Let $\lambda \in C \cup \{\kappa\}$. Let $g \subset \mathbb{P}_\lambda$ be a filter such that $g \cap E \neq \emptyset$ for all dense $E \subset \mathbb{P}_\lambda$ which are definable over $(Q_\lambda; \in, \mathbb{P}_\lambda)$ from elements of Q_λ. Then $\bigcup g$ is a syntactic certificate.*

Proof. Let us first describe how to read off from $\bigcup g$ a candidate

$$\mathfrak{C} = \langle \langle M_i, \pi_{ij}, N_i, \sigma_{ij} \colon i \leq j \leq \omega_1 \rangle, \langle (k_n, \alpha_n) \colon n < \omega \rangle, \langle \lambda_\delta, X_\delta \colon \delta \in K \rangle \rangle$$

for a semantic certificate for $\bigcup g$. A variant of what is to come shows how to derive $\mathfrak{C}$ from a given syntactic certificate Σ, where $\mathfrak{C}$ is unique such that Σ certifies $\mathfrak{C}$, cf. the remark on p. 28.

Some of the formulas to follow simply describe the construction of a direct limit associated with the maps π_{ij} and σ_{ij}.

For i, $j < \omega_1$ and τ, $\sigma \in \{\dot{n} \colon n < \omega\} \cup \{\underline{\xi} \colon \xi < \omega_1\}$ define

$$\begin{array}{rl}
\tau \sim_i \sigma & \text{iff} \quad \ulcorner \dot{N}_i \models \tau = \sigma \urcorner \in \bigcup g \\
(i, \tau) \sim_{\omega_1} (j, \sigma) & \text{iff } i \leq j \wedge \exists \rho \, \{\ulcorner \dot{\sigma}_{ij}(\tau) = \rho \urcorner, \ulcorner \dot{N}_j \models \rho = \sigma \urcorner\} \subset \bigcup g \\
 & \quad \text{or } j \leq i \wedge \exists \rho \, \{\ulcorner \dot{\sigma}_{ji}(\sigma) = \rho \urcorner, \ulcorner \dot{N}_i \models \rho = \tau \urcorner\} \subset \bigcup g \\
[\tau]_i & = \{\sigma \colon \tau \sim_i \sigma\} \\
[(i, \tau)] & = \{(j, \sigma) \colon (i, \tau) \sim_{\omega_1} (j, \sigma)\} \\
M_i & = \{[\tau]_i \colon \tau \in \{\dot{n} \colon n < \omega\} \cup \{\underline{\xi} \colon \xi < \omega_1\} \wedge \ulcorner \dot{N}_i \models \tau \in \dot{M}_i \urcorner \in \bigcup g\} \\
M_{\omega_1} & = (H_{\omega_2})^V \\
N_i & = \{[\tau]_i \colon \tau \in \{\dot{n} \colon n < \omega\} \cup \{\underline{\xi} \colon \xi < \omega_1\}\} \\
N_{\omega_1} & = \{[i, \tau] \colon i < \omega_1 \wedge \ulcorner \dot{N}_i \models \tau = \tau \urcorner \in \bigcup g\} \\
[\tau]_i \,\tilde{\in}_i\, [\sigma]_i & \text{iff} \quad \ulcorner \dot{N}_i \models \tau \in \sigma \urcorner \in \bigcup g \\
[i, \tau] \tilde{\in}_{\omega_1} [j, \sigma] & \text{iff } i \leq j \wedge \exists \rho \, \{\ulcorner \dot{\sigma}_{ij}(\tau) = \rho \urcorner, \ulcorner \dot{N}_j \models \rho \in \sigma \urcorner\} \subset \bigcup g \\
 & \quad \text{or } j \leq i \wedge \exists \rho \, \{\ulcorner \dot{\sigma}_{ji}(\sigma) = \rho \urcorner, \ulcorner \dot{N}_i \models \tau \in \rho \urcorner\} \subset \bigcup g \\
[\tau]_i \in I^{N_i} & \text{iff} \quad \ulcorner \dot{N}_i \models \tau \in \dot{I} \urcorner \in \bigcup g \\
[i, \tau] \in I^{N_{\omega_1}} & \text{iff} \quad [\tau]_i \in I^{N_i} \\
[\tau]_i \in a^{N_i} & \text{iff} \quad \ulcorner \dot{N}_i \models \tau \in \dot{a} \urcorner \in \bigcup g \\
[i, \tau] \in a^{N_{\omega_1}} & \text{iff} \quad [\tau]_i \in I^{N_i} \\
\pi_{ij}([\tau]_i) = [\sigma]_j & \text{iff} \quad \ulcorner \dot{N}_j \models \dot{\pi}_{ij}(\tau) = \sigma \urcorner \in \bigcup g
\end{array}$$

$$\begin{aligned}
\pi_{i\omega_1}([\tau]_i) = x \quad & \text{iff} \quad \ulcorner \dot{\pi}_{i\omega_1}(\tau) = \underline{x} \urcorner \in \bigcup g \\
\pi_{\omega_1\omega_1}(x) = x \quad & \text{iff} \quad x \in (H_{\omega_2})^V \\
\sigma_{ij}([\tau]_i) = [\sigma]_j \quad & \text{iff} \quad \ulcorner \dot{\sigma}_{ij}(\tau) = \sigma \urcorner \in \bigcup g \\
\sigma_{i\omega_1}([\tau]_i) = [i, \tau] \quad & \\
(k, \alpha) = (k_n, \alpha_n) \quad & \text{iff } \exists \vec{u} \exists \vec{\alpha} (\ulcorner (\underline{\vec{u}}, \underline{\vec{\alpha}}) \in \underline{T} \urcorner \in \bigcup g \wedge k = \vec{u}(n) \wedge \alpha = \vec{\alpha}(n)) \\
\delta \in K \quad & \text{iff } \exists \mu \ulcorner \underline{\delta} \mapsto \underline{\mu} \urcorner \in \bigcup g \\
\mu = \lambda_\delta \quad & \text{iff } \delta \in K \wedge \ulcorner \underline{\delta} \mapsto \underline{\mu} \urcorner \in \bigcup g \\
x \in X_\delta \quad & \text{iff } \delta \in K \wedge \ulcorner \underline{x} \in \dot{X}_\delta \urcorner \in \bigcup g
\end{aligned}$$

In order to see that this all works out we have to run a few density arguments. To show that a given subset of $\mathbb{P}$ is dense we frequently make use of Lemma 3.5. We will provide more details in some cases and fewer in others, and we are confident that the reader will be easily able to fill in the straightforward details herself in the latter cases.

Let us first observe that $\tilde{\in}_0$ is wellfounded and that in fact (the transitive collapse of) the structure $N_0 = (N_0; \tilde{\in}_0, a^{N_0}, I^{N_0})$ is an iterable $\mathbb{P}_{\max}$ condition. This is true because of the following.

Claim 3.8 *(C.2) is true, i.e.,* $\langle (k_n, \alpha_n) \colon n < \omega \rangle \in [T]$ *and* $\langle k_n \colon n < \omega \rangle$ *codes the theory of* N_0 *in the sense of Convention 3.2.*

Proof of Claim 3.8. Let $m < \omega$. Writing

$$q_0 = \{ \ulcorner (\underline{(k_n \colon n < m)}, \underline{(\alpha_n \colon n < m)}) \in \dot{T} \urcorner \},$$

we have that $q_0 \in g$. If

$$\mathfrak{C} = \langle \langle M'_i, \pi'_{ij}, N'_i, \sigma'_{ij} \colon i \leq j \leq \omega_1 \rangle, \langle (k'_n, \alpha'_n) \colon n < \omega \rangle, \langle \lambda'_\delta, X'_\delta \colon \delta \in K' \rangle \rangle$$

certifies q_0, then $k'_n = k_n$ and $\alpha'_n = \alpha_n$ for $n < m$ by (Σ.5), and then

$$((k_n \colon n < m), (\alpha_n \colon n < m)) \in T$$

by (C.2).

This shows $\langle (k_n, \alpha_n) \colon n < \omega \rangle \in [T]$.

By (Σ.5) and (C.2), for each $k < \omega$ the sets

$$D^0_k \quad = \quad \{ p \in \mathbb{P} \colon \exists m \, \exists ((k_n \colon n < m), (\alpha_n \colon n < m)) \exists r$$

$$\begin{aligned}&(\ulcorner(\underline{(k_n\colon n<m)},\underline{(\alpha_n\colon n<m)})\in\underline{T}\urcorner\in p\wedge k_r=k\\&\wedge k \text{ is the Gödel number of } \ulcorner\dot{N}_0\vDash\varphi(\dot{n}_1,\dots,\dot{n}_\ell,\dot{a},\dot{I})\urcorner)\to\\&\ulcorner\dot{N}_0\vDash\varphi(\dot{n}_1,\dots,\dot{n}_\ell,\dot{a},\dot{I})\urcorner\in p\}\end{aligned}$$

and

$$\begin{aligned}D^1_k \;=\;& \{p\in\mathbb{P}\colon k \text{ is the Gödel number of } \ulcorner\dot{N}_0\vDash\varphi(\dot{n}_1,\dots,\dot{n}_\ell,\dot{a},\dot{I})\urcorner\wedge\\&\ulcorner\dot{N}_0\vDash\varphi(\dot{n}_1,\dots,\dot{n}_\ell,\dot{a},\dot{I})\urcorner\in p\to\\&\exists m\,\exists((k_n\colon n<m),(\alpha_n\colon n<m))\exists r\;(k_r=k\wedge\\&\ulcorner(\underline{(k_n\colon n<m)},\underline{(\alpha_n\colon n<m)})\in\underline{T}\urcorner\in p\,)\}\end{aligned}$$

are dense in $\mathbb{P}$. This implies that $\langle k_n\colon n<\omega\rangle$ codes the theory of N_0 in the sense of Convention 3.2.

$\square$ (Claim 3.8)

Another set of easy density arguments will give that $(N_i,\sigma_{ij}\colon i\le j\le\omega_1)$ is a generic iteration of N_0, where we identify N_i with the structure $(N_i;\tilde{\in}_i,a^{N_i},I^{N_i})$. To verify this, let us first show:

Claim 3.9 *For each $i<\omega_1$ and for each $\xi\le\omega_1^{N_i}$, $[\underline{\xi}]_i$ represents ξ in (the transitive collapse of the well-founded part of) the term model for N_i; moreover, $a^{N_i}=A\cap\omega_1^{N_i}$. Hence $a^{N_{\omega_1}}=A$.*

Proof of Claim 3.9. The set

$$D^2=\{p\in\mathbb{P}\colon\exists\xi\ulcorner\dot{N}_i\vDash\underline{\xi}\text{ is the least uncountable cardinal }\urcorner\in p\}$$

is easily seen to be dense in $\mathbb{P}$, so that there is some (unique!) $\xi_0<\omega_1$ such that writing

$$p_0=\{\ulcorner\dot{N}_i\vDash\underline{\xi}_0\text{ is the least uncountable cardinal }\urcorner\},$$

$p_0\in g$. Let us now prove by induction on $\xi\le\xi_0$ that $[\underline{\xi}]_i$ must always represent ξ in (the transitive collapse of the well-founded part of) N_i. Fix such ξ.

For all $n<\omega$,

$$D^3_n=\{p\in\mathbb{P}\colon\ulcorner\dot{N}_i\vDash\dot{n}\in\underline{\xi}\urcorner\in p\to\exists\underline{\zeta}<\xi\ulcorner\dot{N}_i\vDash\dot{n}=\zeta\urcorner\in p\}$$

is dense below p_0. Also, for all $\zeta<\xi$,

$$D^4_\zeta=\{p\in\mathbb{P}\colon\ulcorner\dot{N}_i\vDash\underline{\zeta}\in\underline{\xi}\urcorner\in p\}$$

is dense below p_0. This shows that if $\tau \in \{\dot{n} \colon n < \omega\} \cup \{\underline{\xi} \colon \xi \in \xi_0 + 1\}$ is a term, then $[\tau]_i \tilde{\in}_i [\underline{\xi}]_i$ iff $[\tau]_i = [\underline{\zeta}]_i$ for some $\zeta < \xi$. Using the inductive hypothesis, this then implies that $[\underline{\xi}]_i$ represents ξ in (the transitive collapse of the well-founded part of) N_i. In particular, $\xi_0 = \omega_1^{N_i}$.

Now if $\xi \in A \cap \omega_1^{N_i}$, then

$$D^5_\xi = \{p \in \mathbb{P} \colon \ulcorner \dot{N}_i \models \underline{\xi} \in \dot{a} \urcorner\}$$

is dense below p_0, and if $\xi \in \omega_1^{N_i} \setminus A$, then

$$D^6_\xi = \{p \in \mathbb{P} \colon \ulcorner \dot{N}_i \models \underline{\xi} \notin \dot{a} \urcorner\}$$

is dense below p_0. Claim 3.9 then follows. ☐ (Claim 3.9)

Similarly:

Claim 3.10 *Let $i < \omega_1$. N_{i+1} is generated from $\operatorname{ran}(\sigma_{ii+1}) \cup \{\omega_1^{N_i}\}$ in the sense that for every $x \in N_{i+1}$ there is some function $f \in {}^{\omega_1^{N_i}}(N_i) \cap N_i$ such that $x = \sigma_{ii+1}(f)(\omega_1^{N_i})$.*

Proof of Claim 3.10. Let p_0 be as in the proof of Claim 3.9, let $p \leq p_0$, and let $\Sigma \supset p$ be a syntactic certificate for p with associated semantic certificate

$$\mathfrak{C} = \langle \langle M'_i, \pi'_{ij}, N'_i, \sigma'_{ij} \colon i \leq j \leq \omega_1 \rangle, \langle (k'_n, \alpha'_n) \colon n < \omega \rangle, \langle \lambda'_\delta, X'_\delta \colon \delta \in K' \rangle \rangle.$$

Fix $i < \omega_1$, and let $e_i \colon \omega \to N'_i$ and $e_{i+1} \colon \omega \to N'_{i+1}$ be as on p. 27. Notice that $\omega_1^{N'_i} = \omega_1^{N_i}$.

Let $n < \omega$. There must be some $f \in {}^{\omega_1^{N_i}}N'_i \cap N'_i$ with $e_{i+1}(n) = \sigma'_{ii+1}(f)(\omega_1^{N_i})$. Let m, $m' < \omega$ be such that $f = e_i(m)$ and $\sigma'_{ii+1}(f) = e_{i+1}(m')$. Then

$$p \cup \{\ulcorner \dot{N}_{i+1} \models \dot{n} = \dot{m}'(\underline{\omega_1^{N_i}}) \urcorner, \ulcorner \dot{\sigma}_{ii+1}(\dot{m}) = \dot{m}' \urcorner\} \leq p.$$

This argument shows that the set

$$D^7_n = \{p \in \mathbb{P} \colon \exists m \, \exists m' \, \{\ulcorner \dot{N}_{i+1} \models \dot{n} = \dot{m}'(\underline{\omega_1^{N_i}}) \urcorner, \ulcorner \dot{\sigma}_{ii+1}(\dot{m}) = \dot{m}' \urcorner\} \subset p\}$$

is dense below p_0. Claim 3.10 then follows. ☐ (Claim 3.10)

Claim 3.11 *Let $i < \omega_1$. $\{X \in \mathcal{P}(\omega_1^{N_i}) \cap N_i \colon \omega_1^{N_i} \in \sigma_{ii+1}(X)\}$ is generic over N_i for the forcing given by the I^{N_i}-positive sets.*

Proof of Claim 3.11. Let p_0, p, Σ, $\mathfrak{C}$, e_i, and e_{i+1} be as in the previous proof. Let $n < \omega$ be such that $e_i(n)$ is a maximal antichain in N_i' for the forcing given by the $I^{N_i'}$-positive sets. Let m, $m' < \omega$ be such that $e_i(m) \in e_i(n)$ and $\omega_1^{N_i} = \omega_1^{N_i'} \in \sigma_{ii+1}'(e_i(m)) = e_{i+1}(m')$. Then

$$p \cup \{\ulcorner \dot{N}_i \models \dot{m} \in \dot{n} \urcorner, \ulcorner \dot{\sigma}_{ii+1}(\dot{m}) = \dot{m}' \urcorner, \ulcorner \dot{N}_{i+1} \models \underline{\omega_1^{N_i}} \in \dot{m}' \urcorner\} \leq p.$$

This argument shows that the set

$$D_n^8 = \{p \in \mathbb{P} \colon \exists m \, \exists m' \, \{\ulcorner \dot{N}_i \models \dot{m} \in \dot{n} \urcorner, \ulcorner \dot{\sigma}_{ii+1}(\dot{m}) = \dot{m}' \urcorner, \ulcorner \dot{N}_{i+1} \models \underline{\omega_1^{N_i}} \in \dot{m}' \urcorner\} \subset p\}$$

is dense below p_0. Claim 3.11 then follows. □ (Claim 3.11)

Claims 3.10 and 3.11 readily imply that if $i < \omega_1$, then N_{i+1} is a generic ultrapower of N_i. By the next claim, direct limits are taken at limit stages:

Claim 3.12 *Let $i \leq \omega_1$ be a limit ordinal. For every $x \in N_i$ there is some $j < i$ and some $z \in N_j$ such that $x = \sigma_{ji}(z)$.*

Proof of Claim 3.12. This is trivial for $i = \omega_1$. Now let $i < \omega_1$. Let p, Σ, and $\mathfrak{C}$ be as in the previous two proofs. Fix a limit ordinal i. For each $n < \omega$ there are $j < i$ and $n' < \omega$ with $\sigma_{ji}(e_j(n')) = e_i(n)$, where $e_j \colon \omega \to N_j'$ and $e_i \colon \omega \to N_i'$ are as on p. 27. Then

$$p \cup \{\ulcorner \dot{\sigma}_{ji}(\dot{n}') = \dot{n} \urcorner\} \leq p.$$

This argument shows that the set

$$D_n^9 = \{p \in \mathbb{P} \colon \exists n' \, \ulcorner \dot{\sigma}_{ji}(\dot{n}') = \dot{n} \urcorner \in p\}$$

is dense in $\mathbb{P}$. Claim 3.12 then follows. □ (Claim 3.12)

$(N_i, \sigma_{ij} \colon i \leq j \leq \omega_1)$ is then indeed a generic iteration of N_0. As N_0 is iterable, we may and shall identify N_i with its transitive collapse, so that (C.4) holds true.

Another round of density arguments will show that $\mathfrak{C}$ satisfies (C.1), (C.3), (C.5), (C.6), and (C.7), where we identify M_i with the structure $(M_i; \in, (\mathsf{NS}_{\omega_1^{M_i}})^{M_i}, A \cap \omega_1^{M_i})$. Let us now verify (C.8) and (Σ.8), without writing down the relevant dense sets any more.

As for (C.8), its second part, $X_\delta \cap \omega_1 = \delta$ for $\delta \in K$, is easy. We will now use the Tarski-Vaught test to verify the first part of (C.8). Let φ be any formula, and let x_1, …, $x_k \in X_\delta$, $\delta \in K$. Suppose that

$$(Q_{\lambda_\delta}; \in, \mathbb{P}_{\lambda_\delta}, A_{\lambda_\delta}) \models \exists v \, \varphi(v, x_1, \ldots, x_k). \tag{21}$$

Let $p \in g$ be such that $\{\ulcorner \underline{x}_1 \in \dot{X}_\delta \urcorner, \ldots, \ulcorner \underline{x}_k \in \dot{X}_\delta \urcorner, \ulcorner \delta \mapsto \lambda_\delta \urcorner\} \subset p$. Let $q \leq p$, and let Σ be a syntactical certificate for q whose associated semantic certificate is

$$\mathfrak{C}' = \langle \langle M'_i, \pi'_{ij}, N'_i, \sigma'_{ij} \colon i \leq j \leq \omega_1 \rangle, \langle (k'_n, \alpha'_n) \colon n < \omega \rangle, \langle \lambda'_\delta, X'_\delta \colon \delta \in K' \rangle \rangle.$$

Then $\delta \in K'$ and

$$\{x_1, \ldots, x_k\} \subset X'_\delta \prec (Q_{\lambda_\delta}; \in, \mathbb{P}_{\lambda_\delta}, A_{\lambda_\delta}),$$

so that by (21) we may choose some $x \in X'_\delta$ with

$$(Q_{\lambda_\delta}; \in, \mathbb{P}_{\lambda_\delta}, A_{\lambda_\delta}) \models \varphi(x, x_1, \ldots, x_k).$$

Let $r = q \cup \{\ulcorner \underline{x} \in \dot{X}_\delta \urcorner\}$.

By density, there is then some $y \in X_\delta$ such that

$$(Q_{\lambda_\delta}; \in, \mathbb{P}_{\lambda_\delta}, A_{\lambda_\delta}) \models \varphi(y, x_1, \ldots, x_k).$$

The proof of (Σ.8) is similar. Let again $\delta \in K$. Let $E \subset \mathbb{P}_{\lambda_\delta} \cap X^g_\delta$ be dense in $\mathbb{P}_{\lambda_\delta} \cap X_\delta$, and $r \in E$ iff $r \in \mathbb{P}_{\lambda_\delta} \cap X_\delta$ and

$$(Q_{\lambda_\delta}; \in, \mathbb{P}_{\lambda_\delta}, A_{\lambda_\delta}) \models \varphi(r, x_1, \ldots, x_k). \tag{22}$$

Let $p \in g$ be such that $\{\ulcorner \underline{x}_1 \in \dot{X}_\delta \urcorner, \ldots, \ulcorner \underline{x}_k \in \dot{X}_\delta \urcorner, \ulcorner \delta \mapsto \lambda_\delta \urcorner\} \subset p$. Let $q \leq p$, and again let Σ be a syntactical certificate for q whose associated semantic certificate is

$$\mathfrak{C}' = \langle \langle M'_i, \pi'_{ij}, N'_i, \sigma'_{ij} \colon i \leq j \leq \omega_1 \rangle, \langle (k'_n, \alpha'_n) \colon n < \omega \rangle, \langle \lambda'_\delta, X'_\delta \colon \delta \in K' \rangle \rangle.$$

Then $[\Sigma]^{<\omega} \cap X'_\delta$ has an element, say r, such that (22) holds true. Let

$$s = q \cup r \cup \{\ulcorner \underline{r} \in \dot{X}_\delta \urcorner\}$$

By density, then, $g \cap X_\delta \cap E \neq \emptyset$. $\square$ (Lemma 3.7)

Forcing with any $\mathbb{P}_\lambda$ makes ω_2^V ω-cofinal, as the iteration map $\pi_{0\omega_1}$ as being added by the generic filter maps the ordinals of the countable model M_0 cofinally into ω_2^V. If $\lambda < \kappa$ (and $\lambda \in C$), then $\mathbb{P}_\lambda$ has size $\aleph_2$, so that by a result of S. Shelah, see [19, Corollary 23.20], $\mathbb{P}_\lambda$ will collapse ω_1 to become countable. We are now going to prove that $\mathbb{P} = \mathbb{P}_\kappa$, on the other hand, does not collapse ω_1 and in fact preserves stationary subsets of ω_1.

3.3 The forcing preserves stationary sets.

Lemma 3.13 *Let g be $\mathbb{P}$-generic over V. Let*

$$\mathfrak{C} = \langle\langle M_i, \pi_{ij}, N_i, \sigma_{ij} \colon i \leq j \leq \omega_1\rangle, \langle(k_n, \alpha_n) \colon n < \omega\rangle, \langle\lambda_\delta, X_\delta \colon \delta \in K\rangle\rangle$$

be the semantic certificate associated with the syntactic certificate $\bigcup g$. Let

$$N_{\omega_1} = (N_{\omega_1}; \in, A, I^*).$$

Then every element of $(\mathcal{P}(\omega_1) \cap N_{\omega_1}) \setminus I^*$ is stationary in $V[g]$.

Corollary 3.14 *$\mathbb{P}$ preserves stationary subsets of ω_1.*

Proof of Corollary 3.14 from Lemma 3.13. Let $\mathfrak{C}$ be as in the statement of Lemma 3.13, and let us write $M_i = (M_i; \in, I_i, a_i)$ and $N_i = (N_i; \in, I_i^*, a_i^*)$ for $i \leq \omega_1$. In the light of Lemma 3.7, by (C.3) we will have that $I_{\omega_1^{N_0}} = I_0^* \cap M_{\omega_1^{N_0}}$, so that also $I_{\omega_1} = I_{\omega_1}^* \cap M_{\omega_1}$. By (C.5), the universe of M_{ω_1} is $(H_{\omega_2})^V$ and $I_{\omega_1} = (\mathsf{NS}_{\omega_1})^V$, while $I_{\omega_1}^*$ is denoted by I^* in the statement of Lemma 3.13. We thus get that $(\mathsf{NS}_{\omega_1})^V = I^* \cap V$, so that the conclusion of Lemma 3.13 also gives that $\mathbb{P}$ preserves stationary subsets of ω_1. $\square$(Corollary 3.14)

Proof of Lemma 3.13. Let $\dot{N}_{\omega_1} \in V^{\mathbb{P}}$ be a canonical name for N_{ω_1}, and let $\dot{I}^* \in V^{\mathbb{P}}$ be a canonical name for I^*. Let $\bar{p} \in g$, $\dot{C}$, $\dot{S} \in V^{\mathbb{P}}$, and $i_0 < \omega_1$ and $n_0 < \omega$ be such that

(i) $\bar{p} \Vdash$ "$\dot{C} \subset \omega_1$ is club,"

(ii) $\bar{p} \Vdash$ "$\dot{S} \in (\mathcal{P}(\omega_1) \cap \dot{N}_{\omega_1}) \setminus \dot{I}^*$," and

(iii) $\bar{p} \Vdash$ "$\dot{S}$ is represented by $[i_0, \dot{n}_0]$ in the term model producing $\dot{N}_{\omega_1}$."

We may and shall also assume that

$$\ulcorner \dot{N}_{i_0} \models \dot{n}_0 \text{ is a subset of the first uncountable cardinal, yet } \dot{n}_0 \notin \dot{I} \urcorner \in \bar{p}, \tag{23}$$

because the $\mathcal{L}$-formula in (23) must belong to every syntactic certificate for $\bar{p}$, as $\bar{p}$ satisfies (ii) and (iii).

Let $p \leq \bar{p}$ be arbitrary, $p \in \mathbb{P}$. We aim to produce some $q \leq p$ and some $\delta < \omega_1$ such that $q \Vdash \check{\delta} \in \dot{C} \cap \dot{S}$, see Claim 3.17 below.

For $\xi < \omega_1$, let

$$D_\xi = \{q \leq p \colon \exists \eta \geq \xi\, (\eta < \omega_1 \wedge q \Vdash \check{\eta} \in \dot{C})\},$$

so that D_ξ is open dense below p. Let

$$E = \{(q, \eta) \in \mathbb{P} \times \omega_1 \colon q \Vdash \check{\eta} \in \dot{C}\}.$$

Let us write

$$\tau = ((D_\xi \colon \xi < \omega_1), E).$$

We may and shall identify τ with some subset of H_κ which codes τ. Here and in what follows, κ is still equal to ω_3.

By $(\Diamond(\mathbb{P}))$, we may pick some $\lambda \in C$ such that $p \in \mathbb{P}_\lambda$ and

$$(Q_\lambda; \in, \mathbb{P}_\lambda, A_\lambda) \prec (H_\kappa; \in, \mathbb{P}, \tau). \tag{24}$$

Let h be $\mathrm{Col}(\omega, \omega_2)$-generic over V, and let $g' \in V[h]$ be a filter on $\mathbb{P}_\lambda$ such that $p \in g'$ and g' meets every dense set which is definable over $(Q_\lambda; \in, \mathbb{P}_\lambda, A_\lambda)$ from parameters in Q_λ. By Lemma 3.7, $\bigcup g'$ is a syntactic certificate for p, and we may let

$$\langle \langle M'_i, \pi'_{ij}, N'_i, \sigma'_{ij} \colon i \leq j \leq \omega_1 \rangle, \langle (k'_n, \alpha'_n) \colon n < \omega \rangle, \langle \lambda'_\delta, X'_\delta \colon \delta \in K' \rangle \rangle$$

be the associated semantic certificate. In particular, $K' \subset \lambda$.

Let S denote the subset of ω_1 which is represented by $[i_0, \dot{n}_0]$ in the term model giving N'_{ω_1}, so that if $N'_{\omega_1} = (N'_{\omega_1}, \in, A, I')$, then by (23),

$$S \in (\mathcal{P}(\omega_1) \cap N'_{\omega_1}) \setminus I'. \tag{25}$$

Notice that $\omega_1^{V[h]} = \omega_3^V = \kappa$. Inside $V[h]$, we may extend $\langle N'_i, \sigma'_{ij} \colon i \leq j \leq \omega_1 \rangle$ to a generic iteration

$$\langle N'_i, \sigma'_{ij} \colon i \leq j \leq \kappa \rangle$$

such that

$$\omega_1 \in \sigma'_{\omega_1, \omega_1 + 1}(S). \tag{26}$$

This is possible as $\omega_1^{N'_{\omega_1}} = \sup\{\omega_1^{N_j} \colon j < \omega_1\} = \omega_1$ and by (25). Let

$$\langle M'_i, \pi'_{ij} \colon i \leq j \leq \kappa \rangle = \sigma_{0,\kappa}(\langle M'_i, \pi'_{ij} \colon i \leq j \leq \omega_1^{N'_0} \rangle),$$

so that $\langle M'_i, \pi'_{ij} \colon i \leq j \leq \kappa \rangle$ is an extension of $\langle M'_i, \pi'_{ij} \colon i \leq j \leq \omega_1 \rangle$.

Since $M'_{\omega_1} = ((H_{\omega_2})^V; \in, (\mathsf{NS}_{\omega_1})^V, A)$, cf. (16), and $(\mathsf{NS}_{\omega_1})^V$ is assumed to be saturated in V, every maximal antichain in V consisting of stationary subsets of ω_1 is an element of M'_{ω_1}. By [55, Lemma 3.8], we may hence lift the generic ultrapower map $\pi'_{\omega_1\omega_1+1}\colon M'_{\omega_1} \to M'_{\omega_1+1}$ to act on all of V, and inductively we may lift the entire generic iteration $\langle M'_i, \pi'_{ij}\colon \omega_1 \leq i \leq j \leq \kappa\rangle$ to a generic iteration

$$\langle M^+_i, \pi^+_{ij}\colon \omega_1 \leq i \leq j \leq \kappa\rangle$$

of V with all M^+_i, $\omega_1 \leq i \leq \kappa$, being transitive. Let us write $M = M^+_\kappa$ and $\pi = \pi^+_{\omega_1,\kappa}$.

$$\begin{tikzcd}
 & p[T] \subseteq p[\pi(T)] & & \\
 & \Cup & & \\
 & N'_0 \arrow[r, "\sigma'_{0\omega_1}"] & N'_{\omega_1} \arrow[r, "\sigma'_{\omega_1\kappa}"] & N'_\kappa \\
 & \Cup & \Cup & \Cup \\
M'_0 \arrow[r, "{\pi'}_{0\omega_1^{N'_0}}"] & {M'}_{\omega_1^{N_0}} \arrow[r, "{\pi'}_{\omega_1^{N'_0}\omega_1}"] & M'_{\omega_1} \arrow[r, "\pi'_{\omega_1\kappa}"] & M'_\kappa \\
 & & \| & \\
 & & ((H_{\omega_2})^V; \in, (\mathsf{NS}_{\omega_1})^V, A) & \Cap \\
 & & \Cap & \\
 & & V \arrow[r, "\pi"] & M^+_\kappa = M
\end{tikzcd}$$

The key point is now that $\langle M'_i, \pi'_{ij}, N'_i, \sigma'_{ij}\colon i \leq j \leq \kappa\rangle$ may be used to extend $\pi\text{''}\bigcup g'$ to a syntactic certificate

$$\Sigma \supset \pi\text{''}\bigcup g' \tag{27}$$

for $\pi(p)$ in the following manner. Let $K^* = K' \cup \{\omega_1\}$. For $\delta \in K'$, let $\lambda^*_\delta = \pi(\lambda'_\delta)$ and $X^*_\delta = \pi\text{''}X'_\delta$. Also, write $\lambda^*_{\omega_1} = \pi(\lambda)$ and $X^*_{\omega_1} = \pi\text{''}Q_\lambda$. Notice that $\omega_1 \in \pi(C)$, so that $K^* \subset \pi(C)$. Let

$$\mathfrak{C}^* = \langle\langle M'_i, \pi'_{ij}, N'_i, \sigma'_{ij}\colon i \leq j \leq \kappa\rangle, \langle (k'_n, \pi(\alpha'_n))\colon n < \omega\rangle, \langle \lambda^*_\delta, X^*_\delta\colon \delta \in K^*\rangle\rangle.$$

Claim 3.15 *$\mathfrak{C}^*$ is a semantic certificate for $\pi(p)$ relative to the parameters $\pi(A)$, $\pi(T)$, $\pi(H^V_{\omega_2})$, $\pi(\langle A_\mu : \mu \in C \cap \lambda\rangle)$, and $\pi(\langle \mathbb{P}_\mu : \mu \in C \cap \lambda\rangle)$.*

PROOF of Claim 3.15: First notice that

$$\langle (k'_n, \pi(\alpha'_n))\colon n < \omega\rangle \in [\pi(T)]$$

Next, if $\delta \in K'$, then

$$X^*_\delta = \pi\text{''}X'_\delta \prec (\pi(Q_{\lambda'_\delta}); \in, \pi(\mathbb{P}_{\lambda'_\delta}), \pi(A_{\lambda'_\delta})),$$

and $\pi"g' \cap X^*_\delta = \pi"(g' \cap X'_\delta)$; as $\bigcup g'$ is a syntactic certificate for p, we thus have that $\pi"g' \cap X^*_\delta \cap E \neq \emptyset$ for every $E \subset \pi(\mathbb{P}_{\lambda'_\delta})$ which is dense in $\pi(\mathbb{P}_{\lambda'_\delta})$ and definable over the structure $(\pi(Q_{\lambda'_\delta}); \in, \pi(\mathbb{P}_{\lambda'_\delta}), \pi(A_{\lambda'_\delta}))$ from parameters in X^*_δ. Finally, $X^*_{\omega_1} = \pi"Q_\lambda$ and the choice of g' imply that $\pi"g' \cap X^*_{\omega_1} \cap E \neq \emptyset$ for every $E \subset \pi(\mathbb{P}_\lambda)$ which is dense in $\pi(\mathbb{P}_\lambda)$ and definable over the structure

$$(\pi(Q_\lambda); \in, \pi(\mathbb{P}_\lambda), \pi(A_\lambda))$$

from parameters in $X^*_{\omega_1}$. This buys us that $\mathfrak{C}^*$ is indeed a semantic certificate for $\pi(p)$ as an element of $\pi(\mathbb{P})$, and that therefore there is some syntactic certificate Σ as in (27), relative to $\pi(A)$, $\pi(T)$, $\pi(H^V_{\omega_2})$, $\pi(\langle A_\mu : \mu \in C \cap \lambda\rangle)$, and $\pi(\langle \mathbb{P}_\mu : \mu \in C \cap \lambda\rangle)$, such that $\mathfrak{C}^*$ is certified by Σ. $\square$ (Claim 3.15)

Now let $[\dot{m}_0]_{\omega_1+1}$ represent $\sigma'_{\omega_1\omega_1+1}(S)$ in the term model for N'_{ω_1+1} provided by Σ, so that[30]

$$\{\ulcorner \dot{\sigma}_{i_0\omega_1+1}(\dot{n}_0) = \dot{m}_0 \urcorner, \ulcorner \dot{N}_{\omega_1+1} \vDash \underline{\omega_1} \in \dot{m}_0 \urcorner\} \subset \Sigma,$$

in other words,

$$\pi(p) \cup \{\ulcorner \dot{\sigma}_{i_0\omega_1+1}(\dot{n}_0) = \dot{m}_0 \urcorner, \ulcorner \dot{N}_{\omega_1+1} \vDash \underline{\omega_1} \in \dot{m}_0 \urcorner\} \text{ is certified by } \Sigma. \tag{28}$$

Let us now define

$$q^* = \pi(p) \cup \{\ulcorner \dot{\sigma}_{i_0\omega_1+1}(\dot{n}_0) = \dot{m}_0 \urcorner, \ulcorner \dot{N}_{\omega_1+1} \vDash \underline{\omega_1} \in \dot{m}_0 \urcorner, \ulcorner \underline{\omega_1} \mapsto \underline{\pi(\lambda)} \urcorner\}. \tag{29}$$

In the light of Lemma 3.4, we thus established the following.

Claim 3.16 *$q^* \in \pi(\mathbb{P})$, as being certified by Σ.*

The elementarity of $\pi \colon V \to M$ then gives some $\delta < \omega_1$ such that

$$q = p \cup \{\ulcorner \dot{\sigma}_{i_0\delta+1}(\dot{n}_0) = \dot{m}_0 \urcorner, \ulcorner \dot{N}_{\delta+1} \vDash \underline{\delta} \in \dot{m}_0 \urcorner, \ulcorner \underline{\delta} \mapsto \underline{\lambda} \urcorner\} \in \mathbb{P}. \tag{30}$$

Claim 3.17 *$q \Vdash \check{\delta} \in \dot{C} \cap \dot{S}$.*

Proof of Claim 3.17. $q \Vdash \check{\delta} \in \dot{S}$ readily follows from

$$\{\ulcorner \dot{\sigma}_{i_0\delta+1}(\dot{n}_0) = \dot{m}_0 \urcorner, \ulcorner \dot{N}_{\delta+1} \vDash \underline{\delta} \in \dot{m}_0 \urcorner\} \subset q,$$

---

[30] Here, $\dot{\sigma}_{i_0\omega_1+1}$ and $\dot{N}_{\omega_1+1}$ are terms of the language associated with $\pi(\mathbb{P}_\lambda)$, and $\ulcorner \dot{\sigma}_{i_0\omega_1+1}(\dot{n}_0) = \dot{m}_0 \urcorner$ and $\ulcorner \dot{N}_{\omega_1+1} \vDash \underline{\omega_1} \in \dot{m}_0 \urcorner$ are formulae of that language.

the fact that $\bar{p} \geq p$ forces that $\dot{S}$ is represented by $[i_0, \dot{n}_0]$ in the term model giving $\dot{N}_{\omega_1}$, and the fact that by Claim 3.9, $[\underline{\delta}]_{\delta+1}$ represents δ in the model $N_{\delta+1}$ of any semantic certificate for q as being given by a generic which contains q.

Let us now show that $q \Vdash \check{\delta} \in \dot{C}$. We will in fact show that q forces that $\check{\delta}$ is a limit point of $\dot{C}$. Otherwise there is some $r \leq q$ and some $\eta < \delta$ such that

$$r \Vdash \dot{C} \cap \check{\delta} \subset \check{\eta}. \tag{31}$$

Suppose that r is certified by Σ, so that there is some

$$\langle \langle M'_i, \pi'_{ij}, N'_i, \sigma'_{ij} \colon i \leq j \leq \omega_1 \rangle, \langle (k'_n, \alpha'_n) \colon n < \omega \rangle, \langle \lambda'_{\bar{\delta}}, X'_{\bar{\delta}} \colon \bar{\delta} \in K' \rangle \rangle \tag{32}$$

which is certified by Σ and $r \in [\Sigma]^{<\omega}$. We must have that

(a) $\delta \in K'$,

(b) $X'_\delta \prec (Q_\lambda; \in, \mathbb{P}_\lambda, A_\lambda)$,

(c) $X'_\delta \cap \omega_1 = \delta$, and

(d) $[\Sigma]^{<\omega} \cap X'_\delta \cap E \neq \emptyset$ for every $E \subset \mathbb{P}_\lambda$ which is dense in $\mathbb{P}_\lambda \cap X'_\delta$ and definable over the structure

$$(Q_\lambda; \in, \mathbb{P}_\lambda, A_\lambda)$$

from parameters in X'_δ.

Here, (a) is given by $\ulcorner \underline{\delta} \mapsto \underline{\lambda} \urcorner \in r$, (b) and (c) are given by (C.8), while (d) is exactly what (Σ.8) on p. 28 buys us.

We have that $A_\lambda = \tau \cap Q_\lambda$, and hence A_λ may be identified with the ordered pair $((D_\xi \cap Q_\lambda \colon \xi < \omega_1), E \cap Q_\lambda)$. As $\eta < \delta \subset X'_\delta$, D_η is definable over the structure

$$(Q_\lambda; \in, \mathbb{P}_\lambda, A_\lambda)$$

from a parameter in X'_δ. By (24), $D_\eta \cap Q_\lambda$ is dense in $\mathbb{P}_\lambda$. By (d) above, there is then some $s \in [\Sigma]^{<\omega} \cap X'_\delta \cap D_\eta$.

By (24) again, the unique smallest $\eta' \geq \eta$ with $s \Vdash \check{\eta}' \in \dot{C}$ must be in X'_δ, hence $\eta' < \delta$ by (c) above. But now s is compatible with r, as they are both finite subsets of the very same Σ which certifies them (cf. Lemma 3.5). We have reached a contradiction with (31). $\square$ (Claim 3.17)

Now $\dot{C}$, $\dot{S}$, and $\bar{p} \in g$ were such that (i) through (iii) on p. 38 hold true. We showed that the set of all $q \leq \bar{p}$ with $q \Vdash \dot{C} \cap \dot{S} \neq \emptyset$ is dense. As $\dot{C}$ was arbitrary, this buys us that $\dot{S}^g$ will be stationary in $V[g]$. But then as $\dot{S}$ was arbitrary, this means that every element of $(\mathcal{P}(\omega_1) \cap N_{\omega_1}) \setminus I^*$ will be stationary in $V[g]$. $\square$ (Lemma 3.13)

4 Open questions.

Woodin [55] also introduced the axiom $(*)^+$ as a strengthening of $(*)$. $(*)^+$ says that there is some pointclass $\Gamma \subset \mathcal{P}(\mathbb{R})$ and some filter $g \subset \mathbb{P}_{\max}$ such that

(1) $L(\Gamma, \mathbb{R}) \vDash \mathsf{AD}^+$,[31]

(2) g is $\mathbb{P}_{\max}$-generic over $L(\Gamma, \mathbb{R})$, and

(3) $\mathcal{P}(\mathbb{R}) \subset L(\Gamma, \mathbb{R})[g]$.

See [55, p. 908]. While the main result of the current paper gives a new twist to the question if MM is compatible with $(*)^+$, see [55, p. 923, Question (15) a)], it also leaves this question wide open. See [60].

There is a strengthening of MM^{++}, isolated by Viale [53], which has strong completeness properties modulo forcing similar to those of $(*)$. This is the axiom MM^{+++}. It says that a class $\mathbb{T}$ of towers of ideals with certain nice structural properties is dense in the category of stationary set preserving forcings; in other words, for every stationary set preserving forcing $\mathbb{P}$ there is a tower $\mathcal{T}$ in $\mathbb{T}$ such that $\mathbb{P}$ completely embeds into $\mathcal{T}$ in such a way that the quotient forcing preserves stationary sets in $V^{\mathbb{P}}$. MM^{+++} implies MM^{++}, if κ is an almost super-huge cardinal, then there is a partial order $\mathbb{P} \subset V_\kappa$ which forces MM^{+++}, and if there is a proper class of almost super-huge cardinals, then MM^{+++} is complete for the theory of the ω_1-Chang model[32] with respect to stationary set preserving partial orders forcing MM^{+++}.

Schindler [42, Definition 2.10] introduces $\mathsf{MM}^{*,++}$ as a strengthening of MM^{++} by relaxing "forceable by a stationary set preserving forcing" to "honestly consistent" in an appropriate formulation of MM^{++}, see [42].

It remains open if either of MM^{+++} or $\mathsf{MM}^{*,++}$ is really stronger than MM^{++}. While Viale's MM^{+++} is known to be consistent modulo a super-huge cardinal, it is open at this point if $\mathsf{MM}^{*,++}$ is consistent at all relative to large cardinals.

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[31] AD^+ is a natural strengthening of AD which was introduced by H. Woodin, see e.g. [55, Definition 9.6].

[32] The ω_1-Chang model is the $\subseteq$-minimal transitive model of ZF containing all ordinals and closed under ω_1-sequences. It can be construed as $\bigcup_{\alpha \in \mathrm{Ord}} L([\alpha]^{\aleph_1})$ and it includes $L(\mathcal{P}(\omega_1))$ as a definable submodel.

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