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NOTES FOR A TALK AT WUHAN UNIVERSITY, DEC 08, 2020: HOW MANY REAL NUMBERS ARE THERE?

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I would like to thank Yong Cheng for inviting me to give this talk. I would also like to thank my commentator, Qi Feng. And I want to thank all the people who attended my talk with such great interest in the topic. These notes are dedicated to the people of Wuhan who showed so much willpower and strength January 23-April 08, 2020.

People have always reasoned about mathematical objects in a coherent and rational way.

The ancient Greeks did so mostly about geometrical structures, but there is also e.g. the proof in Euclid that there are infinitely many prime numbers:

Assume $p_1, \ldots, p_n$ to be prime. Look at $q = (p_1 \cdot \ldots \cdot p_n) + 1$. None of the p_i, $1 \leq i \leq n$, divides q. As q may be written as a unique product of primes, there must then be a prime which is different from all the p_i, $1 \leq i \leq n$.

This proof presupposes: the natural numbers are out there, they exist and we can reason about them, we have access to their fundamental properties. This attitude towards natural numbers hasn't changed ever since, it is the charm of number theory: we have a very clear intuition about the natural numbers 0, 1, 2, ... and their structure. Same with the rational numbers $\pm\frac{n}{q}$, n, $q \in \mathbb{N}$, $q \neq 0$.

What about more complicated mathematical objects?

While the Greeks essentially had a proof that $\sqrt{2}$ is irrational, they hesitated isolating the general concept of *real numbers*. There is a lot of historical discussion about this.

This was done much later, by R. Dedekind, G. Cantor, and others in the 19^{th} century.

Moving on from natural numbers to real numbers needs a step of abstraction. You have to talk e.g. about arbitrary sequences of natural numbers (or, of rational numbers, for that matter) and equivalence classes thereof.

The step of abstraction is: you see something from inside (e.g. your world is the world of natural numbers), and then you step out to see it as a totality, as a collection, as a set and you *make it an object of mathematical discourse* (e.g. you move on from the elements of $\mathbb{N}$ to $\mathbb{N}$ as a set and you start reasoning about arbitrary sets or sequences of elements of $\mathbb{N}$, etc.) The Greeks were not courageous enough to do so.

Cantor introduced the real numbers via "Fundamentalfolgen," nowadays called Cauchy sequences.

The rational numbers may be identified with pairs of natural numbers (with a sign in front), which is an easy step of abstraction. A sequence $\vec{q} = (q_n \colon n \in \mathbb{N})$ of rational numbers is a Cauchy sequence iff

$$\forall \epsilon > 0 \, \exists n_0 \, \forall m, n \geq n_0 \, |q_m - q_n| < \epsilon.$$

We stipulate $\vec{q} \sim \vec{r}$ iff $\forall \epsilon > 0 \, \exists n_0 \, \forall n \geq n_0 |r_n - q_n| < \epsilon$. This is an equivalence relation by the triangle inequality. We then factorize and let $[\vec{q}] = \{\vec{r} \colon \vec{q} \sim \vec{r}\}$. We may define $+$, $\cdot$, and exponentiation for these, e.g.

$$[\vec{q}] + [\vec{r}] = [n \mapsto q_n + r_n].$$

This is well-defined, i.e., independent from the choice of the representative. The equivalence classes $[\vec{q}]$ of Cauchy sequences of rational numbers "are" the real numbers.

We did a couple of steps in the process "inside $\rightarrow$ outside":

$$\mathbb{N} \to \mathbb{Q} \to \text{sequences of elements of } \mathbb{Q} \to \text{factorize}.$$

Set theoretically speaking, this corresponds to

$$\mathbb{N} \to \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathcal{P}(\mathbb{N})).$$

Using simple set theoretical equipment, one may then *prove* fundamental propertries about the real numbers, e.g. the *nested interval principle*:

If $([a_n, b_n] \colon n \in \mathbb{N})$ is a sequence of closed intervals with real endpoints such that $a_n \leq a_{n+1} \leq b_{n+1} \leq b_n$ for all $n \in \mathbb{N}$, then $\bigcap_{n\in\mathbb{N}}[a_n, b_n] \neq \emptyset$.

Proof: We may easily cook up a Cauchy sequence $\vec{q} = (q_n : n \in \mathbb{N})$ of rational numbers such that $a_n \leq q_n \leq b_n$ for all $n \in \mathbb{N}$. It is then easy to verify that $[\vec{q}] \in [a_n, b_n]$ for all $n \in \mathbb{N}$.

G. Cantor in his paper "Über eine Eigenschaft des Inbegriffes aller reellen algebraischen Zahlen" from 1874 used the nested interval principle to give his first proof of the uncountability of $\mathbb{R}$:[1]

Let $f \colon \mathbb{N} \to \mathbb{R}$. Write $x_n = f(n)$ for $n \in \mathbb{N}$. It is easy to pick a nested sequence $([a_n, b_n] \colon n \in \mathbb{N})$ of closed intervals as above such that for each $n \in \mathbb{N}$, $x_n \notin [a_n, b_n]$. If $x \in \bigcap_{n\in\mathbb{N}}[a_n, b_n]$, then $x \neq x_n$ for all $n \in \mathbb{N}$.

There is thus no surjection $f \colon \mathbb{N} \to \mathbb{R}$.

There was courage involved in looking at things at this level of abstraction, but once somebody tells you the argument, it is *as clear and convincing* as the proof that there are infinitely many primes.

Cantor went ahead and pushed the natural process "inside $\to$ outside" to its limits. He went on to consider *arbitrary* sets of reals. While in order to arrive at a rigorous construction of the reals you need to consider arbitrary sequences of natural (viz., rational) numbers, the next natural step now is to consider arbitrary sets of real numbers, in fact arbitrary sets whatsoever.

Cantor isolated the concept of the *cardinality* of a set, making the "paradox" according to which we may have $A \subsetneq B$, while A and B have the same size, obsolete. E.g., the set of primes $\subsetneq \mathbb{N}$, while there is a bijection from A onto B, as we saw above. Cantor's solution: A and B have the same size (cardinality) iff there is a bijection $f \colon A \to B$.

This immediately leads to the *continuum problem*: can you have an infinite $A \subset \mathbb{R}$ such that A is neither of the same size as $\mathbb{N}$ nor of the same size as $\mathbb{R}$? It is very hard to come up with counterexamples:

The theorem of Cantor-Bendixson says that if $A \subset \mathbb{R}$ is closed, then A is either at most countable or of the same size as $\mathbb{R}$. This was later generalized to Borel sets and in fact analytic sets (by Suslin).

What about more complicated sets? Coanalytic sets? What about arbitrary sets?

The continuum problem has been one of the driving forces of set theory ever since.

Let us summarize and comment on what we saw so far. The natural process "inside $\to$ outside" lets us climb up the hierarchy of sets. We may reason in the same coherent and rational way about sets at more compliacted levels of the set theoretical hierarchy as we may about natural numbers. We have an intuition about the structures involved, some rational way of accessing them. That we cannot solve all the questions on the spot doesn't mean that we don't have a good intuition and that there will never be answers. Sometimes intuition has to grow. Then we move on like in physics where also very abstract entities are considered (particles, quarks, black holes, the beginning of space-time – some of these things being at least as speculative as the objects set theorists talk about) and where the process of trying out theories, looking at their consequences, amalgamating competing theories - brings significant progress. The objects of mathematics are out there to be investigated. Don't confuse truth with provability/with easy accessibility.

[1] G. Cantor, *Über eine Eigenschaft des Inbegriffes aller reellen algebraischen Zahlen*, Crelles Journal f. Mathematik **77** (1874), pp. 258-262.

The objects of mathematics (and of set theory in particular!) are very abstract, but so are many objects in other areas of science and in fact even the ones of our everyday life. Think about what the People's Republic of China (say as a legal entity) is. Pretty abstract! Or think about what money is. Pretty abstract. Even "this piece of wax"[2] is not as concrete as it may look like at first glance.

How about the continuum problem? The continuum hypothesis (CH, for short) states that every uncountable $A \subset \mathbb{R}$ is of the same size as $\mathbb{R}$. K. Gödel showed in the 30'ies that CH is consistent with what had become by then the standard axiom system of set theory, ZFC. Gödel believed (at least for part of his life) that CH is false, in fact that the cardinality of $\mathbb{R}$ is $\aleph_2$.

Cantor's program was: show CH for for and more compliacted sets of reals. Gödel was suspicious that this will eventually prove CH. As he showed CH is consistent but also believed that CH is false, he actually believed CH to be independent from ZFC, a view that was later (in the 60'ies) confirmed by P. Cohen. Combining the work of Gödel and Cohen, there may be a coanalytic counterexample to the continuum hypothesis. The following quote from Gödel states his own program for addressing the continuum problem.[3]

> [...] one may on good reason suspect that the role of the continuum problem in set theory will be this, that it will finally lead to the discovery of new axioms which will make it possible to disprove Cantor's conjecture.

I claim that over the last decades, such new axioms emerged. We have two prominent competing sets of axioms which both negatively settle CH. The two theories around those axioms (the theory of strong forcing axioms and $\mathbb{P}_{\max}$ theory) got partially amalgamated in 2019 by work of David Asperó and myself which you may take as supporting the view that a natural set of axioms has been found which decides how many real numbers there are.

Let me try to explain. I want to stress that I will be biased and ignore arguments in favor of CH (cf. Woodin's "ultimate-L program") or in favor of attempts to show that $2^{\aleph_0}$ is strictly bigger than $\aleph_2$ (there isn't much here - Cohen supported the view that $2^{\aleph_0}$ should be weakly inaccessible, as the operation $\kappa \mapsto \kappa^+$ does not capture the operation $\kappa \mapsto 2^\kappa$).

Luzin early on had proposed that $2^{\aleph_0} = 2^{\aleph_1}$ (a statement which became known as "Luzin's hypothesis"), which implies that CH is false. Gödel (somewhat unsuccessfully) experimented with axioms which he hoped would show $2^{\aleph_0} = \aleph_2$.

Much later Foreman-Magidor-Shelah introduced *Martin's Maximum* (MM, for short) and its strengtehning MM^{++} as an ultimate strengthening of Martin's axiom, MA_{ω_1}. MM is a forcing axiom, and as MA_{ω_1} it may be construed as a generalization of the Baire category theorem: if $(D_n \colon n \in \mathbb{N})$ is a sequence of open dense sets of reals, then $\bigcap_{n\in\mathbb{N}} D_n$ is dense (this may be shown in the same fashion as Cantor's result which we proved above). The forcing axioms result from the Baire category theorem by allowing more dense sets in more complicated spaces than $\mathbb{R}$.

The formal statement of MM^{++} is as follows. Suppose $\mathbb{P}$ is a partial order which as a forcing preserves stationary subsets of ω_1. Let $(D_i \colon i < \omega_1)$ be a collection of dense sets in $\mathbb{P}$, and let $(\tau_i \colon i < \omega_1)$ be a collection of names for stationary subsets of ω_1. There is then a filter g such that $g \cap D_i \neq \emptyset$ for all $i < \omega_1$ and $\tau_i^g = \{\xi \colon \exists p \in g\, p \Vdash_{\mathbb{P}} \check{\xi} \in \tau_i\}$ is stationary for all $i < \omega_1$. MM^{++} may be shown to be consistent by forcing over a model of ZFC which has a supercompact cardinal.

Already MA_{ω_1} proves that $2^{\aleph_0} \geq \aleph_2$, the proof uses Cohen forcing.

MM actually proves that $2^{\aleph_0} = \aleph_2$.

Proof: Let $(T_i \colon i < \omega_1)$ be a maximal antichain of stationary subsets of ω_1, and let $S \subset S^\omega_{\omega_2} = \{\xi < \omega_2 \colon \mathrm{cf}(\xi) = \omega\}$ be stationary and costationary. Let $X \subset \omega_1$. Let $\mathbb{P} = \mathbb{P}_X$ be the set of all p such that for some $\alpha < \omega_1$, $p \colon \alpha + 1 \to S^\omega_{\omega_2}$ is strictly increasing and contiuous and such that for

[2] Hi to René Descartes! See e.g. `https://en.wikipedia.org/wiki/Wax_argument` .

[3] K. Gödel, *What is Cantor's continuum problem?*, Amer. Math. Monthly **54** (1947), pp. 515-525.

all $\xi \leq \alpha$,

$$\xi \in \bigcup_{i \in X} T_i \iff p(\xi) \in S.$$

It is not hard to show that $\mathbb{P}$ preserves stationary subsets of ω_1. Namely, let $T \subset \omega_1$ be stationary, and let $p \Vdash_{\mathbb{P}} \tau$ is club in ω_1. Assume that $T \cap T_i$ is stationary.

Case 1. $i \in X$. Pick Y countable, p, τ, $\mathbb{P} \in Y$, $Y \prec H_\theta$, $\beta = \sup(Y \cap \omega_2) \in S$, and $\alpha = Y \cap \omega_1 \in T \cap T_i$. It is easy to extend p to a condition $q\colon \alpha + 1 \to S^{\omega}_{\omega_2}$ such that $q(\alpha) = \beta$ and $q \Vdash_{\mathbb{P}} \alpha \in \tau$.

Case 2. $i \notin X$. Pick Y countable, p, τ, $\mathbb{P} \in Y$, $Y \prec H_\theta$, $\beta = \sup(Y \cap \omega_2) \notin S$, and $\alpha = Y \cap \omega_1 \in T \cap T_i$. Again it is easy to extend p to a condition $q\colon \alpha + 1 \to S^{\omega}_{\omega_2}$ such that $q(\alpha) = \beta$ and $q \Vdash_{\mathbb{P}} \alpha \in \tau$.

Applying MM, we then get a contionuous increasing function $F_X\colon \omega_1 \to S^{\omega}_{\omega_2}$ such that for all $\xi < \omega_1$,

$$\xi \in \bigcup_{i \in X} T_i \iff F(\xi) \in S.$$

Let us write $\beta_X = \sup F_X{}''\omega_1$.

It is not hard to verify that $X \mapsto \beta_X$ must be injective. This is because if $i \in X \setminus X'$, but $\beta = \beta_X = \beta_{X'}$, then $F_X{}''T_i$ is a stationary subset of $S \cap \beta$, $F_{X'}{}''T_i$ is a stationary subset of $\beta \setminus S$, while $\{\xi\colon F_X(\xi) = F_{X'}(\xi)\}$ is club.

There are many other consequences of MM, e.g. acg (admissible club guessing, which in turn gives $u_2 = \omega_2$) or φ_{AC} and ψ_{AC} (which both give $2^{\aleph_0} = \aleph_2$, ψ_{AC} being a ‘´localized version” of what we got to see that MM $\Longrightarrow 2^{\aleph_0} = \aleph_2$). All these consequences are Π_2 over the structure H_{ω_2}. There are other such statements which had not been known to follow from MM, e.g.

(A) For all stationary and costationary subsets of ω_1 there is a real x and some g which is $\mathrm{Col}(\omega, \omega_1^V)$-generic over $L[x]$ such that $L[x, S] = L[x, g]$.
(B) For all $A \subset \omega_1$ there is an amenably closed $B \subset \omega_1$ (i.e., for all $D \subset \omega_1$, if $D \cap \xi \in L[B]$ for all $\xi < \omega_1$, then $D \in L[B]$) with $A \in L[B]$.
(C) $(*)$.

There is also a whole variety of more global consequences of MM, e.g.: there are no Suslin lines, the union of less than $\mathbb{R}$ may null/meager sets is null/meager, there is a non-free Whitehead group (Shelah), all $\aleph_1$ dense sets of reals are order-isomorphic (Baumgartner), there is a 5-element basis for uncountable linear orders (J. Moore), every automorphism of the Calkin algebra of a separable Hilbert space is inner (I. Farah).

The axiom $(*)$ which was mentioned above looks like a strengthening of local consequences of MM. $(*)$ was introduced by W.H. Woodin, and it had been know to imply all the $\Pi_2^{H_{\omega_2}}$ statements listed above (including (A) and (B)). (*) says that $L(\mathbb{R})$ is a determinacy model and there is a $\mathbb{P}_{\max}$-generic filter g over $L(\mathbb{R})$ such that $\mathcal{P}(\omega_1) \in L(\mathbb{R})[g]$.

MM and $(*)$ lead to competing theories with their own techniques and sets of results. In contrast to MM, it had not been known how to obtain (*) by forcing over a model of ZFC. Rather, the only known models of (*) and strengthenings thereof were obtained by forcing with $\mathbb{P}_{\max}$ over $L(\mathbb{R})$ or over stronger determinacy models.

Forcing axioms like MM spell out the guiding principle of “Maximize!” - the universe of sets should be rich, saturated with generic filters for forcings which consistently may exist. (*) spells out a local version of this, allegedly in a stronger form: in the presence of large cardinals, if a Π_2 statement about H_{ω_2} is Ω-consistent, then it Ω-follows from (*). This is a completeness theorem for (*).

Ω-consistency is a strong form of consistency. While both MM and (*) express maximality principles, $\Pi_2^{H_{\omega_2}}$ maximality as stated in the preceeding paragraph was not known to follow from MM.

We now have the following.

Theorem (2019, with D. Asperó) MM^{++} implies (*).

As a corollary we get that in the presence of large cardinals, (*) is actually *equivalent* to a bounded forcing axiom which follows from MM^{++}, namely to $\mathcal{P}(\mathbb{R}) \cap L(\mathbb{R})$-$\mathrm{BMM}^{++}$. So in the end, (*) turned out to be a forcing axiom.

This is not the end of the story, though, and in fact many open questions remain. My result with Asperó doesn't fully amalgamate forcing axiom theory with (*) theory. For instance, can we force MM over a (strong) model of determinacy? The result of Asperó and mine is asymmetric: it only shows that (*) may be forced over a model of ZFC. Maybe MM may be directly construed as a strong form of (*)? Also, there is a strengthening of (*), namely $(*)^{++}$ (also introduced by W.H. Woodin), which says that there is a determinacy model W and a $\mathbb{P}_{\max}$-generic filter g over W such that $\mathcal{P}(\mathbb{R}) \in W[g]$. If $(*)^{++}$ can be shown to be compatible with MM, then this would require developping new techniques for producing models of MM.

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