Semi-proper forcing, remarkable cardinals, and Bounded Martin's Maximum

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Abstract

We show that $L(\mathbb{R})$ absoluteness for semi-proper forcings is equiconsistent with the existence of a remarkable cardinal, and hence by [6] with $L(\mathbb{R})$ absoluteness for proper forcings. By [7], $L(\mathbb{R})$ absoluteness for stationary set preserving forcings gives an inner model with a strong cardinal.

By [3], the Bounded Semi-Proper Forcing Axiom (BSPFA) is equiconsistent with the Bounded Proper Forcing Axiom (BPFA), which in turn is equiconsistent with a reflecting cardinal. We show that Bounded Martin's Maximum (BMM) is much stronger than BSPFA in that if BMM holds then for every $X \in V$, $X^{\#}$ exists.

1 Introduction.

The present paper is concerned with principles of absoluteness. Let Γ be a class of posets. We say that boldface $L(\mathbb{R})$ absoluteness for Γ holds if and only if for each $\mathbb{P} \in \Gamma$ and for each G which is $\mathbb{P}$-generic over V there is an elementary embedding

$$\pi\colon L(\mathbb{R}^V) \to L(\mathbb{R}^{V[G]})$$

which is the identity on the ordinals. We say that the Bounded Forcing Axiom holds for Γ if and only if for each $\mathbb{P} \in \Gamma$ and for each G which is $\mathbb{P}$-generic over V,

$$H_{\omega_2}^V \prec_{\Sigma_1} H_{\omega_2}^{V[G]}.$$

By [5, Theorem 1], if there is an iterable inner model with infinitely many Woodin cardinals which are all countable in V then boldface $L(\mathbb{R})$ absoluteness holds for proper forcings. On the other hand, boldface $L(\mathbb{R})$ absoluteness for semi-proper forcings need not be true even in the presence of large cardinals (cf. the discussion in [5, p. 801]). It was shown in [6] and [8] that boldface $L(\mathbb{R})$ absoluteness for proper forcings is equiconsistent with a remarkable cardinal, and is hence much weaker than what [5] had used in order to prove boldface $L(\mathbb{R})$ absoluteness for proper forcings.

Recall that a cardinal κ is remarkable iff for every regular cardinal $\theta > \kappa$ there are π, M, $\bar{\kappa}$, σ, N, and $\bar{\theta}$ such that the following hold:

- $\pi\colon M \to H_\theta$ is an elementary embedding,
- M is countable and transitive,
- $\pi(\bar{\kappa}) = \kappa$,
- $\sigma\colon M \to N$ is an elementary embedding with critical point $\bar{\kappa}$,
- N is countable and transitive,
- $\bar{\theta} = M \cap OR$ is a regular cardinal in N, $\sigma(\bar{\kappa}) > \bar{\theta}$, and
- $M = H^N_{\bar{\theta}}$, i.e., $M \in N$ and $N \models$ "M is the set of all sets which are hereditarily smaller than $\bar{\theta}$" (cf. [6, Definition 0.4]).

We shall make use of the key characterization of "remarkability" according to which a cardinal κ is remarkable iff whenever G is $\mathrm{Col}(\omega, < \kappa)$-generic over V then in $V[G]$, for all regular cardinals $\theta > \kappa$ there are stationarily many countable $X \prec (H^V_\theta)[G]$ such that if $\bar{H} \cong X$ is transitive then $\bar{H} = (H^V_{\bar{\theta}})[G \cap V_\alpha]$ for some regular V-cardinal $\bar{\theta} < \kappa$ and $\alpha = \kappa \cap X$ (cf. [6, Lemma 0.7].)

We here show the following.

Theorem 1.1 *If* ZFC+ *"there is a remarkable cardinal" is consistent then so is* ZFC+ *"boldface $L(\mathbb{R})$ absoluteness holds for semi-proper forcings."*

In the light of [6] and [8], we thus get:

Corollary 1.2 *The following theories are equiconsistent.*

(1) ZFC+ *"boldface $L(\mathbb{R})$ absoluteness holds for proper forcings."*
(2) ZFC+ *"boldface $L(\mathbb{R})$ absoluteness holds for semi-proper forcings."*
(3) ZFC+ *"there is a remarkable cardinal."*

The Bounded Proper Forcing Axiom (BPFA) is the Bounded Forcing Axiom for the class of proper forcings, the Bounded Semi-Proper Forcing Axiom (BSPFA) is the Bounded Forcing Axiom for the class of semi-proper forcings, and Bounded Martin's Maximum (BMM) is the Bounded Forcing Axiom for the class of stationary set preserving forcings. By [3], BPFA is equiconsistent with BSPFA (by the way, this is why one might have expected Corollary 1.2 to be true), which in turn is equiconsistent with a reflecting cardinal.

Shelah has shown that the Semi-Proper Forcing Axiom (SPFA) is equivalent with Martin's Maximum (MM) (cf. [10]). Asperó and Welch (cf. [1, Theorem 3.5]) have shown that this is not true for the bounded versions of these forcing axioms: BSPFA does not imply BMM. It was an open problem, however, to decide whether BSPFA does imply BMM at least consistency-wise. It wasn't even known if $\omega_1^L = \omega_1^V$ is consistent with BMM. In this paper we shall prove the following.

Theorem 1.3 *Suppose that* BMM *holds. Then for every $X \in V$, $X^\#$ exists.*

The key technical lemma which will give Theorem 1.3 is Lemma 3.3; this lemma is shown by designing a refined version of Jensen's "reshaping."[1]

Our Theorem 1.3 can also be construed as a negative result on iterating stationary preserving forcings. (Such negative results have also been proven long ago by Shelah.)

Theorem 1.3 still leaves the question on the consistency strength of BMM wide open. Woodin has shown the following (cf. [12]).

Theorem 1.4 (Woodin) *If* ZFC+ *"there are (in order type) $\omega + 1$ many Woodin cardinals" is consistent then so is* ZFC + BMM.

We still do not know, though, how "the" natural model of BMM looks like.

On the other hand, Theorem 1.3 is "optimal" in the following sense: by [4], if V is closed under the #-operation (and hence by Theorem 1.3, if BMM holds) then V is (2-step) Σ_3^1 absolute. On the other hand, the proof of Theorem 1.4 shows that BMM does not imply that V is Σ_4^1 absolute. It is tempting to conjecture that BMM is in fact fairly weak in consistency strength.

The author has shown in subsequent work that BMM implies that there is an inner model with a strong cardinal (cf. [9]).

2 Semi-proper forcing and remarkable cardinals.

The following criterion for the construction of models of absoluteness is part of the folklore (cf. [2, p. 1384f.] or [6, p. 181f.]).

Let Γ be a definable class of posets, and let κ be an inaccessible cardinal. Let us write $W = V^{\mathrm{Col}(\omega,<\kappa)}$. Suppose that if $\mathbb{P} \in \Gamma^W$ then every real in $W^{\mathbb{P}}$ is "small generic" over V, i.e., if $x \in \mathbb{R} \cap W^{\mathbb{P}}$ then there is some poset $\bar{\mathbb{P}} \in V_\kappa$ such that x is $\bar{\mathbb{P}}$-generic over V. Then W is a model of "boldface $L(\mathbb{R})$ absoluteness holds for Γ."

In the light of this criterion, in order to verify Theorem 1.1 it suffices to prove the following. Notice that by [8, Lemma 1.7], if κ is remarkable then κ is remarkable in L. Moreover, it is easy to see that if κ is remarkable and $\lambda > \kappa$ is inaccessible then κ is remarkable in V_λ.

[1]The author would like to thank David Asperó for a pivotal discussion about BMM.

Lemma 2.1 *Suppose that $V = L$. Let κ be remarkable and assume further that there is no inaccessible cardinal above κ. Write $W = V^{\mathrm{Col}(\omega,<\kappa)}$. Let $\mathbb{P}$ be semi-proper in W. Then every real in $W^{\mathbb{P}}$ is "small generic" over V, i.e., if $x \in \mathbb{R} \cap W^{\mathbb{P}}$ then there is some poset $\mathbb{Q} \in V_\kappa = L_\kappa$ such that x is $\mathbb{Q}$-generic over $V = L$.*

PROOF. We may pick some $A \subset \kappa$ such that $W = L[A]$. Notice that there is no inaccessible cardinal in W by our hypothesis.

Let $\tau \in W^{\mathbb{P}}$ be a name for a real. We aim to prove that $\mathbb{P} \Vdash^W$ "there is some $\mathbb{Q} \in L_{\check{\kappa}}$ such that τ is $\mathbb{Q}$-generic over L." Let us assume that this is not the case. We shall eventually derive a contradiction.

Let θ be a large regular cardinal such that all (dense) subsets of $\mathbb{P}$ in W are in $L_\theta[A]$, and let $p \in \mathbb{P}$ be such that $p \Vdash^W$ "there is no $\mathbb{Q} \in L_{\check{\kappa}}$ such that τ is $\mathbb{Q}$-generic over L." As κ is remarkable in L, we may pick some

$$\pi\colon (L_\beta[A \cap \alpha]; \in, \bar{\mathbb{P}}, \bar{p}, \bar{\tau}) \cong X \prec (L_\theta[A]; \in, \mathbb{P}, p, \tau)$$

such that $\beta < \omega_1^{L[A]} = \kappa$ is a cardinal in $L[A \cap \alpha]$. (This uses the key charactrization of remarkable cardinals, cf. [6, Lemma 0.7].) Let $q \leq p$ be $(\mathbb{P}, X)$-semi-generic, and let G be $\mathbb{P}$-generic over $L[A]$ such that $q \in G$.

Setting $X[G] = \{\sigma^G : \sigma \in X\}$, we have that $X[G] \prec L_\theta[A, G]$, because for each $\varphi(v, \vec{\sigma})$ there is some σ_0 such that $\mathbb{P} \Vdash^W \exists v\ \varphi(v, \vec{\sigma}) \to \varphi(\sigma_0, \vec{\sigma})$. Moreover, $X[G] \cap \kappa = X \cap \kappa$, as $q \in G$ is $(\mathbb{P}, X)$-semi-generic. We therefore get

$$\tilde{\pi}\colon L_{\tilde{\beta}}[A \cap \alpha, \bar{G}] \cong X[G] \prec L_\theta[A, G],$$

where $\tilde{\beta} \geq \beta$ and $\bar{G} = \tilde{\pi}^{-1}(G)$. We then also have

$$\tilde{\pi} \restriction L_{\tilde{\beta}}[A \cap \alpha]\colon L_{\tilde{\beta}}[A \cap \alpha] \to L_\theta[A],$$

where of course $X[G] \cap L_\theta[A] = \mathrm{ran}(\tilde{\pi}) \cap L_\theta[A] \supset \mathrm{ran}(\pi) = X$. Setting $\psi = \tilde{\pi}^{-1} \circ \pi$, we may thus write

$$\psi\colon L_\beta[A \cap \alpha] \to L_{\tilde{\beta}}[A \cap \alpha].$$

Claim. ψ is the identity, and $\tilde{\beta} = \beta$.

Let us first show that $\psi = \mathrm{id}$. Suppose otherwise. Set $\mu = \mathrm{crit}(\psi)$.

Then μ is a regular cardinal in $L_\beta[A \cap \alpha]$. As β is a cardinal in $L[A \cap \alpha]$, μ is then also a regular cardinal in $L[A \cap \alpha]$. Assume that μ were a successor cardinal in $L[A \cap \alpha]$, say $\mu = \rho^{+L[A\cap\alpha]}$. Then $\mu = \rho^{+L_\beta[A\cap\alpha]}$, and hence $\mu < \psi(\mu) = \rho^{+L_{\tilde{\beta}}[A\cap\alpha]} \leq \rho^{+L[A\cap\alpha]} = \mu$, using the elementarity of ψ. Contradiction!

We have shown that μ would have to be an inaccessible cardinal in $L[A\cap\alpha]$, and hence in $L_\beta[A\cap\alpha]$ as well. But then by elementarity $\pi(\mu)$ is inaccessible in $L_\theta[A]$, and hence in $L[A]$. However, in $L[A]$ there is no inaccessible cardinal.

Therefore, $\psi=\mathrm{id}$, in other words, $L_\beta[A\cap\alpha]\prec L_{\tilde\beta}[A\cap\alpha]$. Because θ is regular in $L[A]$, and because $L[A]$ doesn't have an inaccessible cardinal, θ must be a successor cardinal. Hence $L_\beta[A\cap\alpha]$ has a largest cardinal, call it ρ. As β is a cardinal in $L[A\cap\alpha]$, we must have $\beta=\rho^{+L[A\cap\alpha]}$.

But ρ must also be the largest cardinal in $L_{\tilde\beta}[A\cap\alpha]$ by $L_\beta[A\cap\alpha]\prec L_{\tilde\beta}[A\cap\alpha]$, and hence $\beta\leq\tilde\beta\leq\rho^{+L[A\cap\alpha]}=\beta$, i.e., $\tilde\beta=\beta$. $\square$ (Claim)

Now $\bar G$ is $\bar{\mathbb{P}}$-generic over $L_{\tilde\beta}[A\cap\alpha]=L_\beta[A\cap\alpha]$ by the elementarity of $\tilde\pi$. But all the (dense) subsets of $\bar{\mathbb{P}}$ in $L[A\cap\alpha]$ are in $L_\beta[A\cap\alpha]$, by the fact that β is a cardinal in $L[A\cap\alpha]$ and by the choice of θ, so that $\bar G$ is in fact $\bar{\mathbb{P}}$-generic over $L[A\cap\alpha]$.

Set $x=\bar\tau^{\bar G}\in L_\beta[A\cap\alpha,\bar G]$. Also, let $\dot{\mathbb{P}}\in X\cap L_\theta$ be a $\mathrm{Col}(\omega,<\kappa)$-name for $\mathbb{P}$, and set $\dot{\bar{\mathbb{P}}}=\tilde\pi^{-1}(\dot{\mathbb{P}})=\pi^{-1}(\dot{\mathbb{P}})$. We have that x is $\mathbb{Q}$-generic over L where $\mathbb{Q}=Col(\omega,<\alpha)*\dot{\bar{\mathbb{P}}}\in L_\kappa$. Because x is a real, $x=\tilde\pi(x)=\tau^G\in L_\theta[A,G]$. Therefore, $L[A,G]\models$ "$x=\tau^G$ is $\mathbb{Q}$-generic over L," in particular, $L[A,G]\models$ "τ^G is $\mathbb{Q}$-generic over L."

But G was arbitrary with $q\in G$, so that $q\Vdash^W$ "τ is $\check{\mathbb{Q}}$-generic over L, where $\check{\mathbb{Q}}\in L_{\check\kappa}$." However, $q\leq p$. Contradiction! $\square$ (Theorem 1.1)

The reader will have noticed that whereas our proof does exploit a minimality assumption on the ground model V, it does not really need $V=L+$ "there is no inaccessible cardinal above κ." We did not attempt to analyse how much one might weaken this smallness assumption, though.

3 BMM and sharps.

Definition 3.1 *Let f, g both be functions from ω_1 to ω_1. We shall write $f<^* g$ iff there is some club $C\subset\omega_1$ such that for all $\nu\in C$, $f(\nu)<g(\nu)$.*

Of course, $<^*$ is a well-founded relation on the set of all $f\colon\omega_1\to\omega_1$. We shall prove Theorem 1.3 by showing that BMM gives an infinite $<^*$-descending chain of such functions unless every set has a sharp.

Definition 3.2 *Let $a\subset\omega$. We aim to define some $\xi_a\leq\omega_1$, a function $f_a\colon\xi_a\to\omega_1$, a sequence $(d_a^i\colon i<\xi_a)$, and some $A_a\subset\xi_a$.*

Suppose that $f_a \upharpoonright \nu$, $(d_a^i : i < \nu)$, and $A_a \cap \nu$ have been defined for some $\nu \leq \omega_1$. If $\nu = \omega_1$ or if $\nu < \omega_1$ and ν is uncountable in $L[A_a \cap \nu]$ then we set $\xi_a = \nu$ and finish the construction. Otherwise we let $f_a(\nu) =$ the least $\beta < \omega_1$ such that

$$L_{\beta+1}[A_a \cap \nu] \models \text{“}\nu \text{ is countable.”}$$

We let d_a^ν be the $L[A_a \cap \nu]$-least $d \subset \omega$ which is almost disjoint from all elements of $\{d_a^i : i < \nu\}$, and we put ν into A_a if and only if $d_a^\nu \cap a$ is finite.

We say that a codes a reshaped subset of ω_1 if and only if $\xi_a = \omega_1$.

Obviously, a given $a \subset \omega$ can only code a reshaped subset of ω_1 if $\omega_1^{L[a]} = \omega_1^V$. Our key lemma is the following.

Lemma 3.3 *Let $a \subset \omega$ be such that a codes a reshaped subset of ω_1. There is then a stationary preserving set-generic extension of V in which there is some $b \subset \omega$ such that b codes a reshaped subset of ω_1 and $f_b <^* f_a$.*

Proof of Theorem 1.3 from Lemma 3.3. Suppose that BMM holds but that for some $X \in V$, $X^\#$ does not exist. We have shown in [7] that there is then a stationary preserving set-generic extension of V in which there is some $a \subset \omega$ with $X \in H_{\omega_2} = L_{\omega_2}[a]$ (where ω_2 denotes the ω_2 of the extension).[2] In this extension, thus

$$\exists a \, \exists \mathcal{M} \; (\mathcal{M} \text{ is a transitive model of } \mathsf{ZFC}^-, \omega_1 \in \mathcal{M}, \text{ and}$$
$$\mathcal{M} \models \text{“}a \text{ codes a reshaped subset of } \omega_1\text{”}).$$

By BMM, the displayed statement holds in V. If a_0, $\mathcal{M} \in V$ witness this then by absoluteness a_0 really codes a reshaped subset of ω_1.

Now let $a \in V$ code a reshaped subset of ω_1. By Lemma 3.3, there is a stationary preserving set-generic extension of V in which there is some $b \subset \omega$ such that b codes a reshaped subset of ω_1 and $f_b <^* f_a$. In this extension, thus

$$\exists b \, \exists \mathcal{M} \; (\mathcal{M} \text{ is a transitive model of } \mathsf{ZFC}^-, \omega_1 \in \mathcal{M}, \text{ and}$$
$$\mathcal{M} \models \text{“}b \text{ codes a reshaped subset of } \omega_1 \text{ and } f_b <^* f_a\text{”}).$$

By BMM, the displayed statement holds in V. If b_0, $\mathcal{M} \in V$ witness this then by absoluteness b_0 really codes a reshaped subset of ω_1 and $f_{b_0} <^* f_a$ really holds true.

[2] The paper [7] in fact just assumes that there is no inner model with a strong cardinal containing X and obtains a stationary preserving set-generic extension of V in which there is some $a \subset \omega$ with $X \in H_{\omega_2} = K(a)||\omega_2$.

But this shows that $<^*$ is not well-founded (in a strong sense: all a start an infinite descending $<^*$ chain). Contradiction! $\square$ (Theorem 1.3)

Proof of Lemma 3.3. Fix $a \subset \omega$ as in the statement of Lemma 3.3. By [7] there is an ω-closed $\mathbb{Q} \in V$ such that in $V^{\mathbb{Q}}$ there is some $A \subset \omega_1$ with $H_{\omega_2} = L_{\omega_2}[A]$ (where ω_2 denotes the ω_2 of $V^{\mathbb{Q}}$). We now want to work in $V^{\mathbb{Q}}$ and "reshape" A in such a way that we shall be able to code the reshaped object by a real b with $f_b <^* f_a$. Of course, the reshaping should better be stationary preserving.

Let us fix some $A \in V^{\mathbb{Q}}$ such that $A \subset \omega_1$ and $H_{\omega_2} = L_{\omega_2}[A]$.

Let $\mathbb{P} \in V^{\mathbb{Q}}$ be the set of all (f, c) such that there is some $\nu < \omega_1$ with:

- $f : \nu \to 2$,
- $c \subset \nu + 1$ is closed,
- for all $\bar{\nu} \leq \nu$,
$$L[A \cap \bar{\nu}, f \restriction \bar{\nu}] \models \text{"}\bar{\nu} \text{ is countable,"}$$
- for all $\bar{\nu} \in c$,
$$L_{f_a(\bar{\nu})}[A \cap \bar{\nu}, f \restriction \bar{\nu}] \models \text{"}\bar{\nu} \text{ is countable."}$$

If $p = (f, c) \in \mathbb{P}$ then we shall write p^ℓ for f and p^ρ for c. A condition q is stronger than p iff $q^\ell \restriction \mathrm{dom}(p^\ell) = p^\ell$ and $q^\rho \cap (\max(p^\rho) + 1) = p^\rho$.

The following is easy to verify.

Claim 1. (Extendability) Let $p \in \mathbb{P}$. If $\nu < \omega_1$ then there is some $q \leq p$ such that $\mathrm{dom}(q^\ell) \geq \nu$. In fact, if $\nu < \omega_1$ then there is some $q \leq p$ such that $q^\rho \setminus \nu \neq \emptyset$.

Whereas it can be shown that $\mathbb{P}$ is not semi-proper in general[3] the following does hold true.

Claim 2. $\mathbb{P}$ is stationary preserving.

Proof of the Claim. Suppose that $p \Vdash \dot{C} \subset \check{\omega}_1$ is club, and let $S \subset \omega_1$ be stationary. We aim to find some $q \leq p$ with $q \Vdash \dot{C} \cap \check{S} \neq \emptyset$.

Let $n_0 \in \omega$ be large enough. Let us first pick a fully elementary embedding

$$\pi : \mathcal{N} \to L_{\omega_2}[A]$$

[3] As we can iterate semi-proper forcings, if $\mathbb{P}$ were semi-proper in general then we would be able to construct a model in which there is some sequence $(a_n : n < \omega)$ with $f_{a_{n+1}} <^* f_{a_n}$ for all n. We do not know if $\mathbb{P}$ can ever be semi-proper.

such that $\mathcal{N}$ is countable and transitive, $\mathrm{crit}(\pi) \in S$, and $\{a, A, \mathbb{P}, p, \dot{C}\} \subset \mathrm{ran}(\pi)$. Set $\nu = \mathrm{crit}(\pi)$, $\bar{\mathbb{P}} = \pi^{-1}(\mathbb{P})$, and $\bar{\dot{C}} = \pi^{-1}(\dot{C})$. Working in $\mathcal{N}$ (a model of ZFC^-), we may pick some

$$\bar{\mathcal{N}} \prec_{\Sigma_{n_0}} \mathcal{N}$$

such that the n_0^{th} projectum $\rho_{n_0}(\bar{\mathcal{N}})$ of $\bar{\mathcal{N}}$ is equal to ν, and $\{a, A \cap \nu, \bar{\mathbb{P}}, p, \bar{\dot{C}}\} \subset \bar{\mathcal{N}}$. (We may for instance let $\bar{\mathcal{N}}$ be the Σ_{n_0} hull of $\nu \cup \{a, A \cap \nu, \bar{\mathbb{P}}, p, \bar{\dot{C}}\}$ formed inside $\mathcal{N}$.)

Set $\beta = \bar{\mathcal{N}} \cap \mathrm{OR}$. Of course, $\bar{\mathcal{N}} = L_\beta[A \cap \nu]$.

Subclaim. $\beta \leq f_a(\nu)$.

Proof of the Subclaim. Of course, ν is uncountable in $\bar{\mathcal{N}}$. Moreover, it is easy to see that $A_a \cap \nu \in \bar{\mathcal{N}}$, so that ν is uncountable in $L_\beta[A_a \cap \nu] \subset \bar{\mathcal{N}}$. Hence $\beta \leq f_a(\nu)$. $\square$ (Subclaim)

We shall now imitate an argument of [11]. Let $(E_i : i < \nu) \in \mathcal{N}$ be an enumeration of all the sets which are club in ν and which exist in $\bar{\mathcal{N}}$, and let E be the diagonal intersection of $(E_i : i < \nu)$. Notice that $E \setminus E_i$ is bounded in ν whenever $i < \nu$. Let us pick an external sequence $(\nu_n : n < \omega)$ of ordinals smaller than ν which is cofinal in ν.[4] Also, let $\{D_n : n < \omega\}$ be the set of all sets in $\bar{\mathcal{N}}$ which are open dense in $\bar{\mathbb{P}}$.

We now construct a sequence $(p_n : n < \omega)$ of conditions such that $p_0 = p$, $p_{n+1} \leq p_n$, and $p_{n+1} \in D_n$ for $n < \omega$. Simultaneously, we'll construct a sequence $(\delta_n : n < \omega)$ of ordinals.

Suppose that p_n is given. Notice that, setting $\gamma = \mathrm{dom}(p_n^\ell)$, $\gamma < \nu$ (as $p_n \in \bar{\mathcal{N}}$). Work inside $\bar{\mathcal{N}}$ for a second. Using Claim 1, for all δ with $\gamma \leq \delta < \nu$ we may easily pick some $p^\delta \leq p_n$ such that: $p^\delta \in D_n$, $\mathrm{dom}((p^\delta)^\ell) > \max(\{\nu_n, \delta\})$, and for all limit ordinals λ with $\gamma \leq \lambda \leq \delta$, $(p^\delta)^\ell(\lambda) = 1$ iff $\lambda = \delta$. There is some $\bar{E} \in \mathcal{P}(\nu) \cap \bar{\mathcal{N}}$ club in ν such that for any $\eta \in \bar{E}$, $\delta < \eta \Rightarrow \mathrm{dom}((p^\delta)^\ell) < \eta$.

Now working inside $\mathcal{N}$, we may pick some $\delta \in E$ such that $E \setminus \bar{E} \subset \delta$. Let us set $p_{n+1} = p^\delta$, and put $\delta_n = \delta$. Of course, $p_{n+1} \leq p_n$ and $p_{n+1} \in D_n$. Moreover, $\mathrm{dom}((p_{n+1})^\ell) < \min(E \setminus (\delta_n + 1))$, so that for all limit ordinals $\lambda \in E \cap (\mathrm{dom}((p_{n+1})^\ell) \setminus \mathrm{dom}((p_n)^\ell))$ we have that $(p_{n+1})^\ell = 1$ iff $\lambda = \delta_n$.

Now let us define an object $q = (q^\ell, q^\rho)$ as follows. We set $q^\ell = \bigcup_{n<\omega} (p_n)^\ell$ and $q^r = \bigcup_{n<\omega} (p_n)^r \cup \{\nu\}$.

Let us verify that $q \in \mathbb{P}$. Well, by Claim 1, $\mathrm{dom}(q^\ell) = \nu$ and $q^\rho \cap \nu$ is unbounded in ν. Hence to prove that $q \in \mathbb{P}$ boils down to having to show that

$$L_{f_a(\nu)}[A \cap \nu, q^\ell] \models \text{“}\nu \text{ is countable.”}$$

[4] I.e., $(\nu_n : n < \omega) \in V$.

However, by the construction of the p_n's we have that

$$\{\lambda \in E \cap (\mathrm{dom}(q^\ell) \setminus \mathrm{dom}(p^\ell)) \colon \lambda \text{ is a limit ordinal and } q^\ell(\lambda) = 1\} = \{\delta_n \colon n < \omega\},$$

which is cofinal in ν. But $E \in \bar{\mathcal{N}} = L_\beta[A \cap \nu]$. Therefore, $L_\beta[A \cap \nu] \cup \{q^\ell\} \subset L_\beta[A \cap \nu, q^\ell]$, and hence $\{\delta_n \colon n < \omega\} \in L_\beta[A \cap \nu, q^\ell]$ witnesses that ν is countable in $L_\beta[A \cap \nu, q^\ell]$. However $\beta \leq f_a(\nu)$ by the above Subclaim.

It is now easy to see that $q \Vdash \check{\nu} \in \dot{C} \cap \check{S}$. □ (Claim 2)

The rest is smooth. Because forcing with $\mathbb{P}$ does not collapse ω_1, it adds a pair B, C such that $B \subset \omega_1$, C is a club subset of ω_1, for all $\nu < \omega_1$,

$$L[A \cap \nu, B \cap \nu] \models \text{“}\nu \text{ is countable,”}$$

and for all $\nu \in C$,

$$L_{f_a(\nu)}[A \cap \nu, B \cap \nu)] \models \text{“}\nu \text{ is countable.”}$$

Let us fix such a pair $(B, C) \in V^{\mathbb{Q} * \mathbb{P}}$, and let us write $D = A \oplus B$. Let us code D down to a real in the usual way (cf. [7]). In order to do this, let us write $(a_\beta \colon \beta < \omega_1)$ for that sequence of subsets of ω such that for each $\beta < \omega_1$, a_β is the $L[D \cap \beta]$-least subset of ω which is almost disjoint from every member of $\{a_{\bar{\beta}} \colon \bar{\beta} < \beta\}$.

Specifically, let $\mathbb{A}$ consist of all pairs $(l(p), r(p))$, where $l(p) \colon n \to 2$ for some $n < \omega$ and $r(p) \subset \omega_1$ is finite. A condition q is stronger than p iff $l(q)$ extends $l(p)$, $r(p)$ is a subset of $r(q)$, and for all $\beta \in r(q)$, if $\beta \in D$ then

$$\{n \in \mathrm{dom}(l(q)) \setminus \mathrm{dom}(l(p)) \colon l(q)(n) = 1\} \cap a_\beta = \emptyset.$$

The forcing $\mathbb{A}$ has the c.c.c., and forcing with $\mathbb{A}$ adds a real b such that for all $\beta < \omega_1$,

$$\beta \in D \text{ iff } b \cap a_\beta \text{ is finite.}$$

Obviously, we have found a $b \in V^{\mathbb{Q} * \mathbb{P} * \mathbb{A}}$ as desired.

□ (Lemma 3.3)

We do not know if BMM implies that $0^\P$, i.e., the sharp for an inner model with a strong cardinal, exists. This might be related to the problem of getting $0^\P$ from the assumption that the theory of $L(\mathbb{R})$ is absolute for stationary preserving forcings (cf. [7]).

References

[1] Asperó, D., and Welch, Ph., *Bounded Martin's Maximum, weak Erdös Cardinals, and ψ_{AC}*, Journal Symb. Logic **67** (2002), pp. 1141-1152.

[2] Bagaria, J., and Woodin W.H., *Δ^1_n sets of reals*, Journal Symb. Logic **62** (1997), pp. 1379-1428.

[3] Goldstern, M., and Shelah, S., *The Bounded Proper Forcing Axiom*, Journal Symb. Logic 60 (1995), pp. 58-73.

[4] Martin, T., and Solovay, B., *A basis theorem for Σ^1_3 sets of reals*, Ann. Math. **89** (1969), pp.. 138-159.

[5] Neeman, I., and Zapletal, J., *Proper forcing and $L(\mathbb{R})$*, Journal Symb. Logic **66** (2001), pp. 801-810.

[6] Schindler, R., *Proper forcing and remarkable cardinals*, Bulletin Symb. Logic **6** (2000), pp. 176-184.

[7] Schindler, R., *Coding into K by reasonable forcing*, Trans. Amer. Math. Soc. **353** (2001), pp. 479-489.

[8] Schindler, R., *Proper forcing and remarkable cardinals II*, Journal Symb. Logic **66** (2001), pp. 1481-1492.

[9] Schindler, R., *Bounded Martin's Maximum and strong cardinals*, preprint, available at `wwwmath.uni-muenster.de/math/inst/logik/org/staff/rds`.

[10] Shelah, S., *Semiproper forcing axiom implies Martin's Maximum but not* PFA^+, Journal Symb. Logic **52** (1987), pp. 360-367.

[11] Shelah, S., and Stanley, L., *Coding and reshaping when there are no sharps*, in: "Set theory of the continuum," (H. Judah et al., eds.), Springer-Verlag 1992, pp. 407-416.

[12] Woodin, W.H., private communication, August 03.