Varsovian models I*†

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Abstract

Let $M_{\rm sw}$ denote the least iterable inner model with a strong cardinal above a Woodin cardinal. By [11], $M_{\rm sw}$ has a fully iterable core model, $K^{M_{\rm sw}}$, and $M_{\rm sw}$ is thus the least iterable extender model which has an iterable core model with a Woodin cardinal. In V, $K^{M_{\rm sw}}$ is an iterate of $M_{\rm sw}$ via its iteration strategy Σ.

We here show that $M_{\rm sw}$ has a bedrock which arises from $K^{M_{\rm sw}}$ by telling $K^{M_{\rm sw}}$ a specific fragment $\bar{\Sigma}$ of its own iteration strategy, which in turn is a tail of Σ. Hence $M_{\rm sw}$ is a generic extension of $L[K^{M_{\rm sw}}, \bar{\Sigma}]$, but the latter model is not a generic extension of any inner model properly contained in it.

These results generalize to models of the form $M_{\rm s}(x)$ for a cone of reals x, where $M_{\rm s}(x)$ denotes the least iterable inner model with a strong cardinal containing x. In particular, the least iterable inner model with a strong cardinal above two (or seven, or boundedly many) Woodin cardinals has a 2–small core model K with a Woodin cardinal and its bedrock is again of the form $L[K, \bar{\Sigma}]$.

1 Introduction.

By a theorem of W. Hugh Woodin, every pure extender model W with a Woodin cardinal has a non–trivial ground,[1] i.e., there is some inner model $\bar{W} \subsetneq W$ such that W is a generic extension of $\bar{W}$. E.g., let $\bar{W} = \mathcal{P}^W(\mathcal{M})$, where $\mathcal{M}$ arises from an

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[1] The terms "ground," "bedrock," and "mantle" are taken from [2]. If $\bar{W} \subset W$ are both inner models, then $\bar{W}$ is a ground of W iff W is a generic extension of $\bar{W}$. W is a bedrock iff W itself is the only ground of W.

$L[E]$–construction inside W up to its first Woodin cardinal and $\mathcal{P}^W(\mathcal{M})$ denotes the $\mathcal{P}$–construction above $\mathcal{M}$ and performed inside W, cf. [13].

The situation is different for hod mice, also called "strategic mice." Woodin showed that there are strategic mice which are bedrocks, i.e., which don't admit any non–trivial grounds, cf. [23]. Strategic mice naturally arise as HODs of models of determinacy, cf. [9].

The current paper produces a minimal example of an extender model with a Woodin cardinal which, when equipped with a fragment of its own iteration strategy, is a bedrock, and it will also be the HOD of a homogeneous generic extension of an extender model.

By a theorem of John Steel, extender models with no strong cardinals cannot have a fully iterable core model with a Woodin cardinal. The paper [3] analyzes the mantle[2] of (tame) extender models with Woodin cardinals but no strong cardinals and shows that it is always a lower part model; in particular, their mantles are not grounds. On the other hand, writing M_{sw} for the least iterable inner model with a strong cardinal above a Woodin cardinal, [11] shows that M_{sw} does have a fully iterable core model $K^{M_{\text{sw}}}$ which in turn has a strong cardinal above a Woodin cardinal, so that the mantle of M_{sw} should contain $K^{M_{\text{sw}}}$ and *not* be a lower part model.

The current paper analyzes the mantle of M_{sw} and shows that it is a ground, hence the smallest ground, and thus a bedrock. The mantle turns out to be $L[K^{M_{\text{sw}}}, \bar{\Sigma}]$, where $\bar{\Sigma}$ is a fragment of the iteration strategy of $K^{M_{\text{sw}}}$ which M_{sw} can see and which in turn is a fragment of the tail of M_{sw}'s own iteration strategy. $K^{M_{\text{sw}}}$ is fully iterable inside $L[K^{M_{\text{sw}}}, \bar{\Sigma}]$.

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[2]The mantle of an inner model is defined to be the intersection of all of its grounds.

2 The mantle of $M_{\rm sw}$.

For the record, a *mouse* is a premouse which is countably iterable, i.e., all transitive collapses of sufficiently elementary countable substructures are supposed to be $(\omega, \omega_1, \omega_1 + 1)$–iterable. Cf. [19, Definition 4.4].

Throughout our paper, we shall assume that V is closed under the operation $a \mapsto a^{\P}$ mapping a to a–pistol, the least active a–mouse with a strong cardinal. For any transitive s.w.o.'d[3] set a, we let $M_{\rm s}(a)$ be the minimal proper class a-mouse with a strong cardinal. $M_{\rm s}(a)$ is obtained from $a^{\P}$ by iterating its top measure out of the universe.

The premice of the current paper are Mitchell–Steel premice, see [8, section 1] and [12, section 2]. For the purposes of the current paper, a premouse $\mathcal{N}$ is called *suitable* if for some $\delta \in \mathcal{N}$,

1. $\mathcal{N} \models$ "δ is a Woodin cardinal,"
2. $\mathcal{N} = M_{\rm s}(\mathcal{N}|\delta)|\delta^{+M_{\rm s}(\mathcal{N})}$,
3. for every $\eta < \delta$, $M_{\rm s}(\mathcal{N}|\eta) \models$ "η is not Woodin," and
4. $\mathcal{N} \models$ "I'm (ω, δ, δ)–iterable."

We shall now also assume that there is a suitable premouse, and more: Let us call a premouse $\mathcal{M}$ sw–*small* iff for all extenders F from $\mathcal{M}$'s sequence,

$$\mathcal{M}|\text{crit}(F) \models \text{"there is no strong cardinal above a Woodin cardinal."}$$

Let us assume that there is a non–sw–small mouse, and let $M_{\rm sw}^{\#}$ be the unique sound non–sw–small mouse $\mathcal{M}$ such that every proper initial segment of $\mathcal{M}$ is sw–small. As we assume V to be closed under $a \mapsto a^{\P}$, the $(\omega, \omega_1, \omega_1)$–iterability of $M_{\rm sw}^{\#}$ implies that $M_{\rm sw}^{\#}$ be fully iterable with respect to arbitrary stacks of normal trees. Let us denote by

$$M_{\rm sw}$$

the result of iterating $M_{\rm sw}^{\#}$'s top measure out of the universe. Let $\delta = \delta^{M_{\rm sw}}$ be the Woodin cardinal of $M_{\rm sw}$, and let $\kappa = \kappa^{M_{\rm sw}}$ be the strong cardinal of $M_{\rm sw}$. We have that $M_{\rm sw} = M_{\rm s}(M_{\rm sw}|\delta)$, and $M_{\rm sw}|\delta^{+M_{\rm sw}}$ is suitable.

By way of notation, if W is any extender model, then we will denote by δ^W the least Woodin cardinal of W (if it exists), we will denote by $\mathbb{B}^W$ the δ–generator version of the extender algebra of W at δ^W (cf. [19, pp. 1657f.] and [13, Lemma

[3] self–well–ordered

1.3]) given by the total extenders of W's sequence up to δ^W (if it exists), and we will denote by κ^W the least strong cardinal of W (if it exists).

In what follows, the relevant W will always be an iterate of $M_{\rm sw}$, so that δ^W will also be the unique Woodin cardinal of W, and κ^W will be the unique strong cardinal of W.

The iteration strategy for $\mathcal{M}$ with respect to finite stacks of normal trees induces an iteration strategy, call it Σ, for $M_{\rm sw}$ with respect to finite stacks of normal trees. We have the following.

(1) Σ satisfies hull condensation, cf. [9, Definition 1.31],

(2) Σ satisfies branch condensation, cf. [9, Definition 2.14], and

(3) Σ is positional, cf. [9, Definition 2.35 (4)].[4]

As suggested by the referee, let us also state the following property of Σ. If $\mathcal{T}$ is a normal iteration tree on $M_{\rm sw}$ which is according to Σ and has limit length, and if b is a cofinal *well–founded* non–dropping branch through $\mathcal{T}$, then $b = \Sigma(\mathcal{T})$. The reason is that if $\delta(\mathcal{T}) \neq \pi^{\mathcal{T}}_{0,b}(\delta^{M_{\rm sw}})$, then if $\mathcal{Q} \triangleleft \mathcal{M}^{\mathcal{T}}_b$ is the least extension of $\mathcal{M}(\mathcal{T})$ such that $\delta(\mathcal{T})$ is not definably Woodin over $\mathcal{Q}$, then $\mathcal{Q}$ is ¶–small above $\delta(\mathcal{T})$ and hence iterable by absoluteness, so that b picks the right $\mathcal{Q}$–structure; and if $\delta(\mathcal{T}) = \pi^{\mathcal{T}}_{0,b}(\delta^{M_{\rm sw}})$, then $\mathcal{M}^{\mathcal{T}}_b$ will also be ¶–small above $\delta(\mathcal{T})$ and hence iterable by absoluteness, so that b moves the theory of any finite set of indiscernibles correctly. This property of Σ may be used to prove (1) through (3) above, and it could also be used to simplify the proofs of Lemma 2.1 as well as parts of the proofs of Lemma 2.9. The reason why we decided to not make use of this property is that it fails for more complicated mice, e.g. the ones studied in [10], and that we try to give arguments which generalize.

We shall need the following slight refinement of (2):

Lemma 2.1 *Let M be a proper class sized Σ–iterate of $M_{\rm sw}$. Let $\mathcal{U}$ be an iteration tree on M living on $M|\delta^M$ with a last model $\mathcal{M}^{\mathcal{U}}_\theta$ such that $[0,\theta]_{\mathcal{U}}$ does not drop and $\mathcal{U}$ is according to Σ_M. Let $\mathcal{T}$ be an iteration tree on M living on $M|\delta^M$ and of limit length which is according to Σ_M. If b and k are in some generic extension of V such that*

(a) b is a cofinal non–dropping branch through $\mathcal{T}$, and

[4] The last "positional" in [9, Definition 2.35 (4)] should read "weakly positional," though.

(b) $k\colon \mathcal{M}_b^{\mathcal{T}}|\delta^{\mathcal{M}_b^{\mathcal{T}}} \to \mathcal{M}_\theta^{\mathcal{U}}|\delta^{\mathcal{M}_\theta^{\mathcal{U}}}$ *is elementary with*

$$\pi_{0,\theta}^{\mathcal{U}} \upharpoonright M|\delta^M = k \circ \pi_{0,b}^{\mathcal{T}} \upharpoonright M|\delta^M, \tag{1}$$

then $b = \Sigma_M(\mathcal{T})$.

Proof. Write $c = \Sigma_M(\mathcal{T})$. If $\delta(\mathcal{U}) \neq \pi_{0,b}^{\mathcal{T}}(\delta^M) = \delta^{\mathcal{M}_b^{\mathcal{T}}}$, then $\mathcal{M}_b^{\mathcal{T}}$ comes with a $\mathcal{Q}$–structure which by the existence of k is iterable, and this gives that $b = c$.

Let us now assume that $\delta(\mathcal{U}) = \pi_{0,b}^{\mathcal{T}}(\delta^M)$. The key fact is that k may be extended to $k^+\colon \mathcal{M}_b^{\mathcal{T}} \to \mathcal{M}_\theta^{\mathcal{U}}$ by setting

$$k^+(\pi_{0,b}^{\mathcal{T}}(f)(a)) = \pi_{0,\theta}^{\mathcal{U}}(f)(k(a)).$$

It is easy to verify that k^+ is well–defined and elementary. Also,

$$\pi_{0,\theta}^{\mathcal{U}} = k^+ \circ \pi_{0,b}^{\mathcal{T}}. \tag{2}$$

Now let λ be a sufficiently large V–cardinal, and let λ^{+n} denote the n^{th} cardinal successor of λ as being computed in V.

We have that

$$X = \text{Hull}^M(\{\lambda^{+n}\colon 0 < n < \omega\}) \cap \delta^M$$

is cofinal in δ^M. Also,

$$\pi_{0,c}^{\mathcal{T}}(\lambda^{+n}) = \lambda^{+n} \text{ for all } n, 0 < n < \omega, \tag{3}$$

and

$$\pi_{0,\theta}^{\mathcal{U}}(\lambda^{+n}) = \lambda^{+n} \text{ for all } n, 0 < n < \omega,$$

and by (2) the latter implies that

$$\pi_{0,b}^{\mathcal{T}}(\lambda^{+n}) = \lambda^{+n} \text{ for all } n, 0 < n < \omega. \tag{4}$$

But (3) and (4) give that

$$\pi_{0,c}^{\mathcal{T}} \upharpoonright X = \pi_{0,b}^{\mathcal{T}} \upharpoonright X,$$

which implies that $b = c$ by the "zipper argument," cf. e.g. [19, p. 1645f.], as desired. $\square$ (Lemma 2.1)

Some of the arguments to follow will look pretty familiar to researchers working in the area of descriptive inner model theory, cf. e.g. [21, Section 3].

Let us consider the set $\mathbb{U}$ consisting of all $\mathcal{U} = (\mathcal{U}_k\colon k \leq n)$, some $n < \omega$, such that either $n = 0$ and $\text{lh}(\mathcal{U}_0) = 1$ (i.e., $\mathcal{U}$ is trivial), or else there is a sequence $\eta_0 < \ldots < \eta_n < \kappa$ of cutpoints of M_{sw} and:

(a) $\mathcal{U} \in M_{\rm sw}|\kappa$,

(b) $\mathcal{U} = (\mathcal{U}_k \colon k \leq n)$ is a finite stack of normal iteration trees $\mathcal{U}_k$,

(c) $\mathcal{U}_0$ is on $M_{\rm sw}$ and lives below δ,

and for every $k \leq n$,

(d) $\mathrm{lh}(\mathcal{U}_k) = (\eta_k)^{+M_{\rm sw}} = \delta(\mathcal{U}_k)$,

(e) $\mathcal{U}_k$ is defiable over $M_{\rm sw}|(\eta_k)^{+M_{\rm sw}}$ and is guided by $\mathcal{Q}$–structures which are obtained via $\mathcal{P}$–constructions, cf. [13, Section 1],

(f) $P(\mathcal{M}(\mathcal{U}_k))$ is a proper class,[5] $\delta(\mathcal{U}_k)$ is a Woodin cardinal of $P(\mathcal{M}(\mathcal{U}))$, and

$$P(\mathcal{M}(\mathcal{U}))[G] = M_{\rm sw}$$

for some G which is $\mathbb{B}^{P(\mathcal{M}(\mathcal{U}))}$–generic over $P(\mathcal{M}(\mathcal{U}))$, and

(g) if $k > 0$, then $\mathcal{U}_k$ is on $P(\mathcal{M}(\mathcal{U}_{k-1}))$ and lives below $\delta(\mathcal{U}_{k-1})$.

Let $\mathcal{U} = (\mathcal{U}_k \colon k \leq n)$ be as above, where $\mathcal{U}_n$ is not trivial. For every $k \leq n$ and inside $M_{\rm sw}$, $P(\mathcal{M}(\mathcal{U}_k))$ is a universal weasel over $\mathcal{M}(\mathcal{U}_k)$ below $\mathcal{M}(\mathcal{U}_k)^{\P}$. Let us write $K(\mathcal{M}(\mathcal{U}_k))$ for the $\mathcal{M}(\mathcal{U}_k)^{\P}$–small core model over $\mathcal{M}(\mathcal{U}_k)$ as constructed inside $M_{\rm sw}$. In V, let $b_k = \Sigma(\mathcal{U}_k)$. We then have:

Lemma 2.2 *Let $\mathcal{U} = (\mathcal{U}_k \colon k \leq n) \in \mathbb{U}$, where $\mathcal{U}_n$ is not trivial. Let I be the class of generating indiscernibles for $M_{\rm sw}$ given by iterating the top measure of $(M_{\rm sw}|\delta)^{\P}$ out of the universe, and let $\pi = \pi_{M_{\rm sw}, P(\mathcal{M}(\mathcal{U}_n))}$ be the map given by $b_0{}^\frown \dots {}^\frown b_n$, i.e., the iteration map from $M_{\rm sw}$ to $P(\mathcal{M}(\mathcal{U}_n))$ which is given by Σ.*

(a) For every $k \leq n$, $P(\mathcal{M}(\mathcal{U}_k)) = K(\mathcal{M}(\mathcal{U}_k)) = M_s(\mathcal{M}(\mathcal{U}_k)) = \mathcal{M}^{\mathcal{U}_k}_{b_k}$.

(b) For every $k \leq n$, I is a class of generating indiscernibles for $P(\mathcal{M}(\mathcal{U}_k))$ relative to $\mathcal{M}(\mathcal{U}_k)$.

(c) $\pi(\eta) = \eta$ for every $\eta \in I$.

Proof. (a) and (b): Let us write $M = \mathcal{M}(\mathcal{U}_k)$. As $P(M)[G] = M_{\rm sw}$ for some generic G, $K(M) = K(M)^{M_{\rm sw}} = K(M)^{P(M)[G]} = K(M)^{P(M)} \subset P(M)$. On the other hand, $P(M)$ is a universal weasel over M, so that there is an elementary embedding $j \colon K(M) \to P(M)$, which, as $K(M)$ and $P(M)$ are below $M^{\P}$, is given by an iteration of $K(M)$. But then $K(M) \subset P(M)$ gives $K(M) = P(M)$.

[5]Here and in what follows we write $P(M)$ for the $\mathcal{P}$–construction over M as being performed inside $M_{\rm sw}$. [13, Section 1] would write $\mathcal{P}(M_{\rm sw}, M, -)$ for this model.

We have that $M_{\mathrm{sw}} = \mathrm{Hull}^{M_{\mathrm{sw}}}(I)$. We claim that

$$P(M) = \mathrm{Hull}^{P(M)}(\delta(\mathcal{U}_k) \cup I). \tag{5}$$

To show (5), notice first that the extender sequence of M_{sw} may be defined over $P(M)[G]$ from the parameter $M_{\mathrm{sw}}|\delta(\mathcal{U}_k) \in P(M)[G]$ and the extender sequence of $P(M)$. The forcing language associated with forcing with $\mathbb{B}^{P(M)}$ over $P(M)$ thus has a term for the extender sequence of M_{sw} and therefore also a term for the canonical Σ_1 Skolem function $h_{M_{\mathrm{sw}}}$ of M_{sw}, cf. [14, Theorem 10.16]. Writing h for this term for $h_{M_{\mathrm{sw}}}$, we have that the function $h^* \colon \mathbb{B}^{P(M)} \times \omega \times [M_{\mathrm{sw}}]^{<\omega} \to P(M)$ with

$$h^*(p, n, \mathsf{a}) = \begin{cases} y & \text{if } p \Vdash^{\mathbb{B}^{P(M)}}_{P(M)} h(\check{n}, \check{\mathsf{a}}) = \check{y}, \text{ and} \\ \emptyset & \text{otherwise.} \end{cases}$$

is definable over $P(M)$ using a name for $M_{\mathrm{sw}}|\delta(\mathcal{U}_k)$. But G and $M_{\mathrm{sw}}|\delta(\mathcal{U}_k)$ are computable from each other, so that $\mathrm{Hull}^{P(M)}(X)$ is closed under h^* for *any* X and by $\mathbb{B}^{P(M)} \subset \mathrm{Hull}^{P(M)}(\delta(\mathcal{U}_k) \cup I)$ and $M_{\mathrm{sw}} = \mathrm{Hull}^{M_{\mathrm{sw}}}(I)$, we obtain (5).

The fact that $P(M)$ is an inner model of M_{sw} which is definable there from M an the extender sequence of M_{sw} above $\delta(\mathcal{U}_k)$ easily implies that I is also a class of indiscernibles for $P(M)$, so that by (5) it is a class of generating indiscernibles relative to $\mathcal{M}(\mathcal{U}_k)$. This shows (b).

But now $M_s(\mathcal{M}(\mathcal{U}_k))$ is also a least inner model with a strong cardinal end–extending $M = \mathcal{M}(\mathcal{U}_k)$ and having a proper class of generating indiscernibles relative to $\mathcal{M}(\mathcal{U}_k)$. It follows that $P(M) = M_s(\mathcal{M}(\mathcal{U}_k))$.

Virtually the same argument shows $P(M) = \mathcal{M}^{\mathcal{U}_k}_{b_k}$ by induction on $k \leq n$. We have shown (a).

(c) In the light of (a), (5) buys us that

$$\mathcal{M}^{\mathcal{U}_n}_{b_n} = \mathrm{Hull}^{\mathcal{M}^{\mathcal{U}_n}_{b_n}}(\delta(\mathcal{U}_n) \cup I). \tag{6}$$

At the same time, $M_{\mathrm{sw}} = \mathrm{Hull}^{M_{\mathrm{sw}}}(I)$ implies that

$$\mathcal{M}^{\mathcal{U}_n}_{b_n} = \mathrm{Hull}^{\mathcal{M}^{\mathcal{U}_n}_{b_n}}(\delta(\mathcal{U}_n) \cup \pi"I), \tag{7}$$

and $\pi"I$ is a class of indiscernibles for $\mathcal{M}^{\mathcal{U}_n}_{b_n}$ relative to $\mathcal{U}_n$.

Let φ be a formula, let τ be a Σ_1 Skolem term, let $x \in \mathcal{M}(\mathcal{U}_n)$, let $\eta_1 < \ldots < \eta_\ell$ be from I, and let $\lambda_1 < \ldots < \lambda_\ell$ be V–cardinals with $\pi(\eta_\ell) < \lambda_1$. We have that $\pi(\lambda_i) = \lambda_i$ for $0 < i \leq \ell$, so that we may conclude that

$$\begin{aligned} \mathcal{M}^{\mathcal{U}_n}_{b_n} \models \varphi(\tau(x,\eta_1,\ldots,\eta_\ell)) &\iff \\ \mathcal{M}^{\mathcal{U}_n}_{b_n} \models \varphi(\tau(x,\lambda_1,\ldots,\lambda_\ell)) &\iff \\ \mathcal{M}^{\mathcal{U}_n}_{b_n} \models \varphi(\tau(x,\pi(\lambda_1),\ldots,\pi(\lambda_\ell))) &\iff \\ \mathcal{M}^{\mathcal{U}_n}_{b_n} \models \varphi(\tau(x,\pi(\eta_1),\ldots,\pi(\eta_\ell))). & \end{aligned}$$

This shows that $\tau^{\mathcal{M}^{\mathcal{U}_n}_{b_n}}(x,\eta_1,\ldots,\eta_\ell) \mapsto \tau^{\mathcal{M}^{\mathcal{U}_n}_{b_n}}(x,\pi(\eta_1),\ldots,\pi(\eta_\ell))$ defines an $\in$–automorphism of $\mathcal{M}^{\mathcal{U}_n}_{b_n}$ and is hence the identity. We have shown (c). □ (Lemma 2.2)

Let $\mathcal{U} = (\mathcal{U}\colon k \leq n) \in \mathbb{U}$. If $\mathcal{U}_n$ is not trivial, then we shall write $\mathcal{M}(\mathcal{U})$ for $\mathcal{M}(\mathcal{U}_n)$. To uniformize the notation, if $n = 0$ and $\mathcal{T}_0$ is trivial, then we shall denote by $P(\mathcal{M}(\mathcal{U}))$ the model M_{sw}. Let us write $\mathcal{F}$ for the family of all proper class mice of the form $P(\mathcal{M}(\mathcal{U}))$, where $\mathcal{U} \in \mathbb{U}$. For the record, $\mathcal{F}$ is definable inside M_{sw} using M_{sw}'s extender sequence as a predicate.

Let $\mathcal{T}, \mathcal{U} \in \mathbb{U}$, and write $N = P(\mathcal{M}(\mathcal{T}))$ and $N' = P(\mathcal{M}(\mathcal{U}))$. By Lemma 2.2, N is a Σ–iterate of M_{sw}. Let Σ_N denote the iteration strategy for N which is induced by Σ. As Σ is positional, Σ_N only depends on N, not on the particular iteration tree which witnesses that N is a Σ–iterate of M_{sw}.

Assume for now that N' is a Σ_N–iterate of N via a finite stack of normal trees, which is tantamount to saying that there is a finite stack $\mathcal{T}_0{}^\frown \ldots {}^\frown\mathcal{T}_k$ of normal trees on M_{sw} such that N is the last model of one of the $\mathcal{T}_i$, $i < k$, and N' is the last model of $\mathcal{T}_k$. As Σ satisfies hull condensation, Σ is commuting, cf. [9, Definition 2.35 (9)], so that Σ_N satisfies the Dodd–Jensen property, cf. [9, Proposition 2.36], and hence there is a *unique* iteration map from N to N'. In what follows, we let $\pi_{N,N'}$ denote this unique iteration map from N to N'.

Let's now drop the assumption that N' be a Σ_N–iterate of N. Let $\eta < \kappa$, $\eta > \max(\delta(\mathcal{T}), \delta(\mathcal{U}))$, be a cutpoint of M_{sw}. Let $\mathcal{T}^*$, $\mathcal{U}^*$ be normal iteration trees on N, N', respectively, such that both start out by iterating the least measurable cardinal and its images $\eta + 1$ times, and from then on $\mathcal{T}^*$ and $\mathcal{U}^*$ result from comparison, simultaneously making an initial segment of the background model generic over the respective iterate; more precisely, if $\mathcal{T}^* \restriction \alpha$ and $\mathcal{U}^* \restriction \alpha$ have already been defined, where $\eta + 2 \leq \alpha \leq \eta^{+M_{\mathrm{sw}}}$, then if α is a successor ordinal, then we let ν be least such that

(a) $E_\nu^{\mathcal{M}^{\mathcal{T}^*}_{\alpha-1}} \neq E_\nu^{\mathcal{M}^{\mathcal{U}^*}_{\alpha-1}}$, or

(b) $E_\nu^{\mathcal{M}^{\mathcal{T}^*}_{\alpha-1}} = E_\nu^{\mathcal{M}^{\mathcal{U}^*}_{\alpha-1}}$, there is no drop along $[0,\alpha-1]_{\mathcal{T}^*}$ and no drop along $[0,\alpha-1]_{\mathcal{U}^*}$, and writing $F = E_\nu^{\mathcal{M}^{\mathcal{T}^*}_{\alpha-1}}$ and $\mu = \mathrm{crit}(F)$, $\nu > \mu^{+\mathcal{M}^{\mathcal{T}^*}_{\alpha-1}} = \mu^{+\mathcal{M}^{\mathcal{U}^*}_{\alpha-1}}$ and there is some sequence $\vec{\varphi} = (\varphi_i \colon i < \mu) \in \mathcal{M}^{\mathcal{T}^*}_{\alpha-1}|\nu = \mathcal{M}^{\mathcal{U}^*}_{\alpha-1}|\nu$ of formulae associated with the δ–version of the extender algebra of the current models such that the extender sequence of M_{sw} satisfies $\bigvee i_F(\vec{\varphi}) \cap \mathcal{M}^{\mathcal{T}^*}_{\alpha-1}|\nu$ but not $\bigvee \vec{\varphi}$,

and then we let $\mathcal{T}^* \restriction (\alpha+1)$ and $\mathcal{U}^* \restriction (\alpha+1)$ arise by applying $E_\nu^{\mathcal{M}^{\mathcal{T}^*}_{\alpha-1}}$ and $E_\nu^{\mathcal{M}^{\mathcal{U}^*}_{\alpha-1}}$ (and padding on one side if ν was chosen according to (a) and on this one side the extender is empty), with the understanding that we stop the construction if there is no such ν; and if α is a limit ordinal, then we pick the unique cofinal branches through $\mathcal{T}^* \restriction \alpha$ and $\mathcal{U}^* \restriction \alpha$ whose limit models have $\mathcal{Q}$–structures as initial segments which are given by $P(\mathcal{M}(\mathcal{T}^* \restriction \alpha)) = P(\mathcal{M}(\mathcal{T}^* \restriction \alpha))$, and we let $\mathcal{T}^* \restriction (\alpha+1)$ and $\mathcal{U}^* \restriction (\alpha+1)$ arise by adding those branches, again with the understanding that we stop the construction if such branches don't exist. Notice that $\mathcal{T}^*$ and U^* are defined inside M_{sw}. By [13, Lemmata 1.3 and 1.5], the construction of $\mathcal{T}^*$ and $\mathcal{U}^*$ will stop exactly at stage $\eta^{+M_{\mathrm{sw}}}$, which means that we produced $P(\mathcal{M}(\mathcal{T}^*)) = P(\mathcal{M}(\mathcal{U}^*)) \in \mathcal{F}$ such that by Lemma 2.2, writing $R = P(\mathcal{M}(\mathcal{T}^*)) = P(\mathcal{M}(\mathcal{U}^*))$, R is a Σ_N–iterate of N as well as a $\Sigma_{N'}$–iterate of N'.

We may now let

$$(\mathcal{M}_\infty, (\pi_{N,\infty} \colon N \in \mathcal{F})) = \mathrm{dirlim}(N, (\pi_{N,N'} \colon N, N' \in \mathcal{F})).$$

Notice that even though $\mathcal{F}$ is a definable collection of classes in M_{sw}, this system is not in M_{sw}, as the maps $\pi_{N,N'}$ are not in M_{sw}.

We are now going to show that we may "catch" $\mathcal{F}$ by a system which does exist in M_{sw}.

In what follows, we shall write $\delta_\infty = \delta^{\mathcal{M}_\infty}$ and $\kappa_\infty = \kappa^{\mathcal{M}_\infty}$.

Let s be a non–empty finite set of ordinals. Write $s^- = s \setminus \max(s)$. For $N = P(\mathcal{M}(\mathcal{U})) \in \mathcal{F}$ we call N *s–iterable* iff for all $\mathcal{T} \in M_{\mathrm{sw}}$ on $\mathcal{M}(\mathcal{U})$ of limit length $\lambda < \kappa$ such that $\mathcal{U}^\frown\mathcal{T} \in \mathbb{U}$, say $\mathcal{T} = (\mathcal{T}_k \colon k < n)$, $n < \omega$, there are for every $i < n$ cofinal branches

$$b_i \in (M_{\mathrm{sw}})^{\mathrm{Col}(\omega,\max(s))}$$

through $\mathcal{T}_i$ such that, writing N_0 for the starting model of $\mathcal{T}_0$ and $N_{i+1} = P(\mathcal{M}(\mathcal{T}_i))$,

$$\pi^{\mathcal{T}_i}_{0,b_i}(s) = s, \text{ and} \tag{8}$$

$$\pi^{\mathcal{T}_i}_{0,b_i}(N_i|\max(s)) = N_{i+1}|\max(s). \tag{9}$$

Writing b for the composition of the branches b_i, $i < n$, and then writing

$$\gamma_s^N = \sup(\delta^N \cap \mathrm{Hull}^{N|\max(s)}(s^-)),$$

the "zipper argument," cf. e.g. the proof of [19, Theorem 6.10], shows that the map

$$\pi_{0,b}^{\mathcal{T}} \restriction \mathrm{Hull}^{N|\max(s)}(\gamma_s^N \cup s^-) \tag{10}$$

is independent from the particular choice of b and hence is in M_{sw}, and moreover if

$$\pi_{N,N'}(s) = s, \text{ and} \tag{11}$$
$$\pi_{N,N'}(N|\max(s)) = N'|\max(s), \tag{12}$$

then

$$\pi_{0,b}^{\mathcal{T}} \restriction \mathrm{Hull}^{N|\max(s)}(\gamma_s^N \cup s^-) = \pi_{N,N'} \restriction \mathrm{Hull}^{N|\max(s)}(\gamma_s^N \cup s^-). \tag{13}$$

We now aim to define $\pi_{N,N'}^s$. For this, we make use of the concept of "strong s–iterability."[6] Let s, s^-, and $N = P(\mathcal{M}(\mathcal{U})) \in \mathcal{F}$ be as before. We call N *strongly s–iterable* iff N is s–iterable and for all $\mathcal{T} \in M_{\mathrm{sw}}$ on $\mathcal{M}(\mathcal{U})$ of limit length $\lambda < \kappa$ such that $\mathcal{U}^\frown\mathcal{T} \in \mathbb{U}$, say $\mathcal{T} = (\mathcal{T}_k \colon k < n)$, $n < \omega$, and for all $\mathcal{T}' \in M_{\mathrm{sw}}$ on $\mathcal{M}(\mathcal{U})$ of limit length $\lambda' < \kappa$ such that $\mathcal{U}^\frown\mathcal{T}' \in \mathbb{U}$, say $\mathcal{T}' = (\mathcal{T}'_k \colon k < n')$, $n' < \omega$, if the $b_i \in (M_{\mathrm{sw}})^{\mathrm{Col}(\omega,\max(s))}$ are cofinal branches through $\mathcal{T}_i$ which "fix s" à la (8) and (9), $i < n$, and if the $b'_i \in (M_{\mathrm{sw}})^{\mathrm{Col}(\omega,\max(s))}$ are cofinal branches through $\mathcal{T}'_i$ which "fix s" à la (8) and (9), $i < n'$, and if b is the composition of the branches b_i, $i < n$, and if b' is the composition of the branches b'_i, $i < n'$, then

$$\pi_{0,b}^{\mathcal{T}} \restriction \mathrm{Hull}^{N|\max(s)}(\gamma_s^N \cup s^-) = \pi_{0,b'}^{\mathcal{T}'} \restriction \mathrm{Hull}^{N|\max(s)}(\gamma_s^N \cup s^-). \tag{14}$$

If (11) and (12) hold true, then by (13) so does (14).

Let us write

$$(N, s) \preceq_{\mathcal{F}} (N', t)$$

to express the fact that $N \in \mathcal{F}$ is strongly s–iterable, $N' \in \mathcal{F}$ is strongly t–iterable, $t \supset s$, and there is a tree $\mathcal{T} \in M_{\mathrm{sw}}$ on N as above such that $N' = P(\mathcal{M}(\mathcal{T}))$. If $(N, s) \preceq_{\mathcal{F}} (N', s)$, then we shall write $\pi_{N,N'}^s$ for the unique map as in (14).

[6] At the cost of making use of [20], we could avoid the concept of "strong s–iterability," as follows. If $N = P(\mathcal{M}(\mathcal{U}))$, $N' \in \mathcal{F}$ and there is some $\mathcal{T}$ with $\mathcal{U}^\frown\mathcal{T} \in \mathbb{U}$ such that $N' = P(\mathcal{M}(\mathcal{T}))$, then by [20], there is a unique *normal* such $\mathcal{T}$ with $\mathcal{U}^\frown\mathcal{T} \in \mathbb{U}$. We may then define $\pi_{N,N'}^s$ as the unique map as in (10) for any cofinal branch $b \in (M_{\mathrm{sw}})^{\mathrm{Col}(\omega,\max(s))}$ through $\mathcal{T}$ which "fixes s" as in (8) and (9).

Notice that for N and s as above, the (strong) s–iterability of N is uniformly defined in a way which is first order over $M_{\rm sw}$.

Let s be a non–empty finite set of ordinals, $N = P(\mathcal{M}(\mathcal{U})) \in \mathcal{F}$, and $\mathcal{U}^\frown\mathcal{T} \in \mathbb{U}$. Write $c = \Sigma_N(\mathcal{T})$. If $\pi^{\mathcal{T}}_{0,c}(s) = s$, then an easy absoluteness argument shows that there is also some $b \in (M_{\rm sw})^{{\rm Col}(\omega,\max(s))}$ with (8) and (9) above.

Lemma 2.3 *Let $N = P(\mathcal{M}(\mathcal{U})) \in \mathcal{F}$.*

(1) Let s be any non–empty finite set of ordinals. There is some $\mathcal{T}$ such that $\mathcal{U}^\frown\mathcal{T} \in \mathbb{U}$ and $N' = P(\mathcal{M}(\mathcal{T}))$ is strongly s–iterable.

(2) Let $\{\eta_1 < \ldots < \eta_\ell\} \subset I$, where I is the class of generating indiscernibles for $M_{\rm sw}$ given by iterating the top measure of $(M_{\rm sw}|\delta)^\P$ out of the universe, and write $s = \{\eta_1, \ldots, \eta_\ell\}$. Then N is strongly s–iterable.

Proof. (1): Otherwise there would some non–empty finite set s of ordinals and some infinite sequence $(N_n \colon n < \omega)$ such that $N_0 = M_{\rm sw}$, and N_{n+1} is a Σ_{N_n}–iterate of N_n via some tree $\mathcal{T}_n$ such that $\mathcal{T}_0{}^\frown \ldots {}^\frown\mathcal{T}_n \in \mathbb{U}$ and $\pi_{N_n,N_{n+1}}(s) > s$ for all $n < \omega$. This contradicts the $(\omega, \omega, {\rm OR})$–iterability of $M_{\rm sw}$ in V.

(2): This follows from Lemma 2.2 (c) by a trivial absoluteness argument. □ (Lemma 2.3)

The collection of all strongly s–iterable $N \in \mathcal{F}$ is finitely directed in that if $N \in F$ is strogly s–iterable and $N' \in \mathcal{F}$ is strongy t–iterable, then there is $N^* \in \mathcal{F}$ which is strongly $(s \cup t)$–iterable and

$$(N, s), (N', t) \preceq_{\mathcal{F}} (N^*, s \cup t).$$

This is true because given (N, s) and (N', t), we may pick some $R \in \mathcal{F}$ which is strongly $s \cup t$–iterable. A joint comparison process as defined above will then produce some strongly $s \cup t$–iterable $N^* \in \mathcal{F}$ which in V is Σ_N–iterate of N, a $\Sigma_{N'}$–iterate of N', as well as a Σ_R–iterate of R.

We may then let

$$(\mathcal{M}'_\infty, (\pi^s_{N,\infty} \colon N \in \mathcal{F}, N \text{ is strongly } s\text{–iterable})) \tag{15}$$

be the direct limit of the system $(N, (\pi^s_{N,N'} \colon (N, s) \preceq_{\mathcal{F}} (N', s))$.

Lemma 2.4

$$\mathcal{M}_\infty = \mathcal{M}'_\infty. \tag{16}$$

Proof. Let ρ' be any ordinal, and let $\rho' = \pi_{N,\infty}(\rho)$, where $N \in \mathcal{F}$. Let $\chi < \delta^N$ and let $\bar{s}$ be a finite set of indiscernibles for $M_{\rm sw}$ such that

$$\rho \in \mathrm{Hull}^N(\chi \cup \{\bar{s}\}).$$

Such χ and $\bar{s}$ exist by Lemma 2.2 (b). As $\mathrm{ran}(\pi_{M_{\rm sw},N}) \cap \delta^N$ is cofinal in δ^N, we may in addition assume (by enlarging χ and $\bar{s}$ if necessary) that

$$[\chi, \delta^N) \cap \mathrm{Hull}^N(\{\bar{s}\}) \neq \emptyset.$$

Let $s = \bar{s} \cup \{\tau\}$, where τ is any V–cardinal strictly above $\max(\bar{s})$. Then N is strongly s–iterable by Lemma 2.3, and $\gamma^N_s > \chi$, so that $\rho \in \mathrm{dom}(\pi^s_{N,\infty})$.

This shows that we may define an elementary embedding $\varphi\colon \mathcal{M}_\infty \to \mathcal{M}'_\infty$ by $\varphi(\pi_{N,\infty}(\rho)) = \pi^s_{N,\infty}(\rho)$ for ρ and s as above. It remains to be shown that φ is surjective.

To this end, let again ρ' be any ordinal, and let $\pi^s_{N,\infty}(\rho) = \rho'$, where $N \in \mathcal{F}$ is strongly s–iterable. Let $N = P(\mathcal{M}(\mathcal{U}))$, and let $\mathcal{T}$ be such that $\mathcal{U}^\frown\mathcal{T} \in \mathbb{U}$ and, setting $N' = P(\mathcal{M}(\mathcal{T}))$,

$$\pi_{N',N''}(s) = s \text{ for all } (N', s) \preceq_{\mathcal{F}} (N'', s), \tag{17}$$

cf. the proof of Lemma 2.3 (1). We may pick a finite set t of indiscernibles for $M_{\rm sw}$ such that

$$s \in \mathrm{Hull}^{N'|\max(t)}(\gamma^{N'}_t \cup t^-),$$

cf. above. We then have that

$$\pi^s_{N,N'}(\rho) \in \mathrm{Hull}^{N'|\max(t)}(\gamma^{N'}_t \cup t^-).$$

Also N' is strongly $s \cup t$–iterable, by (17) and the proof of Lemma 2.3 (2), and because $\pi^s_{N',N''} \subset \pi^{s\cup t}_{N',N''} = \pi_{N',N''} \restriction \mathrm{Hull}^{N'|\max(t)}(\gamma^{N'}_t \cup t^-)$ for $(N', s\cup t) \preceq (N'', s\cup t)$ (which is equivalent to $(N', s) \preceq (N'', s)$), we will get that

$$\rho' = \pi^s_{N,\infty}(\rho) = \pi^{s\cup t}_{N',\infty}(\pi^s_{N,N'}(\rho)) = \pi_{N',\infty}(\pi^s_{N,N'}(\rho)),$$

soi that φ is indeed onto and hence the identity. We showed (16). $\square$ (Lemma 2.4)

The following is straightforward to verify.

Lemma 2.5 *In V, $\mathcal{M}_\infty$ is a Σ–iterate of $M_{\rm sw}$ via an ω–stack of normal trees each of which are individually in $M_{\rm sw}$.*

Moreover, let F be a total extender from the $M_{\rm sw}$–sequence with $\mathrm{crit}(F) = \kappa$*, and write $j\colon M_{\rm sw} \to_F \mathrm{ult}(M_{\rm sw}; F)$. Then $j(\mathcal{M}_\infty)$ is an $\Sigma_{\mathcal{M}_\infty}$– iterate of $\mathcal{M}_\infty$ via using $\pi_{M_{\rm sw},\infty}(F)$, followed by an ω–stack of normal iteration trees which are according to $\Sigma_{\mathrm{ult}(\mathcal{M}_\infty;\pi_{M_{\rm sw},\infty}(F))}$.*

Proof. Let $(\mathcal{U}_k\colon k<\omega)$ be such that $\mathcal{U}_k \in \mathbb{U}$ for all $k<\omega$ and setting $N_k = P(\mathcal{M}(\mathcal{U}_k))$ for $k<\omega$, $(N_k\colon k<\omega)$ is cofinal in $\mathcal{F}$, i.e., if $P(\mathcal{M}(\mathcal{U})) \in \mathcal{F}$, then there is some $k<\omega$ such that N_k is a $\Sigma_{P(\mathcal{M}(\mathcal{U}))}$–iterate of $P(\mathcal{M}(\mathcal{U}))$. The direct limit of the N_k, along with the maps π_{N_k,N_ℓ}, $k \leq \ell < \omega$, must yield $\mathcal{M}_\infty$.

Next, we have for every $N \in \mathcal{F}$, $j(N) \in j(\mathcal{F})$ and $j(N) = \mathrm{ult}(N; F \restriction N)$, where $F \restriction N$ is on the sequence of N. The direct limit of the $\mathrm{ult}(N; E \restriction N)$, along with $j(\pi_{N,N'})$, with N, $N' \in \mathcal{F}$, N' being a Σ_N–iterate of N, is then equal to $\mathrm{ult}(\mathcal{M}_\infty; \pi_{M_{\mathrm{sw}},\infty}(F))$ and canonically embeds into $j(\mathcal{M}_\infty)$. If $N = P(\mathcal{M}(\mathcal{U})) \in \mathcal{F}$, then $\mathrm{ult}(N; E \restriction N)$ is an iterate of M_{sw} via $\mathcal{U}^\frown E \restriction N$, and if N, $N' \in \mathcal{F}$, where N' is a Σ_N–iterate of N via $\mathcal{T}$, and if $\mathcal{T} = \mathcal{U}_0{}^\frown \ldots {}^\frown \mathcal{U}_{k-1}$, where all $\mathcal{U}_i$, $i<k$, are normal, then $j(\mathcal{U}_i)$ has the very same tree structure as $\mathcal{U}_i$, and, as $\mathcal{U}_i$ is a hull of $j(\mathcal{U}_i)$, the fact that Σ satisfies branch condensation implies that $j(\mathcal{U}_i)$ is according to Σ and $\Sigma(\mathcal{U}_i) = \Sigma(j(\mathcal{U}_i))$ for $i<k$.

We may conclude that the collection of all $j(N)$, for $N \in \mathcal{F}$, is definable in $\mathrm{ult}(M_{\mathrm{sw}}; F)$, and for $\eta = \kappa$ which is a cutpoint of $\mathrm{ult}(M_{\mathrm{sw}}; F)$ below $j(\kappa)$ we may work in $\mathrm{ult}(M_{\mathrm{sw}}; F)$ to simultaneously compare all $j(N)$, $N \in \mathcal{F}$, in a fashion as on p. 8f. to produce some $M = P^{\mathrm{ult}(M_{\mathrm{sw}};F)}(\mathcal{M}(\mathcal{U}')) \in j(\mathcal{F})$ with $\delta(\mathcal{U}') = \kappa^{+\mathrm{ult}(M_{\mathrm{sw}};F)} = \kappa^{+M_{\mathrm{sw}}}$ and such that M is a $\Sigma_{j(N)}$–iterate of $j(N)$ for all $N \in \mathcal{F}$.

$\mathrm{ult}(\mathcal{M}_\infty; \pi_{M_{\mathrm{sw}},\infty}(F))$ is a definable inner model of $\mathrm{ult}(M_{\mathrm{sw}}; F)$ and the former must now canonically embed into M. We may then choose some $\eta > \kappa$ which is a cutpoint of $\mathrm{ult}(M_{\mathrm{sw}}; F)$ and work in $\mathrm{ult}(M_{\mathrm{sw}}; F)$ to compare M with $\mathrm{ult}(\mathcal{M}_\infty; \pi_{M_{\mathrm{sw}},\infty}(F))$ in a fashion as on p. 8f. to produce some $M^* = P^{\mathrm{ult}(M_{\mathrm{sw}};F)}(\mathcal{M}(\mathcal{U}^*)) \in j(\mathcal{F})$ with $\delta(\mathcal{U}^*) = \eta^{+\mathrm{ult}(M_{\mathrm{sw}};F)}$ and such that M^* is a Σ_M–iterate of M and also an iterate of $\mathrm{ult}(\mathcal{M}_\infty; \pi_{M_{\mathrm{sw}},\infty}(F))$ via $\Sigma_{\mathrm{ult}(\mathcal{M}_\infty;\pi_{M_{\mathrm{sw}},\infty}(F))}$. We may actually produce an ω–sequence of such M^* which is cofinal in $\mathcal{F}^{\mathrm{ult}(M_{\mathrm{sw}};F)}$.

$j(\mathcal{M}_\infty)$ may thus be represented as an iterate of $\mathcal{M}_\infty$ via using $\pi_{M_{\mathrm{sw}},\infty}(F)$, followed by an ω–stack of normal iteration trees which are according to $\Sigma_{\mathrm{ult}(\mathcal{M}_\infty;\pi_{M_{\mathrm{sw}},\infty}(F))}$. $\square$ (Lemma 2.5)

Inside $\mathcal{M}_\infty$, we may look at the image of the system (15) under the map $\pi_{0,\infty}$. Let us write $\mathcal{M}_\infty^\infty$ for the direct limit model, i.e.,

$$\mathcal{M}_\infty^\infty = \pi_{M_{\mathrm{sw}},\infty}(\mathcal{M}_\infty),$$

which is a definable subclass of $\mathcal{M}_\infty$, defined in the same way over $\mathcal{M}_\infty$ as $\mathcal{M}_\infty$ was defined over M_{sw} by (15). In analogy to Lemma 2.5, we have:

Lemma 2.6 *If $N \in \mathcal{F}^{\mathcal{M}_\infty}$, then N is a $\Sigma_{\mathcal{M}_\infty}$–iterate of $\mathcal{M}_\infty$, and $\mathcal{M}_\infty^\infty$ is a $\Sigma_{\mathcal{M}_\infty}$–iterate of $\mathcal{M}_\infty$ via an ω–stack of normal trees on $\mathcal{M}_\infty^\infty$.*

In particular, we get a unique iteration map, call it $\pi^\infty_{0,\infty}$, from $\mathcal{M}_\infty$ into $\mathcal{M}^\infty_\infty$, which is given by Σ_{M_∞}. A priori, there doesn't seem to be a reason why $\pi^\infty_{0,\infty}$ should be definable in $M_{\rm sw}$.

However, for each ordinal ρ let us denote by ρ^* the minimum of the set of all $\pi_{N,\infty}(\rho)$ for $N \in \mathcal{F}$. The argument for $\mathcal{M}_\infty = \mathcal{M}'_\infty$ we gave above shows that for every ρ and every $N \in \mathcal{F}$ there is some finite set s of ordinals such that N is strongly s–iterable and $\rho \in {\rm dom}(\pi^s_{N,\infty})$. We may then define $\rho \mapsto \rho^*$ *inside* $M_{\rm sw}$ by

$$\rho^* = \min(\{\pi^s_{N,\infty}(\rho) \colon N \text{ is strongly } s\text{–iterable and } \rho \in {\rm dom}(\pi^s_{N,\infty})\}). \tag{18}$$

We have that if $\rho = \pi_{N,\infty}(\bar\rho)$, where N is strongly s–iterable for some s such that $\rho \in {\rm ran}(\pi^s_{N,\infty})$, then

$$\begin{aligned} \pi_{N,\infty}(\rho) &= \pi_{N,\infty}(\pi_{N,\infty}(\bar\rho)) \\ &= \pi_{N,\infty}(\pi^s_{N,\infty}(\bar\rho)) \\ &= \pi_{N,\infty}(\pi^s_{N,\infty})(\pi_{N,\infty}(\bar\rho)) \\ &= \pi^\infty_{0,\infty}(\rho), \end{aligned}$$

which means that

$$\rho^* = \pi^\infty_{0,\infty}(\rho).$$

Notice that $\pi^\infty_{0,\infty}$ is also equal to the ultrapower map produced by applying the long extender derived from $\pi^\infty_{0,\infty} \restriction \mathcal{M}_\infty | \delta_\infty$ to the model $\mathcal{M}_\infty$. In other words,

$$\rho \mapsto \rho^* \text{ may be defined inside the model } L[\mathcal{M}_\infty, (\rho \mapsto \rho^*) \restriction \delta_\infty], \tag{19}$$

and in particular

$$L[\mathcal{M}_\infty, (\rho \mapsto \rho^*)] = L[\mathcal{M}_\infty, (\rho \mapsto \rho^*) \restriction \delta_\infty].$$

Lemma 2.7 *(a) κ is the least measurable cardinal of $\mathcal{M}_\infty$.*

(b) $\delta_\infty = \kappa^{+M_{\rm sw}}$.

(c) $\kappa^{+M_{\rm sw}} < \kappa_\infty < (\kappa_\infty)^{+\mathcal{M}_\infty} < (\kappa_\infty)^{++\mathcal{M}_\infty} = \kappa^{++M_{\rm sw}}$.

Proof. (a): This is easy.

(b): Cf. [21, Lemma 3.38 (2)]. To show that $\delta_\infty \leq \kappa^+$ in $M_{\rm sw}$, let $\eta < \delta_\infty$, say $\eta = \pi^s_{N,\infty}(\bar\eta)$, where $N \in \mathcal{F}$ is strongly s–iterable and $\bar\eta < \gamma^N_s$. Then each ordinal below η is of the form $\pi^s_{N',\infty}(\zeta)$ for some $N' \in \mathcal{F}$ with $(N, s) \preceq_\mathcal{F} (N', s)$ and $\zeta < \pi^s_{N,N'}(\bar\eta)$. As $\mathcal{F}$ has cardinality κ, this shows that $\eta < \kappa^+$ in $M_{\rm sw}$.

Let us now show that $\kappa^{+M_{\rm sw}} \leq \delta_\infty$. Let $\alpha < \kappa^{+M_{\rm sw}}$, and let $f\colon \kappa \to \alpha$, $f \in M_{\rm sw}$, be bijective, say $f = \tau^{M_{\rm sw}|\max(s)}(s^-)$, where τ is a Σ_1–Skolem term and s is a finite set of $M_{\rm sw}$–indiscernibles.

Let $\beta < \alpha$, and let $\lambda < \kappa$ be such that $\beta = f(\lambda)$. Let $N \in \mathcal{F}$ be such that

$$\lambda < \min(\gamma_s^N, \text{ the least measurable cardinal of } N)$$

and $\pi^s_{N,N'}(\beta) = \beta$ for all $N' \in \mathcal{F}$ where $\pi^s_{N,N'}$ is defined. Let

$$S^N = \{\epsilon \colon \exists \mu < \text{the least measurable of } N\, \exists p \in \mathbb{B}^N\, p \Vdash^{\mathbb{B}^N}_N \tau^{N[\dot{G}]|\max(s)}(\check{s}^-)(\check{\mu}) = \check{\epsilon}\}.$$

We have that $\beta \in S^N$ and $\mathrm{otp}(S^N) < \delta^N$. Let γ^N_β be the unique γ such that β is the $\gamma^{\rm th}$ element of S^N. In particular, $\gamma^N_\beta < \delta^N$.

We claim that $\beta \mapsto \pi^s_{N,\infty}(\gamma^N_\beta)$ is well–defined, i.e., that it is independent from the particular choice of an N as above, and that it is also order–preserving. Well, this is because if $\beta \leq \beta' < \alpha$ and γ^N_β and $\gamma^{N'}_{\beta'}$ are defined, then there is some $Q \in \mathcal{F}$ such that $\pi^s_{N,Q}$ and $\pi^s_{N',Q}$ are both defined and $\pi^s_{N,Q}(S^N) = Q^N = \pi^s_{N',Q}(S^{N'})$, and hence $\gamma^Q_\beta \leq \gamma^Q_{\beta'}$.

But now $\beta \mapsto \pi^s_{N,\infty}(\gamma^N_\beta)$ is an injection from α into δ_∞ which exists in $M_{\rm sw}$.

(c): $\kappa^{+M_{\rm sw}} < \kappa_\infty$ is obviously given by (b).

To show that $(\kappa_\infty)^{+\mathcal{M}_\infty} < \kappa^{++M_{\rm sw}}$, we use the argument from the proof of Lemma 2.5 and let $F = E^{M_{\rm sw}}_\nu$ be the least total extender of the $M_{\rm sw}$–sequence which has critical point κ. Write $i_F\colon M_{\rm sw} \to_F W = \mathrm{ult}(M_{\rm sw}; F)$, so that $i_F(\kappa)^{+W} < \kappa^{++M_{\rm sw}} = \kappa^{++W}$. For each $N \in \mathcal{F}$, $F \cap N$ is the least total extender of the N–sequence which has critical point $\kappa = \kappa^N$, and $\mathrm{ult}(N; F \cap N) \in \mathcal{F}^W$. A joint comparison process as defined above on p. 8f. allows us to produce some $N^* \in \mathcal{F}^W$ such that

1. in V, N^* is a $\Sigma_{\mathrm{ult}(N;F\cap N)}$–iterate of $\mathrm{ult}(N; F \cap N)$ for all $N \in \mathcal{F} = \mathcal{F}^{M_{\rm sw}}$, and
2. $\delta^{N^*} = \kappa^{+W} = \kappa^{+M_{\rm sw}}$.

As Σ is commuting, for each $N \in \mathcal{F}$ there is a unique iteration map, call it π_{N,N^*}, from N to N^*, namely the ultrapower map $N \to \mathrm{ult}(N; F \cap N)$ followed by the iteration map from $\mathrm{ult}(N; F \cap N)$ to N^*, and if N, $N' \in \mathcal{F}$ such that $\pi_{N,N'}$ exists, then

$$\pi_{N',N^*} \circ \pi_{N,N'} = \pi_{N,N^*}.$$

Therefore, there is a canonical elementary embedding

$$k\colon \mathcal{M}_\infty \to N^*.$$

But $N^* = P(N^*|\kappa^{+M_{\rm sw}})$, as being constructed inside W. Therefore,

$$k(\kappa_\infty) = \kappa^{N^*} = \kappa^W = i_F(\kappa),$$

and

$$(\kappa_\infty)^{+\mathcal{M}_\infty} \leq i_F(\kappa)^{+W} < \kappa^{++M_{\rm sw}}.$$

Finally, $(\kappa_\infty)^{++\mathcal{M}_\infty} = \pi_{M_{\rm sw},\infty}(\kappa^{++M_{\rm sw}}) \geq \kappa^{++M_{\rm sw}}$. As $\kappa^{++M_{\rm sw}}$ is a cardinal in $\mathcal{M}_\infty$, this gives $(\kappa_\infty)^{++\mathcal{M}_\infty} = \kappa^{++M_{\rm sw}}$. □ (Lemma 2.7)

The following key lemma makes up the first key step in analyzing the mantle of $M_{\rm sw}$.

Lemma 2.8 *Let us write $\kappa^+ = \kappa^{+M_{\rm sw}}$ and $\kappa^{++} = \kappa^{++M_{\rm sw}}$.*[7] *$M_{\rm sw}$ is a forcing extension of $L[M_\infty, \rho \mapsto \rho^*]$ via some $\mathbb{P}$ which satisfies the κ^+–c.c.*

In fact,

$$M_{\rm sw} = L[M_\infty, \rho \mapsto \rho^*][M_{\rm sw}|\kappa^{++}],$$

where $M_{\rm sw}|\kappa^{++}$ is $\mathbb{P}$–generic over $L[M_\infty, \rho \mapsto \rho^*]$ for some $\mathbb{P} \in L[M_\infty, \rho \mapsto \rho^*]$ such that $L[M_\infty, \rho \mapsto \rho^*] \models$ "$\mathbb{P}$ has the κ^+-c.c. and is of size κ^{++}."

Proof. We shall make use of Bukovský's theorem from [1]. For the reader's convenience, we give a proof sketch in the appendix to the current paper, cf. Theorem 3.5, cf. also [15].

We claim that $L[M_\infty, \rho \mapsto \rho^*]$ uniformly κ^+–covers $M_{\rm sw}$, cf. Definition 3.1, i.e., for all functions $f \in M_{\rm sw}$ with $\mathrm{dom}(f) \in L[M_\infty, \rho \mapsto \rho^*]$ and $\mathrm{ran}(f) \subset L[M_\infty, \rho \mapsto \rho^*]$ there is some function $g \in L[M_\infty, \rho \mapsto \rho^*]$ with $\mathrm{dom}(g) = \mathrm{dom}(f)$ such that for all $x \in \mathrm{dom}(g)$,

(a) $f(x) \in g(x)$ and

(b) $\mathrm{Card}(g(x)) < \kappa^+$ for all $x \in \mathrm{dom}(g)$.

It obviously suffices to prove this for all f whose domain is an ordinal and whose range is contained in the class of all ordinals.

Suppose what we claim would not be true. As $L[M_\infty, \rho \mapsto \rho^*]$ is definable inside $M_{\rm sw}$ (from $M_{\rm sw}$'s extender sequence[8]), there is then some counterexample $f\colon \theta \to \mathrm{OR}$ which is parameter–free definable inside $M_{\rm sw}$ (again, from $M_{\rm sw}$'s extender sequence).

[7] Making use of this notation, we will later show that $\kappa^{++} = (\kappa_\infty)^{++\mathcal{M}_\infty}$, cf. Lemma 2.9.

[8] Claim 2.12 (a) will in fact prove a stronger definability fact, but this is not needed here.

Let us fix such an f, $f\colon \theta \to \mathrm{OR}$, and let φ be a formula in the language of M_{sw} such that for all ξ, η, $f(\xi) = \eta$ iff $M_{\mathrm{sw}} \models \varphi(\xi, \eta)$.

If $N \in \mathcal{F}$, then $M_{\mathrm{sw}} = N[h]$ for some h which is $\mathbb{B}^N$–generic over N; in fact, $h = M_{\mathrm{sw}}|\delta^N$. The extender sequence of M_{sw} is then uniformly definable inside $N[h]$ from the extender sequence of N and the parameter $M_{\mathrm{sw}}|\delta^N$. There is then a formula ψ such that for all $N \in \mathcal{F}$, ψ is a formula of the forcing language of N associated to forcing with $\mathbb{B}^N$ over N such that if $M_{\mathrm{sw}} = N[h]$, where h which is $\mathbb{B}^N$–generic over N, then for all ξ, η, $M_{\mathrm{sw}} \models \varphi(\xi, \eta)$ iff there is some $p \in h$ such that $p \Vdash_N^{\mathbb{B}^h} \psi(\check{\xi}, \check{\eta})$. Of course, the formula ψ is also a formula of the forcing language of $\mathcal{M}_\infty$ associated to forcing with $\mathbb{B}^{\mathcal{M}_\infty}$ over $\mathcal{M}_\infty$.

Let $N \in \mathcal{F}$ or $N = \mathcal{M}_\infty$. If $p \in \mathbb{B}^N$, then we write

$$p \Vdash_N^{\mathbb{B}^N} \text{“}\psi \text{ defines a function”}$$

to mean that

$$p \Vdash_N^{\mathbb{B}^N} \forall v \forall w \forall w' \, \psi(v, w) \wedge \psi(v, w') \to w = w'.$$

Let $g_N \in N$ be the function with domain $\pi_{M_{\mathrm{sw}},N}(\theta)$ (in case $N = \mathcal{M}_\infty$ by this we mean $\pi_{M_{\mathrm{sw}},\infty}(\theta)$) such that for all $\xi < \pi_{M_{\mathrm{sw}},N}(\theta)$,

$$g_N(\xi) = \{\eta\colon \exists p \in \mathbb{B}^N \, p \Vdash_N^{\mathbb{B}^N} \text{“}\psi \text{ defines a function and } \psi(\check{\xi}, \check{\eta})\text{”}\} \tag{20}$$

As $\mathbb{B}^N$ has the δ^N–c.c. inside N, $\mathrm{Card}(\tilde{g}(\xi)) < \delta^N$ in N for all $\xi < \pi_{M_{\mathrm{sw}},N}(\theta)$.

Of course, if $N \in \mathcal{F}$, then $\pi_{N,\infty}(g_N) = g_{\mathcal{M}_\infty}$.

Let $g \in L[\mathcal{M}_\infty, \rho \mapsto \rho^*]$ be the function with domain θ such that for all $\xi < \theta$,

$$g(\xi) = \{\eta\colon \eta^* \in g_{\mathcal{M}_\infty}(\xi^*)\}. \tag{21}$$

Obviously, $\mathrm{Card}(g(\xi)) \leq \mathrm{Card}(g_{\mathcal{M}_\infty}(\xi^*)) < \delta_\infty$ in $L[\mathcal{M}_\infty, \rho \mapsto \rho^*]$.

Let $\xi < \theta$ and $\eta = f(\xi)$, i.e., $M_{\mathrm{sw}} \models \varphi(\xi, \eta)$. Pick $N \in \mathcal{F}$ such that $\xi^* = \pi_{N,\infty}(\xi)$ and $\eta^* = \pi_{N,\infty}(\eta)$. As $M_{\mathrm{sw}} = N[h]$, for some h which is $\mathbb{B}^N$–generic over N, there is some $p \in h \subset \mathbb{B}^N$ with

$$p \Vdash_N^{\mathbb{B}^N} \text{“}\psi \text{ defines a function and } \psi(\check{\xi}, \check{\eta}),\text{”} \tag{22}$$

so that $\eta \in g_N(\xi)$. But then

$$\eta^* = \pi_{N,\infty}(\eta) \in \pi_{N,\infty}(g_N)(\pi_{N,\infty}(\xi)) = g_{\mathcal{M}_\infty}(\xi^*),$$

and hence $\eta \in g(\xi)$. Because $\delta_\infty = \kappa^+$ by Lemma 2.7, we have shown that $L[M_\infty, \rho \mapsto \rho^*]$ κ^+–uniformly covers M_{sw}.

The conclusion now follows from Theorem 3.5, letting the λ from the statement of Theorem 3.5 be equal to $\kappa^{+M_{\mathrm{sw}}}$. □ (Lemma 2.8)

Lemma 2.9 *(a) M_∞ is fully iterable inside $M_{\rm sw}$, in fact $\Sigma_{\mathcal{M}_\infty} \restriction M_{\rm sw}$ is definable in $M_{\rm sw}$.*

(b) If $\mathbb{P}$ is a poset in $M_{\rm sw}$ and if $g \in V$ is $\mathbb{P}$–generic over $M_{\rm sw}$, then $\mathcal{M}_\infty$ is fully iterable inside $M_{\rm sw}[g]$, in fact $\Sigma_{\mathcal{M}_\infty} \restriction M_{\rm sw}[g]$ is definable in $M_{\rm sw}[g]$.

(c) $\kappa^{+M_{\rm sw}} = \delta_\infty < (\delta_\infty)^{+L[\mathcal{M}_\infty,\rho\mapsto\rho^*]} = \kappa^{++M_{\rm sw}}$.

(d) If λ is a cardinal of $L[\mathcal{M}_\infty, \rho \mapsto \rho^*]$ with $\lambda \geq \delta_\infty$, then λ is also a cardinal of $M_{\rm sw}$.

Proof. (a): Cf. [11]. We aim to show that $\Sigma_{\mathcal{M}_\infty} \restriction M_{\rm sw}$ is definable in $M_{\rm sw}$. To this end, let $\mathcal{T} \in M_{\rm sw}$ be a tree of limit length on $\mathcal{M}_\infty$ which is according to $\Sigma_{\mathcal{M}_\infty}$. Let $c = \Sigma_{\mathcal{M}_\infty}(\mathcal{T})$.

If there is a drop along c, or if there is no drop along c and $\delta(\mathcal{T}) \neq \delta^{\mathcal{M}_c^\mathcal{T}}$, then there is a $\mathcal{Q}$–structure $\mathcal{Q} \trianglelefteq \mathcal{M}_c^\mathcal{T}$ which is ¶–small above $\delta(\mathcal{T})$. But then $\mathcal{Q} \in M_{\rm sw}$, as $\mathcal{Q}$ may be found inside W by stacking sound mice which are ¶–small above $\delta(\mathcal{T})$ and project to $\delta(\mathcal{T})$ on top of $\mathcal{M}(\mathcal{T})$.

Let us now assume that there is no drop along c and $\delta(\mathcal{T}) = \delta^{\mathcal{M}_c^\mathcal{T}}$. We have that $\mathcal{M}_c^\mathcal{T}$ is an iterate of $K(\mathcal{M}(\mathcal{T}))^{M_{\rm sw}}$. Let us assume that $\mathcal{M}_c^\mathcal{T} = K(\mathcal{M}(\mathcal{T}))^{M_{\rm sw}}$ and leave the other case to the reader's discretion.

We then have that $\mathcal{M}_c^\mathcal{T}$ is definable in $\mathcal{M}_{\rm sw}$. Let E be a total extender on the $M_{\rm sw}$–sequence such that ${\rm crit}(E) = \kappa$ and $\mathcal{T} \in {\rm ult}(M_{\rm sw}; E)$. Let us write

$$j\colon M_{\rm sw} \to_E W = {\rm ult}(M_{\rm sw}; E).$$

We may produce some $N \in \mathcal{F}^W$ such that in V, $N|\delta^N$ is a normal iterate of $\mathcal{M}_c^\mathcal{T}|\delta(\mathcal{T})$. There is hence some elementary

$$k'\colon \mathcal{M}_c^\mathcal{T}|\delta(\mathcal{T}) \to j(\mathcal{M}_\infty|\delta_\infty) = (\mathcal{M}_\infty)^W|\delta^{\mathcal{M}_\infty^W}. \tag{23}$$

Let g be ${\rm Col}(\omega, \delta(\mathcal{T}))$–generic over V. Inside $M_{\rm sw}[g]$ let us consider a tree T searching for a cofinal branch b through $\mathcal{T}$ such that b does not drop and there is an elementary embedding

$$k\colon \mathcal{M}_b^\mathcal{T}|\delta(\mathcal{T}) \to j(\mathcal{M}_\infty|\delta_\infty)$$

such that

$$k \circ \pi_{0,b}^\mathcal{T} \restriction \mathcal{M}_\infty|\delta_\infty = j \restriction \mathcal{M}_\infty|\delta_\infty \tag{24}$$

We claim that $c = \Sigma_{\mathcal{M}_\infty}(\mathcal{T})$ is given by a branch through T. To see this, let $x \in \mathcal{M}_\infty|\delta_\infty$. Let $x \in {\rm ran}(\pi_{N,\infty})$, where $N \in \mathcal{F}$, and write $\bar{x} = {\pi_{N,\infty}}^{-1}(x)$. Pick s, a finite set of $M_{\rm sw}$–indiscernibles which is moved neither by $\pi_{M_{\rm sw},\infty}$ nor by j and such that $\bar{x} \in \mathrm{Hull}^{N|\max(s)}(\gamma_s^N \cup s^-) = \mathrm{dom}(\pi^s_{N,\infty})$. Notice that $j(\bar{x}) = \bar{x}$, and $j(N) = \mathrm{ult}(N; E \cap N) \in \mathcal{F}^W$. We may copy $\mathcal{T}$ onto $\mathrm{ult}(\mathcal{M}_\infty; \pi_{M_{\mathrm{sw}},\infty}(E))$ via the map $i = i_{\pi_{M_{\mathrm{sw}},\infty}(E)}$, write $i\mathcal{T}$ for the resulting tree. Let

$$i^* \colon \mathcal{M}_c^{\mathcal{T}} \to \mathrm{ult}(\mathcal{M}_c^{\mathcal{T}}; i_c^{\mathcal{T}} \circ i(E)) = \mathcal{M}_c^{i\mathcal{T}}.$$

We may produce some $N^* \in \mathcal{F}^W$ such that in V, N^* is a $\Sigma_{j(N)}$–iterate of $j(N)$ as well as a $\Sigma_{\mathcal{M}_c^{i\mathcal{T}}}$–iterate of $\mathcal{M}_c^{i\mathcal{T}}$. We write $\pi_{j(N),N^*}$ and $\pi_{\mathcal{M}_c^{i\mathcal{T}},N^*}$ for the iteration maps, and we also write $\pi_{N^*,j(\mathcal{M}_\infty)}$ for the iteration map from N^* to $j(\mathcal{M}_\infty)$.

We now get that

$$\begin{aligned} j(x) &= j(\pi_{N,\infty}(\bar{x})) \\ &= j(\pi^s_{N,\infty}(\bar{x})) \\ &= j(\pi^s_{N,\infty})(j(\bar{x})) \\ &= \pi^s_{j(N),j(\mathcal{M}_\infty)}(\bar{x}) \\ &= \pi_{N^*,j(\mathcal{M}_\infty)} \circ \pi_{\mathcal{M}_c^{i\mathcal{T}},N^*} \circ \pi^{i\mathcal{T}}_{0,c} \circ \pi_{j(N),\mathrm{ult}(\mathcal{M}_\infty;\pi_{M_{\mathrm{sw}},\infty}(E))}(\bar{x}) \\ &= \pi_{N^*,j(\mathcal{M}_\infty)} \circ \pi_{\mathcal{M}_c^{i\mathcal{T}},N^*} \circ \pi^{i\mathcal{T}}_{0,c} \circ i^* \circ \pi^{\mathcal{T}}_{0,c}(x), \end{aligned}$$

so that $k = \pi_{N^*,j(\mathcal{M}_\infty)} \circ \pi_{\mathcal{M}_c^{i\mathcal{T}},N^*} \circ \pi^{i\mathcal{T}}_{0,c} \circ i^*$ witnesses that c is indeed given by a branch through T.

Notice that (24) implies that

$$k \circ \pi^{\mathcal{T}}_{0,b} \circ \pi_{M_{\mathrm{sw}},\infty} \restriction M_{\mathrm{sw}}|\delta = j \circ \pi_{M_{\mathrm{sw}},\infty} \restriction M_{\mathrm{sw}}|\delta. \tag{25}$$

Let $x \in M_{\mathrm{sw}}|\delta$, and let s be a finite set of M_{sw}–indiscernibles which are moved neither by $\pi_{M_{\mathrm{sw}},\infty}$ nor by j and such that $x \in \mathrm{Hull}^{M_{\mathrm{sw}}|\max(s)}(\gamma_s^{M_{\mathrm{sw}}} \cup s^-) = \mathrm{dom}(\pi^s_{M_{\mathrm{sw}},\infty})$. Then $\pi^s_{M_{\mathrm{sw}},\infty} \in M_{\mathrm{sw}}$ and $j \circ \pi_{M_{\mathrm{sw}},\infty}(x) = j \circ \pi^s_{M_{\mathrm{sw}},\infty}(x) = j(\pi^s_{M_{\mathrm{sw}},\infty})(j(x)) = \pi^s_{M_{\mathrm{sw}},j(\mathcal{M}_\infty)}(x) = \pi_{M_{\mathrm{sw}},j(\mathcal{M}_\infty)}(x)$, where $\pi_{M_{\mathrm{sw}},j(\mathcal{M}_\infty)}$ is the iteration map from M_{sw} to $j(\mathcal{M}_\infty)$. Hence the right hand side of (25) is equal to $\pi_{M_{\mathrm{sw}},j(\mathcal{M}_\infty)}$. The left hand side of (25) is equal to the iteration map $\pi^{\mathcal{T}}_{0,b} \circ \pi_{M_{\mathrm{sw}},\infty} \restriction M_{\mathrm{sw}}|\delta$ followed by k.

By Lemmas 2.5 and 2.1, b must therefore be equal to c, so that in fact $c \in M_{\mathrm{sw}}$.

We have shown that $\Sigma_{\mathcal{M}_\infty}(\mathcal{T}) \in M_{\mathrm{sw}}$ for every $\mathcal{T} \in M_{\mathrm{sw}}$. But recall that $\delta_\infty = \kappa^{+M_{\mathrm{sw}}}$, cf. Lemma 2.7 (b), and δ_∞ is hence regular in M_{sw}. Hence if $\mathcal{T}$ is a tree on $\mathcal{M}_\infty$ with $\delta(\mathcal{T}) = \pi_{0,\Sigma(\mathcal{T})}(\delta_\infty)$, then M_{sw} will have exactly one cofinal branch through $\mathcal{T}$, namely $\Sigma(\mathcal{T})$. $\Sigma_{\mathcal{M}_\infty} \restriction M_{\mathrm{sw}}$ is therefore definable in M_{sw}.

(b): Let $\mathcal{T} \in M_{\mathrm{sw}}[g]$ be a tree of limit length on $\mathcal{M}_\infty$ which is according to $\Sigma_{\mathcal{M}_\infty}$. Let $c = \Sigma_{\mathcal{M}_\infty}(\mathcal{T})$. Assume that there is no drop along c and $\delta(\mathcal{T}) = \delta^{\mathcal{M}_c^{\mathcal{T}}}$.

Let θ be an appropriate ordinal, and let h be $\mathrm{Col}(\omega,\theta)$–generic over V such that $M_{\mathrm{sw}}[g] \subset M_{\mathrm{sw}}[h]$. Say $p \Vdash^{\mathrm{Col}(\omega,\theta)}_{M_{\mathrm{sw}}}$ "$\dot{\mathcal{T}}$ is a tree of limit length on $\mathcal{M}_\infty$ which is guided by ¶–small iterable $\mathcal{Q}$–structures, and $\delta(\dot{\mathcal{T}})$ is Woodin in $K(\mathcal{M}(\dot{\mathcal{T}}))$."

For any $q \leq_{\mathrm{Col}(\omega,\theta)} p$ let h_q denote the unique $\mathrm{Col}(\omega,\theta)$–generic filter over N such that for $n < \omega$,

$$(\bigcup h_q)(n) = \begin{cases} q(n) & \text{if } n \in \mathrm{dom}(q), \text{ and} \\ (\bigcup h)(n) & \text{otherwise.} \end{cases}$$

Inside $M_{\mathrm{sw}}[h]$, we may pseudo–compare all $K(\mathcal{M}(\dot{\mathcal{T}}^{h_q}))$, $q \leq_{\mathrm{Col}(\omega,\theta)} p$, so as to produce $K(\mathcal{M})$ for some $\mathcal{M}$. As $\mathcal{M}$ is definable inside $M_{\mathrm{sw}}[h]$ from $\{h_q \colon q \leq_{\mathrm{Col}(\omega,\theta)} p\}$ and some parameters from M_{sw}, $\mathcal{M}$ will actually be an element of M_{sw}, and in $V[h]$, $K(\mathcal{M})$ is a $\Sigma_{\mathcal{M}_c^{\mathcal{T}}}$–iterate of $\mathcal{M}_c^{\mathcal{T}}$, a fact which will give rise to the existence of the natural iteration map from $\mathcal{M}_c^{\mathcal{T}} = K(\mathcal{M}(\mathcal{T}))$ into $K(\mathcal{M})$.

Inside M_{sw}, we may now pseudo–compare $\mathcal{M}_\infty$ with $K(\mathcal{M})$, producing a $\Sigma_{\mathcal{M}_\infty}$–iterate $\mathcal{M}^*$ of $\mathcal{M}_\infty$ such that in V, $K(\mathcal{M})$ is also a $\Sigma_{K(\mathcal{M})}$–iterate of $K(\mathcal{M})$, a fact which will give rise to the existence of the natural iteration map from $K(\mathcal{M})$ into $\mathcal{M}^*$. As $\mathcal{M}_\infty$ is iterable in M_{sw} by (a), the iteration map

$$i \colon \mathcal{M}_\infty \to M^*$$

is definable inside M_{sw}. Inside $M_{\mathrm{sw}}[h]$, we may now construct a tree T searching for a cofinal branch b through $\mathcal{T}$ together with an elementary embedding $k \colon \mathcal{M}_b^{\mathcal{T}}|\delta(\mathcal{T}) \to \mathcal{M}^*|\delta^{\mathcal{M}^*}$ such that

$$k \circ \pi_{0,c}^{\mathcal{T}} \restriction \mathcal{M}_\infty|\delta_\infty = i \restriction \mathcal{M}_\infty|\delta_\infty.$$

T is ill–founded in $V[h]$, hence in $M_{\mathrm{sw}}[h]$, and by Lemma 2.1 there is a unique b given by a branch through T, so that $b \in M_{\mathrm{sw}}[g]$.

This argument shows that $\Sigma_{\mathcal{M}_\infty} \restriction M_{\mathrm{sw}}[g]$ is definable in $M_{\mathrm{sw}}[g]$.

(c): Let E be the least extender on the $\mathcal{M}_\infty$–sequence such that E is total and $\mathrm{crit}(E) = \kappa_\infty$. Inside $\mathrm{ult}(\mathcal{M}_\infty; E)$, we may pick some $N = P(\mathcal{M}(\mathcal{U})) \in \mathcal{F}^{\mathrm{ult}(\mathcal{M}_\infty;E)}$ such that $\delta(\mathcal{U}) = (\kappa_\infty)^{+\mathrm{ult}(\mathcal{M}_\infty;E)} = (\kappa_\infty)^{+\mathcal{M}_\infty}$. Let $c = \Sigma_{\mathcal{M}_\infty}(\mathcal{U})$.

By the proof of Lemma 2.2, $N = \mathcal{M}_c^{\mathcal{U}}$. But $c \in L[\mathcal{M}_\infty, \rho \mapsto \rho^*]$ by (b), and hence $\pi_{0,c}``\delta_\infty \in L[\mathcal{M}_\infty, \rho \mapsto \rho^*]$ witnesses that $(\kappa_\infty)^{+\mathcal{M}_\infty}$ has cofinality δ_∞ inside $L[\mathcal{M}_\infty, \rho \mapsto \rho^*]$.

Because N is also the ¶–small core model over $\mathcal{M}(\mathcal{U})$ inside $\mathrm{ult}(\mathcal{M}_\infty; E)$, again by the proof of Lemma 2.2, the Weak Covering Lemma (cf. e.g. [4]) therefore gives that $\mathrm{Card}((\kappa_\infty)^{+\mathcal{M}_\infty}) = \delta_\infty$ inside $L[\mathcal{M}_\infty, \rho \mapsto \rho^*]$. By Lemma 2.7 (c), $(\kappa_\infty)^{++\mathcal{M}_\infty} = \kappa^{++M_{\mathrm{sw}}}$, so that now $(\delta_\infty)^{+L[\mathcal{M}_\infty,\rho\mapsto\rho^*]} = \kappa^{++M_{\mathrm{sw}}}$.

(d): This now immediately follows from (c) and Lemma 2.8. $\square$ (Lemma 2.9)

Let us define the meaning of "the core model of M_{sw}." One way to make sense of this phrase is to define the core model as a hull of K^c, essentially as Steel did it in [18]. To this end, let us work in $M_{\rm sw}$. Let K^c be as defined in [5, Definition 2.7],[9] but with the following additivity adjustment: the critical point of an extender added (i.e., $\mathrm{crit}(G)$ for G as in [5, Definition 2.7 (a)]) is supposed to be above $\kappa^{+M_{\rm sw}}$. In the light of Lemma 2.9 (a), the paper [11] shows that K^c is fully iterable (inside $M_{\rm sw}$). The core model K may then be isolated as the unique weasel W such that for every α, $W|\alpha$ is isomorphic to an initial segment of

$$\bigcap\{\mathrm{Hull}^{K^c}(\Gamma)\colon \Gamma \text{ is } A_0\text{–thick in } K^c\},$$

where A_0 is defined as in [18, p. 8] and the notion of an "S–thick class" of ordinals is defined as in [18, Definition 3.8] (but with Ω being replaced by the class of all odinals in both cases). The paper [11] verifies that the core model K of $M_{\rm sw}$, thus defined, exists and is fully iterable inside $M_{\rm sw}$.

In our context, there is a shortcut, though, which will serve our purposes. We may let $\mathcal{M}_\infty$ play the role of K^c, as follows. Inside $M_{\rm sw}$, we define $\Gamma \subset \mathrm{OR}$ to be *thick* iff for all but nonstationary many inaccessibles α, $\Gamma \cap \alpha^+$ contains an α–club. As $M_{\rm sw}^\#$ exists but all mice in $M_{\rm sw}$ are sw–small, $M_{\rm sw}$ thinks that for all but nonstationary many α, α is inaccessible, $\alpha^{+\mathcal{M}_\infty} = \alpha^+$, and α is not the critical point of an $\mathcal{M}_\infty$–measure. (Cf. [18, Definition 3.8].) By Lemma 2.9 (a), the arguments of [18, section 5] then go through to show that definably over $M_{\rm sw}$ there is a unique weasel W such that for some thick class Γ_0, whenever $\Gamma \subset \Gamma_0$ is a thick class, then

$$W \cong \mathrm{Hull}^{\mathcal{M}_\infty}(\Gamma). \tag{26}$$

We call this weasel the *core model of* $M_{\rm sw}$, abbreviated by K. As K elementarily embeds into $\mathcal{M}_\infty$ (by (26), Lemma 2.9 (a) implies that K is fully iterable inside $M_{\rm sw}$. Also, $M_{\rm sw}$ thinks that for all but nonstationary many α, α is inaccessible and $\alpha^{+\mathcal{M}_\infty} = \alpha^+$.

We are now going to verify that K is actually *equal to* $\mathcal{M}_\infty$.

Lemma 2.10 *$\mathcal{M}_\infty = K$.*

Proof. Let us fix g which is $\mathrm{Col}(\omega, <\kappa)$–generic over $M_{\rm sw}$. Let us write[10]

$$H = \mathsf{HOD}^{M_{\rm sw}[g]}.$$

[9]This definition is a variant of the one presented in [7, section 2], but with the smallness assumption on the premice showing up in the K^c construction being relaxed, and it builds upon the definition which is given in [18, p. 6f.].

[10]Ordinal definability here is taken as definability in the usual language of set theory with $\in$ as the only non–logical predicate, in particular excluding a predicate for the extender sequence of $M_{\rm sw}$.

Claim 2.11 *$L[M_\infty, \rho \mapsto \rho^*] \subset H$.*

Proof. Let us write $\mathcal{C}$ for the collection, as being defined inside $M_{\rm sw}[g]$, of all extender models N with a Woodin cardinal, δ^N, and a strong cardinal, κ^N, such that the following conditions (1) through (6) are met.

(1) $N|(\delta^N)^{+N}$ is suitable,

(2) $\kappa^N = \kappa$,

(3) $N[h] = M_{\rm sw}[g]$ for some h which is $\mathrm{Col}(\omega, <\kappa)$–generic over N,

(4) $N = K(N|\delta^N)$ is the ¶–small core model over $N|\delta^N$,

(5) N is *pseudo–iterable* in the following sense. Let $\mathbb{T}(N)$ be the collection of all $\mathcal{U} = (\mathcal{U}_k \colon k \leq n) \in N$, some $n < \omega$, such that either $n = 0$ and $\mathrm{lh}(\mathcal{U}_0) = 1$ (i.e., $\mathcal{U}$ is trivial), or else there is a sequence $\eta_0 < \ldots < \eta_n < \kappa$ of cutpoints of N and:

   (a) $\mathcal{U} \in N|\kappa$,

   (b) $\mathcal{U} = (\mathcal{U}_k \colon k \leq n)$ is a finite stack of normal iteration trees $\mathcal{U}_k$,

   (c) $\mathcal{U}_0$ is on N and lives below δ^N,

   and for every $k < n$,

   (d) if $k < n$, then $\mathrm{lh}(\mathcal{U}_k) = (\eta_k)^{+N} = \delta(\mathcal{U}_k)$, and $\mathrm{lh}(\mathcal{U}_n) = (\eta_n)^{+N} = \delta(\mathcal{U}_n)$,

   (e) $\mathcal{U}_k$ is definable over $N|(\eta_k)^{+N}$ and is guided by $\mathcal{Q}$–structures which are obtained via $\mathcal{P}$–constructions inside N, cf. [13, Section 1],

   (f) if $k < n$, then $P^N(\mathcal{M}(\mathcal{U}_k))$ is a proper class, $\delta(\mathcal{U}_k)$ is a Woodin cardinal of $P^N(\mathcal{M}(\mathcal{U}))$, and
   $$P^N(\mathcal{M}(\mathcal{U}))[G] = N$$
   for some G which is $\mathbb{B}^{P(\mathcal{M}(\mathcal{U}))}$–generic over $P(\mathcal{M}(\mathcal{U}))$, and

   (g) if $k > 0$, then $\mathcal{U}_k$ is on $P^N(\mathcal{M}(\mathcal{U}_{k-1}))$ and lives below $\delta(\mathcal{U}_{k-1})$. (We allow $\mathcal{U}_n$ to consist of only one model, namely $P^N(\mathcal{M}(\mathcal{U}_{n-1}))$.)

   For N to be pseudo–iterable we demand that if $\mathcal{U} = (\mathcal{U}_k \colon k \leq n) \in \mathbb{T}(N)$, then

   (a) if $\mathcal{U}_n$ has a last model, say $\mathcal{M}_\theta^{\mathcal{U}_n}$ and if F is an extender from the sequence of $\mathcal{M}_\theta^{\mathcal{U}_n}$ such that if $[0, \theta]_{\mathcal{U}_n}$ does not drop, then the index of F is below $\delta^{\mathcal{M}_\theta^{\mathcal{U}_n}}$, then $(\mathcal{U}_k \colon k < n)^\frown(\mathcal{U}_n{}^\frown F) \in \mathbb{T}(N)$, where $(\mathcal{U}_n{}^\frown F)$ is the normal extension of $\mathcal{U}_n$, and

(b) if $\mathcal{U}_n$ is of limit length, then there is either a cofinal branch b through $\mathcal{U}_n$ such that $(\mathcal{U}_k \colon k < n)^\frown(\mathcal{U}_n{}^\frown b) \in \mathbb{T}(N)$, or else letting $\mathcal{U}^*$ be the trivial tree consisting only of the model $P^N(\mathcal{U}_n)$, $(\mathcal{U}_k \colon k \leq n)^\frown \mathcal{U}^* \in \mathbb{T}(N)$.

Before stating condition (6) let us say that we call M a *pseudo–iterate* of N iff there is some $\mathcal{U} = (\mathcal{U}_k \colon k \leq n) \in \mathbb{T}(N)$ such that $\mathcal{U}_n$ consists of only one model, namely M. We will write $\mathcal{F}^N$ for the collection of all pseudo–iterates of N.[11] Let s be a non–empty finite set of ordinals. For $M \in \mathcal{F}^N$ we call $M \in \mathcal{F}^N$ *s–iterable inside* N iff for all $\mathcal{U} = (\mathcal{U}_k \colon k \leq n) \in \mathbb{T}(N)$, writing M_k for the starting model of $\mathcal{U}_k$, $k \leq n$, if $M = M_{k_0}$ for some $k_0 < n$, there are for every $i \geq k_0$, $i < n+1$, cofinal branches

$$b_i \in (M_{\mathrm{sw}})^{\mathrm{Col}(\omega, \max(s))}$$

through $\mathcal{U}_i$ such that

(1) $\pi^{\mathcal{U}_i}_{0,b_i}(s) = s$, and

(2) $\pi^{\mathcal{U}_i}_{0,b_i}(N_i|\max(s)) = N_{i+1}|\max(s)$.[12]

In this situation, we may write b for the composition of the branches b_i, $k_0 \leq i < n+1$, and we may consider the map

$$\pi^{\mathcal{U}_{k_0}^\frown \ldots ^\frown \mathcal{U}_n}_{0,b} \restriction \mathrm{Hull}^{M_{k_0}|\max(s)}(\gamma_s^{M_{k_0}} \cup s^-). \tag{27}$$

We call M *strongly s–iterable inside* N iff the map in (27) doesn't depend on the particular choice of $\mathcal{U}$.

Our last condition on N now runs:

(6) For every finite set s of ordinals there is some $M \in \mathcal{F}^N$ such that M is strongly s–iterable in N.

Given $N \in \mathcal{C}$, we may define a direct limit system inside N in much the same way as the system was defined in M_{sw} to give rise to $\mathcal{M}_\infty$. We write $(\mathcal{M}_\infty)^N$ for the direct limit of that system as being defined in N.

We claim that if $N \in \mathcal{C}$, then

$$(\mathcal{M}_\infty)^N = \mathcal{M}_\infty$$

[11] We have that $\mathcal{F}^{M_{\mathrm{sw}}}$, defined this way, is equal to $\mathcal{F}$ as being defined earlier.

[12] The two notions of being s–iterable in M_{sw} we have now defined, cf. p. 9, coincide with each other.

and that in fact the systems giving rise to $\mathcal{M}_\infty$ and $(\mathcal{M}_\infty)^N$, respectively, have cofinally many common points. As $\mathcal{C}$ is ordinal definable inside $M_{\text{sw}}[g]$, this immediately establishes Claim 2.11.

Let us thus fix some $N \in \mathcal{C}$. Let $\xi < \kappa$ be least such that $N|\delta^N \in M_{\text{sw}}[g \restriction \xi]$. We have, by the forcing absoluteness of the ¶–small K over $N|\delta^N$,

$$N = (K(N|\delta^N))^N = (K(N|\delta^N))^{N[h]} = (K(N|\delta^N))^{M_{\text{sw}}[g]} = (K(N|\delta^N))^{M_{\text{sw}}[g \restriction \xi]}, \quad (28)$$

so that in particular N exists in $M_{\text{sw}}[g \restriction \xi]$ as a subclass which is definable there from the parameter $N|\delta^N$. Symmetrically, if $\xi' < \kappa$ is least such that $M_{\text{sw}}|\delta \in N[h \restriction \xi']$, then

$$M_{\text{sw}} = (K(M_{\text{sw}}|\delta))^{N[h \restriction \xi']} \quad (29)$$

and M_{sw} exists in $N[h \restriction \xi']$ as a subclass which is definable there from the parameter $M_{\text{sw}}|\delta$.

Let us denote by F_1 the M_{sw}–extender of Mitchell order 0 and with critical point κ, and let us denote by F_2 the N–extender of Mitchell order 0 with critical point κ. Let $\pi_1 \colon M_{\text{sw}} \to \text{ult}(M_{\text{sw}}; E_1)$ and $\pi_2 \colon N \to \text{ult}(N; E_2)$ denote the ultrapower maps. Let us write

$$\bar{H} = (H_{\kappa^+})^{\text{ult}(M_{\text{sw}};E_1)[g]} = (H_{\kappa^+})^{M_{\text{sw}}[g]} = (H_{\kappa^+})^{N[h]} = (H_{\kappa^+})^{\text{ult}(N;E_2)[h]}.$$

We have that

$$\text{ult}(M_{\text{sw}}; E_1)[g] = K(\bar{H})^{M_{\text{sw}}[g]} = K(\bar{H})^{\text{ult}(M_{\text{sw}};E_1)[g]},$$

and

$$\text{ult}(N; E_2)[h] = K(\bar{H})^{N[h]} = K(\bar{H})^{\text{ult}(N;E_2)[h]}$$

Let us write $K(\bar{H})$ for this common value of the ¶–small K over $\bar{H}$. Then

$$\text{ult}(M_{\text{sw}}; E_1)[g] = K(\bar{H}) = \text{ult}(N; E_2)[h]. \quad (30)$$

This immediately gives

$$\pi_1(\kappa) = \pi_2(\kappa). \quad (31)$$

But also, $M_{\text{sw}}|\kappa^{+M_{\text{sw}}}$ may be defined over $\bar{H}$ from the parameter $M_{\text{sw}}|\kappa$ as the stack of all ¶–small sound mice end–extending $M_{\text{sw}}|\kappa$ and projecting to κ, and

$$\text{ult}(M_{\text{sw}}; E_1) = \mathcal{P}^{\text{ult}(M_{\text{sw}};E_1)[g]}(M_{\text{sw}}|\kappa^{+M_{\text{sw}}}) = \mathcal{P}^{K(\bar{H})}(M_{\text{sw}}|\kappa^{+M_{\text{sw}}}). \quad (32)$$

In the same way, $N|\kappa^{+N}$ may be defined over $\bar{H}$ from the parameter $N|\kappa$ as the stack of all ¶–small sound mice end–extending $N|\kappa$ and projecting to κ, and

$$\mathrm{ult}(N;E_2)=\mathcal{P}^{\mathrm{ult}(N;E_2)[h]}(N|\kappa^{+N})=\mathcal{P}^{K(\bar{H})}(N|\kappa^{+N}). \tag{33}$$

Let k be $\mathrm{Col}(\omega,[\kappa,\pi_1(\kappa)))$–generic over the common model from (30), cf. (31). Then π_1 and π_2 lift to

$$\tilde{\pi}_1\colon M_{\mathrm{sw}}[g]\to \mathrm{ult}(M_{\mathrm{sw}};E_1)[g^\frown k]=K(\bar{H})[k]$$

and

$$\tilde{\pi}_2\colon N[h]\to \mathrm{ult}(N;E_2)[h^\frown k]=K(\bar{H})[k],$$

respectively. The maps $\tilde{\pi}_1$ and $\tilde{\pi}_2$ might be different, but the universes of their domains and target models are the same, and by (31), any objects defined in $M_{\mathrm{sw}}[g]=N[h]$ from parameters in $(H_\kappa)^{M_{\mathrm{sw}}[g]}\cup\{\kappa\}=(H_\kappa)^{N[h]}\cup\{\kappa\}$ will be moved the same way.

In particular, $\tilde{\pi}_1$ maps $N=(K(N|\delta^N))^{M_{\mathrm{sw}}[g]}$ to

$$\begin{aligned}(K(N|\delta^N))^{\mathrm{ult}(M_{\mathrm{sw}};E_1)[g^\frown k]}=(K(N|\delta^N))^{\mathrm{ult}(N;E_2)[h^\frown k]} &= \tilde{\pi}_2(K(N|\delta^N)^{N[h]})\\ &= \tilde{\pi}_2(N)=\mathrm{ult}(N;E_2),\end{aligned}$$

i.e.,

$$\tilde{\pi}_1(N)=\mathrm{ult}(N;E_2). \tag{34}$$

Let $\rho<\kappa$ be arbitrary. We have that $\mathrm{ult}(M_{\mathrm{sw}};E_1)[g^\frown k]$ thinks that there is some strong cutpoint $\eta<\tilde{\pi}_1(\kappa)$ of both $\mathrm{ult}(M_{\mathrm{sw}};E_1)=\tilde{\pi}_1(M_{\mathrm{sw}})=K(M_{\mathrm{sw}}|\delta)$ and $\mathrm{ult}(N;E_2)=\tilde{\pi}_1(N)=K(N|\delta^N)$ with $\eta>\rho$ (namely, $\eta=\kappa$) such that setting

$$H'=(H_{\eta^+})^{\tilde{\pi}_1(M_{\mathrm{sw}})[g^\frown k\restriction\eta]}$$

(so $H'=\bar{H}$ for $\eta=\kappa$), $\tilde{\pi}_1(M_{\mathrm{sw}})|\eta^{+\tilde{\pi}_1(M_{\mathrm{sw}})}$ may be defined over H' from the parameter $\tilde{\pi}_1(M_{\mathrm{sw}})|\eta$ as the stack of ¶–small sound mice end–extending $\tilde{\pi}_1(M_{\mathrm{sw}})|\eta$ and projecting to η,

$$\tilde{\pi}_1(M_{\mathrm{sw}})=\mathcal{P}^{\tilde{\pi}_1(M_{\mathrm{sw}})[g^\frown k\restriction\eta]}(\tilde{\pi}_1(M_{\mathrm{sw}})|\eta^{+\tilde{\pi}_1(M_{\mathrm{sw}})})=\mathcal{P}^{K(H')}(\tilde{\pi}_1(M_{\mathrm{sw}})|\eta^{+\tilde{\pi}_1(M_{\mathrm{sw}})}),$$

$\tilde{\pi}_1(N)|\eta^{+\tilde{\pi}_1(N)}$ may be defined over H' from the parameter $\tilde{\pi}_1(N)|\eta$ as the stack of all ¶–small sound mice end–extending $\tilde{\pi}_1(N)|\eta$ and projecting to η, and finally there is some h^* which is $\mathrm{Col}(\omega, <\eta)$–generic over $\tilde{\pi}_2(N)$ (namely, $h^* = h$) such that $\tilde{\pi}_1(M_{\mathrm{sw}})[g \frown k \restriction \eta] = \tilde{\pi}_1(N)[h^*]$ and

$$\tilde{\pi}_1(N) = \mathcal{P}^{\tilde{\pi}_1(N)[h^*]}(\tilde{\pi}_1(N)|\eta^{+\tilde{\pi}_1(N)}) = \mathcal{P}^{K(H')}(\tilde{\pi}_1(N)|\eta^{+\tilde{\pi}_1(N)}).$$

By the elementarity of $\tilde{\pi}_1$ and because $\rho < \kappa$ was arbitrary, we then get arbitrarily large $\eta < \kappa$ which are strong cutpoints of both M_{sw} and N such that setting

$$H'' = (H_{\eta^+})^{M_{\mathrm{sw}}[g \restriction \eta]}, \tag{35}$$

$M_{\mathrm{sw}}|\eta^{+M_{\mathrm{sw}}}$ may be defined over H'' from the parameter $M_{\mathrm{sw}}|\eta$ as the stack of all ¶–small sound mice end–extending $M_{\mathrm{sw}}|\eta$ and projecting to η,

$$M_{\mathrm{sw}} = \mathcal{P}^{M_{\mathrm{sw}}[g \restriction \eta]}(M_{\mathrm{sw}}|\eta^{+M_{\mathrm{sw}}}) = \mathcal{P}^{K(H'')}(M_{\mathrm{sw}}|\eta^{+M_{\mathrm{sw}}}),$$

$N|\eta^{+N}$ may be defined over H'' from the parameter $N|\eta$ as the stack of all ¶–small sound mice end–extending $N|\eta$ and projecting to η, and there is some h^* which is $\mathrm{Col}(\omega, <\eta)$–generic over N such that

$$N = \mathcal{P}^{N[h^*]}(N|\eta^{+N}) = \mathcal{P}^{K(H'')}(N|\eta^{+N}), \tag{36}$$

where $K(H'')$ is the ¶–small core model over H'' inside the model

$$M_{\mathrm{sw}}[g \restriction \eta] = N[h^*].$$

Let us write $S \subset \kappa$ for the set all of $\eta < \kappa$ with the properties as above, so that S is unbounded in κ.

Let us now suppose that $\mathcal{M}$ is a premouse with a largest limit ordinal $\delta^{\mathcal{M}}$ such that

1. $\eta^{+M_{\mathrm{sw}}} < \delta^{\mathcal{M}} \leq \eta^{++M_{\mathrm{sw}}}$ for some $\eta \in S$,
2. $\mathcal{M} \in M_{\mathrm{sw}} \cap N$,
3. $\mathcal{M} \models$ "$\delta^{\mathcal{M}}$ is a Woodin cardinal," and
4. both $M_{\mathrm{sw}}|\delta^{\mathcal{M}}$ and $N|\delta^M$ are $\mathbb{B}^{\mathcal{M}}$–generic over $\mathcal{M}$.

We then have, for H'' as in (35) and h^* being $\mathrm{Col}(\omega, <\eta)$–generic over N with (36),

$$\begin{aligned} \mathcal{P}^{M_{\mathrm{sw}}}(\mathcal{M}) &= \mathcal{P}^{M_{\mathrm{sw}}[g \restriction \eta]}(\mathcal{M}) \\ &= \mathcal{P}^{K(H'')}(\mathcal{M}) \\ &= \mathcal{P}^{N[h^*]}(\mathcal{M}) \\ &= \mathcal{P}^{N}(\mathcal{M}), \end{aligned} \tag{37}$$

where $K(H'')$ is the ¶–small K over H'' in $M_{\mathrm{sw}}[g \restriction \eta] = N[h^*]$.

Now let $s \in \mathrm{OR}^{<\omega}$, and let $M \in \mathcal{F} = \mathcal{F}^{M_{\mathrm{sw}}}$ be strongly s–iterable in M_{sw}, and let $M' \in \mathcal{F}^N$ be strongly s–iterable in N. We aim to find $M^* \in \mathcal{F} \cap \mathcal{F}^N$ such that

$$(M, s) \preceq_{\mathcal{F}} (M^*, s) \text{ and } (M', s) \preceq_{\mathcal{F}^N} (M^*, s).$$

Let $\xi' \leq \xi'' < \kappa$ be such that $g \restriction \xi \in N[h \restriction \xi'']$, so that by (28) and (29)

$$N \subset M_{\mathrm{sw}}[g \restriction \xi] \subset N[h \restriction \xi''],$$

which implies that N is a ground of $M_{\mathrm{sw}}[g \restriction \xi]$, and in fact both M_{sw} and N grounds of $M_{\mathrm{sw}}[g \restriction \xi]$ via posets of size less than κ. Therefore, by [22, Proposition 5.1], there is an inner model $P \subset M_{\mathrm{sw}} \cap N$ such that P is a ground of $M_{\mathrm{sw}}[g \restriction \xi]$ via a poset of size less than κ. We may then pick some $\theta < \kappa$ such that for some $\ell \in M_{\mathrm{sw}}[g]$ which is $\mathrm{Col}(\omega, \theta)$–generic over P,

$$\{M_{\mathrm{sw}}|\delta, N|\delta^N, M|\delta^M, M'|\delta^{M'}\} \subset P[\ell], \tag{38}$$

and in fact all of M_{sw}, N, M, M' exist in $P[\ell]$ as subclasses which are definable there as $K(M_{\mathrm{sw}}|\delta)$, $K(N|\delta^N)$, $K(M|\delta^M)$, and $K(M'|\delta^{M'})$, respectively.

Let τ_0, τ_1, σ_0, $\sigma_1 \in P^{\mathrm{Col}(\omega,\theta)}$ be such that

$$\tau_0^\ell = M_{\mathrm{sw}}|\delta^{+M_{\mathrm{sw}}},\ \tau_1^\ell = N|(\delta^N)^{+N},\ \sigma_0^\ell = M|(\delta^M)^{+M}, \text{ and } \sigma_1^\ell = M'|(\delta^{M'})^{+M'}. \tag{39}$$

Let $p \in \mathrm{Col}(\omega, \theta)$ force over P all the relevant properties about τ_0,τ_1, σ_0,σ_1 for the following to go through. For any $q \leq_{\mathrm{Col}(\omega,\theta)} p$ let ℓ_q denote the unique $\mathrm{Col}(\omega, \theta)$–generic filter over N such that for $n < \omega$,

$$(\bigcup \ell_q)(n) = \begin{cases} q(n) & \text{if } n \in \mathrm{dom}(q), \text{ and} \\ (\bigcup \ell)(n) & \text{otherwise.} \end{cases}$$

Let $\eta \in S$, $\eta > \max\{\xi, \xi'\}$. Notice that $\eta^{++N} \leq \eta^{++M_{\mathrm{sw}}[g \restriction \xi]} = \eta^{++M_{\mathrm{sw}}} \leq \eta^{++N[h \restriction \xi}= \eta^{++N}$ by (28) and (29), so that

$$\eta^{++M_{\mathrm{sw}}} = \eta^{++N}.$$

This is then also the common η^{++} of all $K(\tau_0^{\ell_q})$, $K(\tau_1^{\ell_q})$. Working in $P[\ell]$, let for $q \leq_{\mathrm{Col}(\omega,\theta)} p$,

$$\mathcal{U}_q \text{ and } U'_q \text{ be normal iteration trees on } \sigma_0^{\ell_q} \text{ and } \sigma_1^{\ell_q}, \text{ respectively,}$$

such that

1. $\mathrm{lh}(\mathcal{U}_q) = \mathrm{lh}(\mathcal{U}'_q) = \eta^{++M_{\mathrm{sw}}} = \delta(\mathcal{U}_q) = \delta(U'_q)$ for all $q \leq_{\mathrm{Col}(\omega,\theta)} p$,
2. $\mathcal{M}(\mathcal{U}_q) = \mathcal{M}(\mathcal{U}'_{q'})$ for all q, $q' \leq_{\mathrm{Col}(\omega,\theta)} p$,
3. every $\mathcal{U}_q$ as well as every $\mathcal{U}'_q$ is guided by ¶–small $\mathcal{Q}$–structures,
4. $K(\tau_0^{\ell_q})|\delta(\mathcal{U}_q)$ is generic over $\mathcal{M}(\mathcal{U}_q)$ for all $q \leq_{\mathrm{Col}(\omega,\theta)} p$, and
5. $K(\tau_1^{\ell_q})|\delta(\mathcal{U}'_q)$ is generic over $\mathcal{M}(\mathcal{U}'_q)$ for all $q \leq_{\mathrm{Col}(\omega,\theta)} p$.

Let us write $\mathcal{M}$ for the common value of all $\mathcal{M}(\mathcal{U}_q)$ and $\mathcal{M}(\mathcal{U}'_q)$. Notice that $\mathcal{M} \in P \subset M_{\mathrm{sw}} \cap N$. Set

$$M^* = (K(\mathcal{M}))^P.$$

By (37), we have that

$$\mathcal{M}^* = (\mathcal{P}(\mathcal{M}))^{M_{\mathrm{sw}}} = (\mathcal{P}(\mathcal{M}))^N. \tag{40}$$

Also, $\mathcal{U}_p$ is normal and is a tree on M which produces $\mathcal{M}^*$, so that (modulo potential padding) $\mathcal{U}_p$ can be computed in M_{sw} via the comparison process which tries to coiterate M and $\mathcal{M}^*$. Similarly, $\mathcal{U}'_p$ is normal and is a tree on M' which produces $\mathcal{M}^*$, so that (again modulo potential padding) $\mathcal{U}'_p \in N$. As M is strongly s–iterable in M_{sw} and M' is strongly s–iterable in N, we therefore get that

$$M^* \in \mathcal{F} \cap \mathcal{F}^N,\ (M,s) \preceq_{\mathcal{F}} (M^*,s), \text{ and } (M',s) \preceq_{\mathcal{F}^N} (M^*,s),$$

as desired. □ (Claim 2.11)

Claim 2.12 *(a) $H \subset L[M_\infty, \rho \mapsto \rho^*]$. Hence, $H = L[M_\infty, \rho \mapsto \rho^*]$.*

(b) If $\gamma < \delta_\infty$ and $X \in H \cap \mathcal{P}(\gamma)$, then $X \in \mathcal{M}_\infty$. In particular, $(H_{\delta_\infty})^H = \mathcal{M}_\infty|\delta_\infty$.

Proof. (a): Let us fix X, a set of ordinals, such that $X \in H$, say $X \subset \gamma$ and $\xi \in X$ iff

$$\Vdash^{\mathrm{Col}(\omega,<\kappa)}_{M_{\mathrm{sw}}} \varphi(\check{\xi}, \check{\alpha}_1, \ldots, \check{\alpha}_k). \tag{41}$$

If $N \in \mathcal{F}$, then there is some h which is $\mathrm{Col}(\omega,<\kappa)$–generic over N such that $N[h] = M_{\mathrm{sw}}[g]$, so that (41) is equivalent with

$$\Vdash^{\mathrm{Col}(\omega,<\kappa)}_{N} \varphi(\check{\xi}, \check{\alpha}_1, \ldots, \check{\alpha}_k). \tag{42}$$

In particular, $X \in \bigcap \mathcal{F}$ and $\pi_{N,N'}(X) = X$ for all N, $N' \in \mathcal{F}$ such that $\pi_{N,N'}$ exists and

$$\pi_{N,N'}(\alpha_1, \ldots, \alpha_k) = \alpha_1, \ldots, \alpha_k. \tag{43}$$

Let $N \in \mathcal{F}$ be such that (43) holds true for all $N' \in \mathcal{F}$ such that $\pi_{N,N'}$ exists, and set $\tilde{X} = \pi_{N,\infty}(X) \in \mathcal{M}_\infty$. Then for any $\xi < \gamma$, if $N' \in \mathcal{F}$ is such that $\pi_{N,N'}$ exists and $\pi_{N',N''}(\xi) = \xi$ for all $N'' \in \mathcal{F}$ for which $\pi_{N',N''}$ exists, we have that $\xi \in X$ iff

$$\xi^* = \pi_{N',\infty}(\xi) \in \pi_{N',\infty}(X) = \pi_{N,\infty}(X) = \tilde{X},$$

so that $X \in L[\mathcal{M}_\infty, \rho \mapsto \rho^*]$.

We have shown (a). (b): Let $\gamma < \delta_\infty$, say $\gamma \leq \pi_{M_{\rm sw},\infty}(\bar{\gamma})$. Pick a finite set s of ordinals such that $M_{\rm sw}$ is strongly s–iterable and $\bar{\gamma} < \gamma_s^{M_{\rm sw}}$, cf. the argument on p. 12. We have that $\pi^s_{M_{\rm sw},\infty} \restriction \gamma_s^{M_{\rm sw}} \in M_{\rm sw}$, so that

$$(\rho \mapsto \rho^*) \restriction \gamma = \pi^\infty_{0,\infty} \restriction \gamma = \pi_{M_{\rm sw},\infty}(\pi^s_{M_{\rm sw},\infty} \restriction \gamma_s^{M_{\rm sw}}) \restriction \gamma$$

is an element of $\mathcal{M}_\infty$. The above argument then shows (b). $\square$ (Claim 2.12)

Claim 2.12 (a) has the following remarkable consequence.

Lemma 2.13 *$\mathcal{M}_\infty|\delta_\infty$ is fully iterable inside $L[M_\infty, \rho \mapsto \rho^*]$, in fact $\Sigma_{\mathcal{M}_\infty} \restriction L[M_\infty, \rho \mapsto \rho^*]$ is definable inside $L[M_\infty, \rho \mapsto \rho^*]$.*

Proof. Let $\mathcal{T} \in L[M_\infty, \rho \mapsto \rho^*]$ be a tree on $\mathcal{M}_\infty|\delta_\infty$ of limit length which is according to $\Sigma_{\mathcal{M}_\infty}$. Write $b = \Sigma_{\mathcal{M}_\infty}(\mathcal{T})$. By Lemma 2.9 (a), $b \in M_{\rm sw}$. If there is a (necessarily, ¶–small) $\mathcal{Q}$–structure $\mathcal{Q} \trianglelefteq \mathcal{M}_b^\mathcal{T}$, then $\mathcal{Q} \in L[M_\infty, \rho \mapsto \rho^*]$ and hence also $b \in L[M_\infty, \rho \mapsto \rho^*]$. So let us assume that there is no such $\mathcal{Q}$–structure.

Then $\delta(\mathcal{T}) = \mathcal{M}_b^\mathcal{T} \cap {\rm OR}$, and hence ${\rm cf}({\rm lh}(\mathcal{T})) = {\rm cf}(\delta(\mathcal{T})) = {\rm cf}(\mathcal{M}_b^\mathcal{T} \cap {\rm OR}) = \delta_\infty = \kappa^+$ inside $M_{\rm sw}$. Let g be ${\rm Col}(\omega, <\kappa)$–generic over $M_{\rm sw}$. Then $\delta_\infty = \aleph_2$ in $M_{\rm sw}[g]$, so that inside $M_{\rm sw}[g]$, b is the unique cofinal branch through $\mathcal{T}$. As $\mathcal{T} \in L[M_\infty, \rho \mapsto \rho^*] = H = \mathsf{HOD}^{M_{\rm sw}[g]}$ by Claim 2.12 (a), we get $b \in \mathsf{HOD}^{M_{\rm sw}[g]}$, and hence $b \in L[M_\infty, \rho \mapsto \rho^*]$.

The argument we gave shows that $\Sigma_{\mathcal{M}_\infty} \restriction L[M_\infty, \rho \mapsto \rho^*]$ is definable inside $L[M_\infty, \rho \mapsto \rho^*]$. $\square$ (Lemma 2.13)

We are now ready to finish the proof of Lemma 2.10.

As $L[\mathcal{M}_\infty, \rho \mapsto \rho^*]$ is a ground of $M_{\rm sw}$ by Lemma 2.8 and $\mathcal{M}_\infty$ is fully iterable inside both $M_{\rm sw}$ as well as $L[\mathcal{M}_\infty, \rho \mapsto \rho^*]$ by Lemma 2.9 (a) and Lemma 2.13, we may define the core model $K^{L[\mathcal{M}_\infty,\rho\mapsto\rho^*]}$ of $L[\mathcal{M}_\infty,\rho\mapsto\rho^*]$ in much the same way as we defined the core model $K=K^{M_{\mathrm{sw}}}$ of M_{sw} on p. 21 and $K=K^{M_{\mathrm{sw}}}=K^{L[\mathcal{M}_\infty,\rho\mapsto\rho^*]}$. Inside $L[\mathcal{M}_\infty,\rho\mapsto\rho^*]$, there is a canonical elementary embedding $j\colon K\to\mathcal{M}_\infty$ given by (26). We aim to show that $j=\mathrm{id}$.

Let us assume that $j\neq\mathrm{id}$, and set $\lambda=\mathrm{crit}(j)$. Inside $L[\mathcal{M}_\infty,\rho\mapsto\rho^*]$, K and $\mathcal{M}_\infty$ coiterate to a common weasel, $\mathcal{Q}$, such that if $\pi_{K,\mathcal{Q}}$ and $\pi_{\mathcal{M}_\infty,\mathcal{Q}}$ denote the canonical iteration maps,

$$\pi_{\mathcal{M}_\infty,\mathcal{Q}}\circ j=\pi_{K,\mathcal{Q}}. \tag{44}$$

If $j(\lambda)<\delta_\infty$, then by (44) $j\restriction\lambda^{+K}$ is cofinal in $j(\lambda)^{+\mathcal{M}_\infty}$ and witnesses that $j(\lambda)^{+\mathcal{M}_\infty}$ is singular. However, this contradicts Claim 2.12 (b). If $j(\lambda)=\delta_\infty$, then λ is the Woodin cardinal of K, but there is some initial segement $\mathcal{N}$ of $\mathcal{M}_\infty$ projecting to λ which defines a counterexample to the Woodinness of λ. However, by universality, $\mathcal{N}$ would have to be an initial segment of K. Finally, if $j(\lambda)>\delta_\infty$, then j comes from an iteration of K strictly above δ_∞, the common Woodin cardinal of K and $\mathcal{M}_\infty$. But $\mathcal{M}_\infty$ is generated from δ_∞ together with a club class of indiscernibles above κ_∞, which immediately gives $j\restriction\kappa_\infty=\mathrm{id}$ and then $j=\mathrm{id}$. $\square$ (Lemma 2.10)

Theorem 2.14 *$L[M_\infty,\rho\mapsto\rho^*]$ is the mantle of M_{sw}.*

Proof. As $L[M_\infty,\rho\mapsto\rho^*]$ is a ground of M_{sw} by Lemma 2.8, if suffices to prove that $L[M_\infty,\rho\mapsto\rho^*]\subset W$ for every ground W of M_{sw}.

So let us fix W, a ground of M_{sw}. Let $\mathbb{P}\in W$ be a poset such that for some $g\in M_{\mathrm{sw}}$ which is $\mathbb{P}$–generic over W, $M_{\mathrm{sw}}=W[g]$. Let λ be the cardinality of $\mathbb{P}$ inside W, so that $\mathbb{P}*\mathrm{Col}(\omega,\lambda)\cong\mathrm{Col}(\omega,\lambda)$. Let $\bar h$ be $\mathrm{Col}(\omega,\lambda)$–generic over M_{sw}, and let h be $\mathrm{Col}(\omega,\lambda)$–generic over W such that $W[h]=M_{\mathrm{sw}}[\bar h]$.

$W[h]$ contains $\mathcal{M}_\infty|\delta_\infty$ as an element, and it can define $\mathcal{M}_\infty$ as $K(\mathcal{M}_\infty|\delta_\infty)$. Let $\tau\in W^{\mathrm{Col}(\omega,\lambda)}$ be such that $\mathcal{M}_\infty|\delta_\infty=\tau^h$. By Lemma 2.9 (b), $\mathcal{M}_\infty$ is fully iterable inside $W[h]$, so that we may pick some $p\in h$ such that

$$p\Vdash_W^{\mathrm{Col}(\omega,\lambda)} K(\tau)\text{ is sw–small, has a strong cardinal above}$$
$$\text{the Woodin cardinal }\tau\cap\mathrm{OR}\text{, and is fully iterable.}$$

For any $q\leq_{\mathrm{Col}(\omega,\lambda)}p$ let h_q denote the unique $\mathrm{Col}(\omega,\lambda)$–generic filter over W such that for $n<\omega$,

$$(\bigcup h_q)(n)=\begin{cases}q(n) & \text{if } n\in\mathrm{dom}(q)\text{, and}\\ (\bigcup h)(n) & \text{otherwise,}\end{cases}$$

and let us write M^q for $K(\tau^{h_q})$, as being computed inside $W[h] = W[h_q]$. By (45), every M^q, $q \leq_{\mathrm{Col}(\omega,\lambda)} p$, is fully iterable inside $W[h]$, and it is straightforward to see that all M^q, $q \leq_{\mathrm{Col}(\omega,\lambda)} p$, coiterate to a common coiterate, say $\mathcal{Q}$. We have that $\mathcal{Q}$ is a definable inner model of W.

Let $\Gamma \subset \mathrm{OR}$ be the class of all ordinal fixed points under all the iteration maps from an M^q, $q \leq_{\mathrm{Col}(\omega,\lambda)} p$, to $\mathcal{Q}$. Γ is then a definable class in W, and also Γ is easily verified to be thick in the sense of the definition given on p. 21. We must then have that

$$\mathcal{M}_\infty \cong \mathrm{Hull}^{\mathcal{Q}}(\Gamma),$$

so that $\mathcal{M}_\infty \subset W$.

In order to show that the map $\rho \mapsto \rho^*$ is in W, it suffices to show that $\Sigma_{\mathcal{M}_\infty}$ is amenable to and definable over W.

Let $\mathcal{T} \in W$ be an iteration tree on $\mathcal{M}_\infty$ of limit length which is according to $\Sigma_{\mathcal{M}_\infty}$. Write $b = \Sigma_{\mathcal{M}_\infty}(\mathcal{T})$. We have that $b \in W[h]$ by Lemma 2.9 (c). If $\mathcal{M}_b^{\mathcal{T}}$ has an initial segment $\mathcal{Q}$ end–extending $\mathcal{M}(\mathcal{T})$ such that $\delta(\mathcal{T})$ is not definably Woodin over $\mathcal{Q}$, then the unique least such $\mathcal{Q}$ may be found inside W by stacking sound mice which are ¶–small above $\delta(\mathcal{T})$ and project to $\delta(\mathcal{T})$ on top of $\mathcal{M}(\mathcal{T})$, so that $b \in W$. Otherwise b does not drop and $\delta(\mathcal{T}) = \pi^{\mathcal{T}}_{0,b}(\delta_\infty)$. We then have that inside $W[h]$, b is the only cofinal branch c through $\mathcal{T}$ such that $\delta(\mathcal{T}) = \pi^{\mathcal{T}}_{0,c}(\delta_\infty)$ and $\mathcal{M}^{\mathcal{T}}_c$ is iterable above $\delta(\mathcal{T})$. (In fact, inside $W[h]$, b is the only cofinal branch c through $\mathcal{T}$ such that $\delta(\mathcal{T}) = \pi^{\mathcal{T}}_{0,c}(\delta_\infty)$ and $\mathcal{M}^{\mathcal{T}}_c$ is well–founded, cf. the remark on p. 4.) Therefore $b \in W$.

But the argument we gave also shows that $\Sigma_{\mathcal{M}_\infty}$ is amenable to and definable over W. $\square$ (Theorem 2.14)

We call $L[M_\infty, \rho \mapsto \rho^*]$ the *Varsovian model derived from* M_{sw}. If M is a model which is elementarily equivalent to M_{sw}, then the *Varsovian model derived from* M is that inner model of M which is defined over M as $L[M_\infty, \rho \mapsto \rho^*]$ is defined over M_{sw}.

Lemma 2.15 (F. Schlutzenberg)

(a) $\mathrm{ran}(\pi_{M_{\mathrm{sw}},\infty})$ is closed under both $\pi^\infty_{0,\infty}$ and $(\pi^\infty_{0,\infty})^{-1}$.

(b) $\mathrm{Hull}^{L[\mathcal{M}_\infty,\rho\mapsto\rho^*]}(\mathrm{ran}(\pi_{M_{\mathrm{sw}},\infty})) \cap \mathrm{OR} = \mathrm{ran}(\pi_{M_{\mathrm{sw}},\infty}) \cap \mathrm{OR}$.

Proof. (a) Let ρ be such that $\{\rho, \rho^*\} \cap \mathrm{ran}(\pi_{M_{\mathrm{sw}},\infty}) \neq \emptyset$. Let s be a finite set of M_{sw}–indiscernibles such that

$$\rho \in \mathrm{Hull}^{M_{\mathrm{sw}}|\max(s)}(\gamma_s^{M_{\mathrm{sw}}} \cup s^-).$$

We have that $\pi^{\infty}_{0,\infty} \restriction \mathrm{Hull}^{\mathcal{M}_\infty|\max(s)}(\gamma_s^{\mathcal{M}_\infty} \cup s^-) \in \mathcal{M}_\infty$ and in fact

$$\pi^{\infty}_{0,\infty} \restriction \mathrm{Hull}^{\mathcal{M}_\infty|\max(s)}(\gamma_s^{\mathcal{M}_\infty} \cup s^-) = \pi_{M_{\mathrm{sw}},\infty}(\pi_{M_{\mathrm{sw}},\infty} \restriction \mathrm{Hull}^{M_{\mathrm{sw}}|\max(s)}(\gamma_s^{M_{\mathrm{sw}}} \cup s^-),$$

where $\pi_{M_{\mathrm{sw}},\infty} \restriction \mathrm{Hull}^{\mathcal{M}_\infty|\max(s)}(\gamma_s^{\mathcal{M}_\infty} \cup s^-) \in M_{\mathrm{sw}}$. Then if $\rho \in \mathrm{ran}(\pi_{M_{\mathrm{sw}},\infty})$, then $\rho^* = (\pi^{\infty}_{0,\infty} \restriction \mathrm{Hull}^{\mathcal{M}_\infty|\max(s)}(\gamma_s^{\mathcal{M}_\infty} \cup s^-))(\rho) \in \mathrm{ran}(\pi_{M_{\mathrm{sw}},\infty})$, and if $\rho^* \in \mathrm{ran}(\pi_{M_{\mathrm{sw}},\infty})$, then $\rho = (\pi^{\infty}_{0,\infty} \restriction \mathrm{Hull}^{\mathcal{M}_\infty|\max(s)}(\gamma_s^{\mathcal{M}_\infty} \cup s^-))^{-1}(\rho^*) \in \mathrm{ran}(\pi_{M_{\mathrm{sw}},\infty})$.

(b) Let $\rho \in \mathrm{Hull}^{L[\mathcal{M}_\infty,\rho\mapsto\rho^*]}(\mathrm{ran}(\pi_{M_{\mathrm{sw}},\infty})) \cap \mathrm{OR}$. By (a), it suffices to prove that $\rho^* \in \mathrm{ran}(\pi_{M_{\mathrm{sw}},\infty})$.

We may pick a finite set s of M_{sw}–indiscernibles such that

$$\rho \in \mathrm{Hull}^{L[\mathcal{M}_\infty,\rho\mapsto\rho^*]}(s). \tag{45}$$

Let $N \in \mathcal{F}$ be strongly s–iterable such that $\pi_{N,N'}(\rho) = \rho$ for all $N' \in \mathcal{F}$ with $\pi_{N,N'}\downarrow$. As $L[\mathcal{M}_\infty, \rho \mapsto \rho^*] = \mathrm{HOD}^{N[h]}$ for some/all h which are $\mathrm{Col}(\omega, <\kappa)$–generic over N, cf. Claim 2.12 (a), (45) implies that

$$\rho \in \mathrm{Hull}^N(s).$$

But then

$$\rho^* \in \mathrm{Hull}^{\mathcal{M}_\infty}(s) \subset \mathrm{ran}(\pi_{M_{\mathrm{sw}},\infty}).$$

□ (Lemma 2.15)

Corollary 2.16 *Let $\sigma\colon \mathcal{V} \cong \mathrm{Hull}^{L[\mathcal{M}_\infty,\rho\mapsto\rho^*]}(\mathrm{ran}(\pi_{M_{\mathrm{sw}},\infty}))$, where $\mathcal{V}$ is transitive. $\mathcal{V} = L[M_{\mathrm{sw}}, \rho \mapsto \pi_{M_{\mathrm{sw}},\infty}(\rho)]$, and $\sigma \supset \pi_{M_{\mathrm{sw}},\infty}$.*

Proof. By Lemma 2.15 (b) and by (19), it remains to be seen that

$$\sigma^{-1}((\rho \mapsto \rho^*) \restriction \delta_\infty) = \pi_{M_{\mathrm{sw}},\infty} \restriction \delta. \tag{46}$$

For $n < \omega$ let us write $s_n = \{\aleph_1^V, \dots, \aleph_{n+1}^V\}$. Then for each $n < \omega$, $\pi_{M_{\mathrm{sw}},\infty} \restriction \gamma_{s_n}^{M_{\mathrm{sw}}} = \pi^{s_n}_{M_{\mathrm{sw}},\infty} \restriction \gamma_{s_n}^{M_{\mathrm{sw}}} \in M_{\mathrm{sw}}$ and $\sigma(\pi^{s_n}_{M_{\mathrm{sw}},\infty} \restriction \gamma_{s_n}^{M_{\mathrm{sw}}}) = \pi^{s_n}_{\mathcal{M}_\infty,\mathcal{M}_\infty^\infty}$, by the elementarity of σ and $\sigma(s_n) = s_n$, and the latter is equal to $\pi^\infty_{0,\infty} \restriction \gamma_{s_n}^{\mathcal{M}_\infty}$ which is hence in $\mathcal{M}_\infty$. But then $\sigma^{-1}((\rho \mapsto \rho^*) = \sigma^{-1}(\bigcup_{n<\omega} \pi^\infty_{0,\infty} \restriction \gamma_{s_n}^{\mathcal{M}_\infty}) = \bigcup_{n<\omega} \sigma^{-1}(\pi^\infty_{0,\infty} \restriction \gamma_{s_n}^{\mathcal{M}_\infty}) = \bigcup_{n<\omega} \pi^{s_n}_{M_{\mathrm{sw}},\infty} \restriction \gamma_{s_n}^{M_{\mathrm{sw}}} = \pi_{M_{\mathrm{sw}},\infty} \restriction \delta$, which shows (46). □ (Corollary 2.16)

Lemma 2.17 *Let $\sigma\colon \mathcal{V} = L[M_{\mathrm{sw}}, \rho \mapsto \pi_{M_{\mathrm{sw}},\infty}(\rho)] \cong \mathrm{Hull}^{L[\mathcal{M}_\infty,\rho\mapsto\rho^*]}(\mathrm{ran}(\pi_{M_{\mathrm{sw}},\infty}))$. $\mathcal{V}$ is iterable via iteration trees which live on $M_{\mathrm{sw}}|\delta$.*

Proof. Implicitly, [21] contains a simplified version of the argument to follow, cf. [21, Lemma 3.46]. This was pointed out to the authors by Farmer Schlutzenberg who then independently arrived at a proof of Lemma 2.17.

We claim that Σ may serve as an iteration strategy for iteration trees on $\mathcal{V}$ which live on $M_{\mathrm{sw}}|\delta$. This makes sense by Claim 2.12 (b), Corollary 2.16, and the elementarity of σ.

Let $\mathcal{T}$ be a putative tree on $\mathcal{V}$ which lives on $M_{\mathrm{sw}}|\delta$ and is according to Σ. If $\mathcal{M}_\alpha^{\mathcal{T}}$ is a transitive proper class, $\alpha < \mathrm{lh}(\mathcal{T})$, then we may write $\mathcal{M}_\alpha^{\mathcal{T}} = L[M_\alpha, \pi_\alpha]$. The tree $\mathcal{T}$ induces a canonical tree, which we shall denote by $\bar{\mathcal{T}}$, on M_{sw} which is according to Σ.

Let us write Π for the set of all $\alpha < \mathrm{lh}(\mathcal{T})$ such that $\mathcal{M}_\alpha^{\mathcal{T}}$ is a proper class. If $\alpha \in \mathrm{lh}(\mathcal{T}) \setminus \Pi$, then $\mathcal{M}_\alpha^{\bar{\mathcal{T}}} = \mathcal{M}_\alpha^{\mathcal{T}}$. We claim that we may define a sequence

$$((M_\alpha, \pi_\alpha, M_\alpha^*, \pi_\alpha^*, \mathcal{V}_\alpha, \tilde{\pi}_\alpha) \colon \alpha \in \Pi)$$

such that

(a) $M_0 = M_{\mathrm{sw}}$, $\pi_0 = \pi_{M_{\mathrm{sw}},\infty}$, $M_0^* = \mathcal{M}_\infty$, $\pi_0^* = (\rho \mapsto \rho^*)$

and for all $\alpha \leq_{\mathcal{T}} \beta < \mathrm{lh}(\mathcal{T})$ with α, $\beta \in \Pi$:

(b) $M_\alpha = \mathcal{M}_\alpha^{\bar{\mathcal{T}}}$,

(c) $L[M_\alpha, \pi_\alpha \restriction \mathrm{OR}] = \mathcal{M}_\alpha^{\mathcal{T}}$,

(d) $\mathcal{V}_\alpha = L[M_\alpha^*, \pi_\alpha^*]$ is the Varsovian model derived from M_α,

(e) $\pi_\alpha \colon M_\alpha \to M_\alpha^*$ is an elementary embedding,

(f) $\tilde{\pi}_\alpha \colon L[M_\alpha, \pi_\alpha \restriction \mathrm{OR}] \to L[M_\alpha^*, \pi_\alpha^*]$ is an elementary embedding,

(g) $\tilde{\pi}_\beta \restriction \mathrm{lh}(E_\gamma) = \tilde{\pi}_\alpha \restriction \mathrm{lh}(E_\gamma)$ for $\alpha <_{\mathcal{T}} \gamma + 1 \leq_{\mathcal{T}} \beta$,

(h) $\tilde{\pi}_\alpha \supset \pi_\alpha$, and

(i) $\pi_{\alpha,\beta}^{\mathcal{T}} \supset \pi_{\alpha,\beta}^{\bar{\mathcal{T}}}$.

Let us present the successor steps of the construction, leaving the limit steps to the reader's discretion. Let $\alpha = \mathcal{T}\text{–prec}(\beta + 1)$, where $\beta + 1 \in \Pi$, and write $F = E_\beta^{\mathcal{T}} = E_\beta^{\bar{\mathcal{T}}}$.

We may define an elementary embedding

$$\tilde{\pi}_{\beta+1} \colon \mathrm{ult}(L[M_\alpha, \pi_\alpha \restriction \mathrm{OR}]; F) \to \mathcal{V}_{\beta+1}$$

by setting

$$\tilde{\pi}_{\beta+1}([a,f]_F^{\mathcal{M}_\alpha^{\mathcal{T}}}) = [a, u \mapsto \tilde{\pi}_\alpha(f)(\pi_\alpha(u))]_F^{M_\alpha}.$$

$$\begin{tikzcd}
 & & M_\alpha \arrow[dl, phantom, "\in" sloped] \arrow[ddd, bend left=60, "\pi^{\bar{\mathcal{T}}}_{\alpha,\beta+1}"] \\
L[M_\alpha, \pi_\alpha \restriction \mathrm{OR}] \arrow[r, "\tilde{\pi}_\alpha"] \arrow[d, "\pi^{\mathcal{T}}_{\alpha,\beta+1}"'] & L[M^*_\alpha, \pi^*_\alpha] & \\
L[M_{\beta+1}, \pi_{\beta+1} \restriction \mathrm{OR}] \arrow[r, "\tilde{\pi}_{\beta+1}"'] & L[M^*_{\beta+1}, \pi^*_{\beta+1}] & \\
 & & M_{\beta+1} \arrow[ul, phantom, "\in" sloped]
\end{tikzcd}$$

This is indeed well–defined and elementary, as we may use $(\pi_\alpha \restriction [\mathrm{crit}(F)]^{\mathrm{Card}(a)}) \in M_\alpha$ and compute as follows. Let φ be a formula, let us assume for notational convenience that φ has only one free variable, and let $a \in [\mathrm{lh}(F)]^{<\omega}$ and $f\colon [\mathrm{crit}(F)]^{\mathrm{Card}(a)} \to \mathcal{M}_\alpha^{\mathcal{T}}$, $f \in \mathcal{M}_\alpha^{\mathcal{T}}$.

$$\begin{aligned}
& \mathcal{M}^{\mathcal{T}}_{\beta+1} \models \varphi([a,f]^{\mathcal{M}_\alpha^{\mathcal{T}}}) \\
\iff\ & \{u \in [\mathrm{crit}(F)]^{\mathrm{Card}(a)} \colon \mathcal{M}_\alpha^{\mathcal{T}} \models \varphi(f(u))\} \in F_a \\
\iff\ & \{u \in [\mathrm{crit}(F)]^{\mathrm{Card}(a)} \colon L[M^*_\alpha, \pi^*_\alpha] \models \varphi(\tilde{\pi}_\alpha(f)(\tilde{\pi}_\alpha(u)))\} \in F_a \\
\iff\ & \{u \in [\mathrm{crit}(F)]^{\mathrm{Card}(a)} \colon L[M^*_\alpha, \pi^*_\alpha] \models \varphi(\tilde{\pi}_\alpha(f)((\pi_\alpha \restriction [\mathrm{crit}(F)]^{\mathrm{Card}(a)})(u)))\} \in F_a \\
\iff\ & a \in \pi^{\bar{\mathcal{T}}}_{\alpha,\beta+1}(\{u \in [\mathrm{crit}(F)]^{\mathrm{Card}(a)} \colon L[M^*_\alpha, \pi^*_\alpha] \models \varphi(\tilde{\pi}_\alpha(f)((\pi_\alpha \restriction [\mathrm{crit}(F)]^{\mathrm{Card}(a)})(u)))\}) \\
\iff\ & L[M^*_{\beta+1}, \pi^*_{\beta+1}] \models \varphi(\pi^{\bar{\mathcal{T}}}_{\alpha,\beta+1}(\tilde{\pi}_\alpha(f))(((\pi_\alpha \restriction [\mathrm{crit}(F)]^{\mathrm{Card}(a)})(a))) \\
\iff\ & L[M^*_{\beta+1}, \pi^*_{\beta+1}] \models \varphi(\pi^{\bar{\mathcal{T}}}_{\alpha,\beta+1}(\tilde{\pi}_\alpha(f))((\pi_\alpha(a))).
\end{aligned}$$

Notice that $\tilde{\pi}_{\beta+1} \restriction \mathrm{lh}(F) = \tilde{\pi}_\alpha \restriction \mathrm{lh}(F)$, as required by (g).

The key point is now that

$$M^*_{\beta+1} \cap \mathrm{ran}(\tilde{\pi}_{\beta+1}) \cong \mathcal{M}^{\bar{\mathcal{T}}}_{\beta+1}. \tag{47}$$

(47) is established by the argument which gave Schlutzenberg's Lemma 2.15. Let I denote the class of all M_{sw}–indiscernibles, and let us assume for notational convenience that all embeddings which we consider fix all the points in I.

In order to show (47), let $x \in M^*_{\beta+1} \cap \mathrm{ran}(\tilde{\pi}_{\beta+1})$, say $x = \tilde{\pi}_{\beta+1}(\bar{x}) \in M^*_{\beta+1}$. We have that $\bar{x} \in \mathrm{Hull}^{\mathcal{M}^{\mathcal{T}}_{\beta+1}}(\mathrm{lh}(F) \cup I)$, so that $x \in \mathrm{Hull}^{L[M^*_{\beta+1}, \pi^*_{\beta+1}]}(\tilde{\pi}_{\beta+1}"\mathrm{lh}(F) \cup I) \cap M^*_{\beta+1}$. By the elementarity of $\pi^{\mathcal{T}}_{0,\beta+1}$, $L[M^*_{\beta+1}, \pi^*_{\beta+1}]$ is the Varsovian model derived from $M_{\beta+1}$ which in turn is equal to $\mathsf{HOD}^{P[h]}$ for all $P \in \mathcal{F}^{M_{\beta+1}}$ and all h which are $\mathrm{Col}(\omega, < \kappa^P)$–generic over P, cf. Claim 2.12 (a). We thus have $x \in \mathrm{Hull}^P(\tilde{\pi}_{\beta+1}"\mathrm{lh}(F) \cup I)$ for all $P \in \mathcal{F}^{M_{\beta+1}}$. By picking P sufficiently far out in the system, we thus get that

$$\pi^*_{\beta+1}(x) \in \mathrm{Hull}^{M^*_{\beta+1}}(\pi^*_{\beta+1} \circ \tilde{\pi}_{\beta+1}"\mathrm{lh}(F) \cup I). \tag{48}$$

However, for each ordinal ρ we may pick some $s \in [I]^{<\omega}$ such that $\rho \in \mathrm{dom}(\pi^*_{\beta+1} \restriction \mathrm{Hull}^{M^*_{\beta+1}|\max(s)}(\gamma_s^{M^*_{\beta+1}}) \cup \{s^-\})$, i.e., $\pi^*_{\beta+1}(\rho) = (\pi^*_{\beta+1} \restriction \mathrm{Hull}^{M^*_{\beta+1}|\max(s)}(\gamma_s^{M^*_{\beta+1}}) \cup \{s^-\}))(\rho)$, and then

$$\begin{aligned}
\pi^*_{\beta+1}(\rho) &= (\pi^*_{\beta+1} \restriction \mathrm{Hull}^{M^*_{\beta+1}|\max(s)}(\gamma_s^{M^*_{\beta+1}}) \cup \{s^-\}))(\rho) \\
&= \pi^{\mathcal{T}}_{0,\beta+1}(\pi^*_0 \restriction \mathrm{Hull}^{M^*_0|\max(s)}(\gamma_s^{M^*_0}) \cup \{s^-\}))(\rho) \\
&= \pi^{\mathcal{T}}_{0,\beta+1}(\pi_0(\pi_0 \restriction \mathrm{Hull}^{M^*_0|\max(s)}(\gamma_s^{M^*_0}) \cup \{s^-\})))(\rho).
\end{aligned}$$

But $\pi_0 \restriction \mathrm{Hull}^{M^*_0|\max(s)}(\gamma_s^{M^*_0}) \cup \{s^-\}) \in \mathrm{Hull}^{M_0}(I)$, hence $\pi_0(\pi_0 \restriction \mathrm{Hull}^{M^*_0|\max(s)}(\gamma_s^{M^*_0}) \cup \{s^-\})) \in \mathrm{Hull}^{M^*_0}(I)$, hence $\pi^{\mathcal{T}}_{0,\beta+1}(\pi_0(\pi_0 \restriction \mathrm{Hull}^{M^*_0|\max(s)}(\gamma_s^{M^*_0}) \cup \{s^-\}))) \in \mathrm{Hull}^{M^*_{\beta+1}}(I)$. This shows that $\mathrm{Hull}^{M^*_{\beta+1}}(\tilde{\pi}_{\beta+1}"\mathrm{lh}(F) \cup I)$ is closed under $\rho \mapsto \pi^*_{\beta+1}(\rho)$ as well as under $\rho \mapsto (\pi^*_{\beta+1})^{-1}(\rho)$, so that by $x \in M^*_{\beta+1}$, (48) is tantamount to saying that

$$x \in \mathrm{Hull}^{M^*_{\beta+1}}(\tilde{\pi}_{\beta+1}"\mathrm{lh}(F) \cup I). \tag{49}$$

We have shown that $x \in M^*_{\beta+1} \cap \mathrm{ran}(\tilde{\pi}_{\beta+1})$ implies (49). This gives (47).

By (47), we may let $\pi_{\beta+1} = \tilde{\pi}_{\beta+1} \restriction M_{\beta+1}$. It remains to be verified that

$$\pi^{\mathcal{T}}_{\alpha,\beta+1}(\pi_\alpha) = \tilde{\pi}_{\beta+1} \restriction \mathrm{OR}. \tag{50}$$

Let $\xi = \pi^{\mathcal{T}}_{\alpha,\beta+1}(f)(a)$, where $a \in [\mathrm{lh}(F)]^{<\omega}$ and $f \colon [\mathrm{crit}(F)]^{\mathrm{Card}(a)} \to \mathrm{OR}$, $f \in \mathcal{M}^{\mathcal{T}}_\alpha$. Then

$$\begin{aligned}
\pi^{\mathcal{T}}_{\alpha,\beta+1}(\pi_\alpha)(\xi) &= \pi^{\mathcal{T}}_{\alpha,\beta+1}(\pi_\alpha)(\pi^{\mathcal{T}}_{\alpha,\beta+1}(f)(a)) \\
&= \pi^{\mathcal{T}}_{\alpha,\beta+1}(\pi_\alpha \circ f)(\pi^{\mathcal{T}}_{\alpha,\beta+1}(a)) \\
&= \pi^{\mathcal{T}}_{\alpha,\beta+1}(u \mapsto \tilde{\pi}_\alpha(f)((\pi_\alpha \restriction [\mathrm{crit}(F)]^{<\omega})(u))(a) \\
&= \tilde{\pi}_{\beta+1}(\pi^{\mathcal{T}}_{\alpha,\beta+1}(f)(a)) \\
&= \tilde{\pi}_{\beta+1}(\xi).
\end{aligned}$$

$\square$ (Theorem 2.17)

The proof of Theorem 2.19 makes use of the following result. We know that $\mathcal{M}_\infty$ is an iterate of M_{sw} via an ω–stack of normal trees, $(\mathcal{T}_n : n < \omega)$. The normalizing procedure which is developed in the papers [16], [17], and [20] produces a normal iteration tree $X(\mathcal{T}_n : n < \omega)$ on M_{sw} with last model $\mathcal{M}_\infty$.

Theorem 2.18 (F. Schlutzenberg, J. Steel) ([16], [17], [20]) $\mathcal{M}_\infty$ is a Σ–iterate of M_{sw} via a *normal* iteration tree on M_{sw} which lives on $M_{\text{sw}}|\delta$ and with iteration map $\pi_{M_{\text{sw}},\infty}$.

Theorem 2.19 *δ is a Woodin cardinal inside $L[M_{\text{sw}}, \rho \mapsto \pi_{M_{\text{sw}},\infty}(\rho)]$.*

Proof. The proof we are about to present was also found independently by Farmer Schlutzenberg following a hint by John Steel.

Let $\mathcal{T}$ be the (unique) tree on M_{sw} which witnesses the statement of Theorem 2.18. By Corollary 2.16 (b), we may construe $\mathcal{T}$ as a tree on $L[M_{\text{sw}}, \rho \mapsto \pi_{M_{\text{sw}},\infty}(\rho)]$, and we may lift the iteration map $\pi_{M_{\text{sw}},\infty}$ to an iteration map

$$\tilde{\pi} \colon L[M_{\text{sw}}, \rho \mapsto \pi_{M_{\text{sw}},\infty}(\rho)] \to L[\mathcal{M}_\infty, \sigma],$$

where σ is the image of $\rho \mapsto \pi_{M_{\text{sw}},\infty}(\rho)$ under $\tilde{\pi}$. However, the same argument as in the proof of Corollary 2.16 (a) shows that

$$\pi_{M_{\text{sw}},\infty}(\pi_{M_{\text{sw}},\infty} \restriction \delta) = \pi^\infty_{0,\infty} \restriction \delta_\infty. \tag{51}$$

This is true because if again $s_n = \{\aleph_1, \ldots, \aleph_{n+1}\}$ for $n < \omega$, then $\pi_{M_{\text{sw}},\infty}(\pi_{M_{\text{sw}},\infty} \restriction \delta) = \pi_{M_{\text{sw}},\infty}(\bigcup_{n<\omega} \pi^{s_n}_{M_{\text{sw}},\infty} \restriction \gamma^{M_{\text{sw}}}_{s_n}) = \bigcup_{n<\omega} \pi_{M_{\text{sw}},\infty}(\pi^{s_n}_{M_{\text{sw}},\infty} \restriction \gamma^{M_{\text{sw}}}_{s_n}) = \bigcup_{n<\omega} \pi^\infty_{0,\infty} \restriction \gamma^{\mathcal{M}_\infty}_{s_n} = \pi^\infty_{0,\infty} \restriction \delta_\infty$.

We therefore have that

$$\tilde{\pi} \colon L[M_{\text{sw}}, \rho \mapsto \pi_{M_{\text{sw}},\infty}(\rho)] \to L[\mathcal{M}_\infty, \rho \mapsto \rho^*]$$

is given by the normal iteration tree $\mathcal{T}$.

Let us now suppose that δ is not a Woodin cardinal in $L[M_{\text{sw}}, \rho \mapsto \pi_{M_{\text{sw}},\infty}(\rho)]$ which implies that δ_∞ is not a Woodin cardinal in $L[\mathcal{M}_\infty, \rho \mapsto \rho^*]$. Notice that $\mathcal{T}$ must have length $\delta_\infty + 1 = \kappa^{+M_{\text{sw}}} + 1$, and $\mathcal{T} \restriction \kappa^{+M_{\text{sw}}}$ is guided by ¶–small $\mathcal{Q}$–structures, so that $\mathcal{T} \restriction \kappa^{+M_{\text{sw}}} \in M_{\text{sw}}$.

Write $\lambda = \kappa^{++M_{\text{sw}}}$, and $\mathcal{V} = L[\mathcal{M}_\infty, \rho \mapsto \rho^*]$. Let $g \in V$ be $\text{Col}(\omega, \lambda)$–generic over M_{sw}. Inside $M_{\text{sw}}[g]$, let T be a tree of height ω searching for a $\mathcal{Q}$ and b such that

(α) $\mathcal{Q}$ is a transitive model of ZFC^- of height λ such that δ is a cardinal in $\mathcal{Q}$ and $H_\delta^{\mathcal{Q}} = M_{\rm sw}|\delta$,

(β) b is a cofinal branch through $\mathcal{T} \restriction \kappa^{+M_{\rm sw}}$ such that when $\mathcal{T}'$ is $\mathcal{T} \restriction \kappa^{+M_{\rm sw}}$, being construed as a tree on $\mathcal{Q}$,[13] then all the models $\mathcal{M}_\alpha^{\mathcal{T}'}$, $\alpha < \kappa^{+M_{\rm sw}}$, are well–founded, and

$$\pi_{0,b}^{\mathcal{T}'} \colon \mathcal{Q} \to H_\lambda^{\mathcal{V}}.$$

T is ill–founded in V, as we may set $\mathcal{Q} = H_\lambda^{L[M_{\rm sw}, \pi_{M_{\rm sw},\infty} \restriction {\rm OR}]}$ and $b = [0, \kappa^{+M_{\rm sw}})_\mathcal{T}$. Therefore, T is ill–founded in $M_{\rm sw}[g] \subset V$ as well. Let $\mathcal{Q}$ and b in $M_{\rm sw}[g]$ be given by a branch through T. Suppose that $b \neq [0, \kappa^{+M_{\rm sw}})_\mathcal{T}$. As $\mathcal{T} \restriction \kappa^{+M_{\rm sw}}$ is normal, the "zipper argument," cf. e.g. [19, p. 1645f.], then shows that $\delta(\mathcal{T} \restriction \kappa^{+M_{\rm sw}}) = \delta_\infty$ must be Woodin in $H_\lambda^{\mathcal{V}}$ which is against our current hypothesis.

Therefore, $[0, \kappa^{+M_{\rm sw}})_\mathcal{T} = b \in M_{\rm sw}[g]$. As this was shown to be true for *any* b such that $\mathcal{Q}$ and b come from a branch through T for some $\mathcal{Q}$, we must have that $[0, \kappa^{+M_{\rm sw}})_\mathcal{T} \in M_{\rm sw}$ by the homogeneity of ${\rm Col}(\omega, \lambda)$. But this gives that

$$\pi_{M_{\rm sw},\infty} \restriction \delta = \pi_{0,[0,\kappa^{+M_{\rm sw}})_\mathcal{T}}^{\mathcal{T} \restriction \kappa^{+M_{\rm sw}}} \in M_{\rm sw},$$

which is a map which sends $\delta < \kappa$ cofinally into $\delta_\infty = \kappa^{+M_{\rm sw}}$. Hence $\kappa^{+M_{\rm sw}}$ is singular in $M_{\rm sw}$. Contradiction! $\square$ (Theorem 2.19)

J. Steel observed that if g is ${\rm Col}(\omega, < \kappa)$–generic over $M_{\rm sw}$, then $M_{\rm sw}[g]$ is *not* a model of "every OD–set of reals is determined," so that one canot use [6] to deduce the conclusion of Lemma 2.19.

Lemma 2.20 $L[\mathcal{M}_\infty, \rho \mapsto \rho^*] = L[\mathcal{M}_\infty|\delta_\infty, \Sigma_{\mathcal{M}_\infty|\delta_\infty}]$.

Proof sketch. "$\supset$": By Lemma 2.13, $\Sigma_{\mathcal{M}_\infty} \restriction L[M_\infty, \rho \mapsto \rho^*]$ is definable inside $L[M_\infty, \rho \mapsto \rho^*]$.

"$\subset$": Let us write W for $K(\mathcal{M}_\infty|\delta_\infty)$ as being constructed inside $L[\mathcal{M}_\infty|\delta_\infty, \Sigma_{\mathcal{M}_\infty|\delta_\infty}]$. Inside $L[\mathcal{M}_\infty|\delta_\infty, \Sigma_{\mathcal{M}_\infty|\delta_\infty}]$, W is fully iterable, W satisfies weak covering above δ_∞, and W has a Woodin cardinal. By an unpublished theorem of Steel, W must then have a strong cardinal above δ_∞. From the point of view of $L[\mathcal{M}_\infty, \rho \mapsto \rho^*]$, W must then be a universal weasel.

We thus get an elementary embedding $j : \mathcal{M}_\infty \to W$. Suppose $j \neq {\rm id}$. Using an argument from [11], we may then reconstruct $j \restriction \mathcal{M}_\infty|{\rm crit}(j)^+$ inside $L[\mathcal{M}_\infty|\delta_\infty, \Sigma_{\mathcal{M}_\infty|\delta_\infty}]$ as follows.

[13]This is possible by item (α).

Write $\lambda = \text{crit}(j)^{+\mathcal{M}_\infty}$ and $\lambda' = j(\lambda)$. There are trees $\mathcal{T}$ and $\mathcal{T}'$, both on $\mathcal{M}_\infty$ and inside $L[\mathcal{M}_\infty|\delta_\infty, \Sigma_{\mathcal{M}_\infty|\delta_\infty}]$ of length $\lambda+1$ and $\lambda'+1$, respectively, such that $\lambda = \pi^{\mathcal{T}}_{0\lambda}(\delta_\infty)$ and $\lambda' = \pi^{\mathcal{T}'}_{0\lambda'}(\delta_\infty)$. $j \restriction \mathcal{M}_\infty|\text{crit}(j)^+$ is then the unique map which sends $\pi^{\mathcal{T}}_{0\lambda}{}''\delta_\infty$ to $\pi^{\mathcal{T}'}_{0\lambda'}{}''\delta_\infty$.

Contradiction! $\square$ (Lemma 2.20)

In a sequel to this paper, cf. [10], we will study Varsovian models in more generality.

The attentive reader will notice that the preceding arguments actually produced the following statement.

Theorem 2.21 *For a cone of reals x, $M_s(x)$ has a 2–small core model $K = K^{M_s(x)}$ which in V is an iterate of M_{sw}, and the mantle of $M_s(x)$ is the Varsovian model $L[K, \Sigma_K]$, where Σ_K is the tail of Σ.*

3 Appendix: Bukovský's theorem.

Definition 3.1 *Let W be an inner model of V. Let λ be an infinite cardinal. We say that W* uniformly λ–covers V *iff for all functions $f \in V$ with $\text{dom}(f) \in W$ and $\text{ran}(f) \subset W$ there is some function $g \in W$ with $\text{dom}(g) = \text{dom}(f)$ such that $f(x) \in g(x)$ and $\text{Card}(g(x)) < \lambda$ for all $x \in \text{dom}(g)$.*

If there is some poset $\mathbb{P} \in W$ having the λ–c.c. in W and some g which is $\mathbb{P}$–generic over W such that $V = W[g]$, then W uniformly λ–covers V. Bukovský's Theorem 3.5 will say that the converse is true also.

The following is probably part of the folklore.

Theorem 3.2 *Let W be an inner model of V, and let λ be an infinite regular cardinal. Assume that W uniformly λ–covers V, and assume also that $\mathcal{P}(2^{<\lambda}) \cap V \subset W$. Then $W = V$.*

Proof. Let us call any set Γ of functions an *antichain* iff for all a, $b \in \Gamma$ with $a \neq b$ there is some $i \in \text{dom}(a) \cap \text{dom}(b)$ with $a(i) \neq b(i)$.

It is easily seen that the hypotheses on W give that

$$^{2^{<\lambda}}W \subset W. \tag{52}$$

To verify (52), notice first that by $\mathcal{P}(2^{<\lambda}) \cap V \subset W$, W computes the cardinal successor of $2^{<\lambda}$ correctly and for every $\gamma < (2^{<\lambda})^+$, $\mathcal{P}(\gamma) \cap V \subset W$.

Now let $f\colon 2^{<\lambda} \to \mathrm{OR}$, $f \in V$. Using the fact that W uniformly λ–covers V, let $g \in W$ be a function with $\mathrm{dom}(g) = 2^{<\lambda}$ such that $g(\xi)$ is a set of ordinals, $f(\xi) \in g(\xi)$, and $\mathrm{Card}(g(\xi)) < \lambda$ for all $\xi < 2^{<\lambda}$. Let $e\colon \gamma \cong \bigcup \mathrm{ran}(g)$ be the (inverse of the) transitive collapse of $\bigcup \mathrm{ran}(g)$, so that $e \in W$ and $\gamma < (2^{<\lambda})^+$. As $\mathcal{P}(\gamma) \cap V \subset W$, the function $e^{-1} \circ f\colon 2^{<\lambda} \to \gamma$ is in W, which gives that $f = e \circ (e^{-1} \circ f) \in W$. We showed (52).

Assume that $A\colon \alpha \to 2$, for some ordinal α, is such that $A \in V \setminus W$. Let us write $\mathcal{F}$ for the collection of all functions a such that there is some $x \subset \alpha$ of size $< \lambda$ such that $a\colon x \to 2$. Using again the fact that W uniformly λ–covers V,[14] we may pick a function g in W such that if $\Gamma \subset \mathcal{F}$ is an antichain with $\Gamma \in W$, then

(i) $g(\Gamma) \in W$ is a subset of Γ of size $< \lambda$, and

(ii) if there is some (unique!) $a \in \Gamma$ with $a = A \restriction \mathrm{dom}(a)$, then $a \in g(\Gamma)$.

We call $a \in \mathcal{F}$ *legal* iff for no antichain $\Gamma \in W$, $a \in \Gamma \setminus g(\Gamma)$. Notice that being legal is defined inside W (from the parameter $g \in W$).

Every $A \restriction x$, where $x \subset \alpha$ has size $< \lambda$, is legal.

If $\Gamma \subset \mathcal{F}$ is an antichain with $\Gamma \in W$, and if every $a \in \Gamma$ is legal, then we must have $g(\Gamma) = \Gamma$, from which it follows that Γ has size $< \lambda$.

Let $\theta \gg \alpha$ be such that $\theta^{<\lambda} = \theta$. Let

$$X \prec (H_\theta; \in, \{A\}, \mathcal{F}, g, H_\theta \cap W)$$

be such that ${}^{<\lambda}X \subset X$ and $\mathrm{Card}(X) = 2^{<\lambda}$. By (52), $X \cap W \in W$, and of course

$$X \cap W \prec (H_\theta \cap W; \in, \mathcal{F}, g) \in W. \tag{53}$$

Write $\sigma\colon \bar{W} \cong X \cap W$ for the (inverse of the) transitive collapse of $X \cap W$, so that $\sigma \in W$. σ extends to $\tilde{\sigma}\colon H \cong X$, the (inverse of the) transitive collapse of X.

Notice that $\mathcal{P}(2^{<\lambda}) \cap V \subset W$ gives that $\bar{A} = \tilde{\sigma}^{-1}(A) \in W$, which in turn yields that

$$A \restriction (X \cap \alpha) = \sigma\text{"}\bar{A} \in W. \tag{54}$$

We are now going to derive a contradiction from (54).

Using (54), we may work inside W and define a sequence $(a_i\colon i < \lambda)$ of elements of $\mathcal{F}$ such that $a_i \in X$ and $\mathrm{dom}(a_i) \supset \mathrm{dom}(a_j)$ for all $j < i < \lambda$ as follows. Assume $(a_j\colon j < i)$ has already been chosen. Notice that $(a_j\colon j < i) \in X$ by

[14] This use is now substantial, in contrast to the previous one.

${}^{<\lambda}X \subset X$. Write $x = \bigcup_{j<i} \mathrm{dom}(a_j)$, so that $x \in X$. Clearly, for every $\xi < \alpha$ there is some legal $a \in \mathcal{F}$ such that $x \cup \{\xi\} \subset \mathrm{dom}(a)$ and $a = A \restriction \mathrm{dom}(a)$ (just pick $A \restriction (x \cup \{\xi\})$). There must then be some $\xi < \alpha$ such that there are legal a and b in $\mathcal{F}$ with $x \cup \{\xi\} \subset \mathrm{dom}(a) \cap \mathrm{dom}(b)$ and $a(\xi) \neq b(\xi)$, as otherwise A would be the union of all legal $a \in \mathcal{F}$ with $a \supset A \restriction x$ and thus A would be in W.

By (53) we must then have inside X some $\xi < \alpha$ and some legal a and b in $\mathcal{F}$ with $x \cup \{\xi\} \subset \mathrm{dom}(a) \cap \mathrm{dom}(b)$ and $a(\xi) \neq b(\xi)$. By (54), we may then choose in W some $\xi \in \alpha \cap X$ and some $a \in \mathcal{F} \cap X$ such that $x \cup \{\xi\} \subset \mathrm{dom}(a)$, $a \restriction x = (A \restriction (X \cap \alpha)) \restriction x$ $(= A \restriction x)$, and $a(\xi) \neq (A \restriction (X \cap \alpha))(\xi)$ $(= A(\xi))$. Let $a_i = a$.

Writing $\Gamma = \{a_i \colon i < \lambda\}$, $\Gamma \in W$, and Γ is an antichain consisting of legal functions. But this is a contradiction! □ (Theorem 3.2)

Let us fix $W \subset V$, an inner model, and let λ and μ be infinite cardinals, $\lambda \leq \mu$. We aim to define a poset in W which will be a candidate for generically adding a given subset of μ.

Working in W, let $\mathcal{L}$ be the infinitary language with atomic fomulae "$\check{\xi} \in \dot{a}$," for $\xi < \mu$, and such that the set of formulae is closed under negation and infinite disjunctions of the form $\bigvee\!\!\!\bigvee \Gamma$ for all well–ordered sets Γ of fomulae with $\mathrm{Card}(\Gamma) < \lambda$. Writing $\mu^{<\lambda} = (\mu^{<\lambda})^W$, $\mathcal{L}$ has size $\mu^{<\lambda}$.

For $A \subset \mu$, $A \in V^{\mathrm{Col}(\omega,\mu^{<\lambda})}$, and $\varphi \in \mathcal{L}$, we may define the meaning of "$A \vDash \varphi$" in the obvious recursive fashion: $A \vDash$ "$\check{\xi} \in \dot{a}$" iff $\xi \in A$, $A \vDash \neg\varphi$ iff $A \nvDash \varphi$, and $A \vDash \bigvee\!\!\!\bigvee \Gamma$ iff $A \vDash \varphi$ for some $\varphi \in \Gamma$. Inside $V^{\mathrm{Col}(\omega,\mu^{<\lambda})}$, the relation "$A \vDash \varphi$" is Borel in the codes. For $\Gamma \subset \mathcal{L}$, $A \vDash \Gamma$ means $A \vDash \varphi$ for all $\varphi \in \Gamma$. For $\Gamma \cup \{\varphi\} \in \mathcal{P}(\mathcal{L}) \cap W$, we write

$$\Gamma \vdash \varphi \tag{55}$$

iff in $W^{\mathrm{Col}(\omega,\mu^{<\lambda})}$, for all $A \subset \mu$, if $A \vDash \Gamma$, then $A \vDash \varphi$. (55) is thus defined over W, and inside $W^{\mathrm{Col}(\omega,\mu^{<\lambda})}$, (55) is Π^1_1 in the codes By absoluteness, (55) is thus equivalent with the fact that in $V^{\mathrm{Col}(\omega,\mu^{<\lambda})}$, for all $A \subset \mu$, if $A \vDash \Gamma$, then $A \vDash \varphi$. For $\Gamma \in \mathcal{P}(\mathcal{L}) \cap W$, Γ is called *consistent* iff there is no $\varphi \in \mathcal{L}$ such that $\Gamma \vdash \varphi$ and $\Gamma \vdash \neg\varphi$, which in turn is easily seen to be equivalent with the fact that in $W^{\mathrm{Col}(\omega,\mu^{<\lambda})}$ (equivalently, in $V^{\mathrm{Col}(\omega,\mu^{<\lambda})}$) there is some $A \subset \mu$ with $A \vDash \Gamma$.

Now let

$$g \colon [\mathcal{L}]^{\lambda} \cap W \to [\mathcal{L}]^{<\lambda} \cap W,\ g \in W$$

be a function such that

(i) $g(\Gamma) \subset \Gamma$, and

(ii) $\mathrm{Card}(g(\Gamma)) < \lambda$

for all $\Gamma \in [\mathcal{L}]^\lambda \cap W$. Let us call $\varphi \in \mathcal{L}$ *illegal* iff there is some $\Gamma \in [\mathcal{L}]^\lambda \cap W$ such that $\varphi \in \Gamma \setminus g(\Gamma)$, and let us write T^g for the set of all formulae of the form[15]

$$\varphi \to \bigvee\!\!\!\!\bigvee g(\Gamma), \tag{56}$$

where φ is illegal, $\Gamma \in [\mathcal{L}]^\lambda \cap W$, and $\varphi \in \Gamma \setminus g(\Gamma)$.

Let us write $\mathbb{P}^g$ for the set of all $\varphi \in \mathcal{L}$ such that $T^g \cup \{\varphi\}$ is consistent. We also write

$$\varphi \leq_{\mathbb{P}^g} \varphi' \tag{57}$$

for $T^g \cup \{\varphi\} \vdash \varphi'$.

Claim 3.3 *$\mathbb{P}^g$ has the λ–c.c. inside W.*

Proof. Let $\Gamma \in [\mathbb{P}^g]^\lambda \cap W$. Let $\varphi \in \Gamma \setminus g(\Gamma)$. By (56), $\varphi \leq_{\mathbb{P}^g} \bigvee\!\!\!\!\bigvee g(\Gamma)$, so that Γ cannot be an antichain. $\square$ (Claim 3.3)

For an arbitrary choice of g, we might have that $\mathbb{P}^g$ is quite trivial, or even $\mathbb{P}^g = \emptyset$. Let $A \subset \mu$, $A \in V$. We set

$$G_A = \{\varphi \in \mathbb{P}^g \colon A \vDash \varphi\}.$$

Claim 3.4 *Assume that $A \vDash T^g$. Then $G_A \subset \mathbb{P}^g$ is a $\mathbb{P}^g$–generic filter over W and*

$$A = \{\xi < \mu \colon \text{ “}\check{\xi} \in \dot{a}\text{”} \in G_A\} \in W[G_A].$$

Proof. If φ, $\varphi' \in \mathbb{P}^g$, $A \vDash \varphi$, and $\varphi \leq_{\mathbb{P}^g} \varphi'$, then $A \vDash \varphi'$ using absoluteness. If φ, $\varphi' \in \mathbb{P}^g$, $A \vDash \varphi$, and $A \vDash \varphi'$, then $A \vDash \varphi \wedge \varphi'$,[16] $\varphi \wedge \varphi' \in \mathbb{P}^g$ by $A \vDash T^g$, and clearly $\varphi \wedge \varphi' \leq_{\mathbb{P}^g} \varphi$ and $\varphi \wedge \varphi' \leq_{\mathbb{P}^g} \varphi'$. Hence G_A is a filter.

Now let $\Gamma \in W$ be a maximal antichain in $\mathbb{P}^g$. By Claim 3.3, $\Gamma \in [\mathbb{P}^g]^{<\lambda}$. If $G_A \cap \Gamma = \emptyset$, then $A \vDash \neg \bigvee\!\!\!\!\bigvee \Gamma$. By $A \vDash T^g$, $\neg \bigvee\!\!\!\!\bigvee \Gamma \in \mathbb{P}^g$, and

$$\Gamma \cup \{\neg \bigvee\!\!\!\!\bigvee \Gamma\} \supsetneq \Gamma$$

is an antichain. Contradiction!

The rest is easy. $\square$ (Claim 3.4)

[15] $\varphi \to \varphi'$ is short for $\bigvee\!\!\!\!\bigvee\{\neg\varphi, \varphi'\}$.
[16] $\varphi \wedge \varphi'$ is short for $\neg \bigvee\!\!\!\!\bigvee\{\neg\varphi, \neg\varphi'\}$.

Theorem 3.5 (Lev Bukovský) *Let $W \subset V$ be an inner model, and let λ be an infinite regular cardinal such that W uniformly λ–covers V. Let $e\colon 2^{2^{<\lambda}} \to \mathcal{P}(2^{<\lambda})$ be a bijection, and let*

$$A = \{2^{<\lambda} \cdot \eta + \xi \colon \eta < 2^{2^{<\lambda}} \wedge \xi \in e(\eta)\}.$$

There is then some poset $\mathbb{P} \in W$ such that

(a) $\mathbb{P}$ has the λ–c.c. in W,

(b) $\mathbb{P}$ has size $2^{2^{<\lambda}}$ in W,

(c) A is $\mathbb{P}$–generic over W, and

(d) $V = W[A]$.

Proof. Let us write

$$\mu = 2^{2^{<\lambda}},$$

as being computed in V.

By the fact that W uniformly λ–covers V, we may find a function

$$g\colon [\mathcal{L}]^{\lambda} \to [\mathcal{L}]^{<\lambda},\ g \in W$$

such that for all $\Gamma \in [\mathcal{L}]^{\lambda} \cap W$,

(i) $g(\Gamma) \subset \Gamma$,

(ii) $\mathrm{Card}(g(\Gamma)) < \lambda$, and

(iii) if $A \vDash \varphi$ for *some* $\varphi \in \Gamma$, then $A \vDash \bigvee\!\!\!\bigvee g(\Gamma)$.

For this choice of g, $A \vDash T^g$. Hence by Claim 3.4, G_A is $\mathbb{P}^g$–generic over W, and $A \in W[G_A]$. This gives (a), (b), and (c). Clearly, $W[G_A]$ inherits from W the fact that it uniformly λ–covers V, so that (d) is given by Theorem 3.2. $\square$ (Theorem 3.5)

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