The self-iterability of $L[E]$

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Abstract

Let $L[E]$ be an iterable tame extender model. We analyze to which extent $L[E]$ knows fragments of its own iteration strategy. Specifically, we prove that inside $L[E]$, for every cardinal κ which is not a limit of Woodin cardinals there is some cutpoint $t < \kappa$ such that $J_\kappa[E]$ is iterable above t with respect to iteration trees of length less than κ.

As an application we show $L[E]$ to be a model of the following two cardinals versions of the diamond principle. If $\lambda > \kappa > \omega_1$ are cardinals, then $\diamondsuit^*_{\kappa,\lambda}$ holds true, and if in addition λ is regular, then $\diamondsuit^+_{\kappa,\lambda}$ holds true.

0 Introduction.

If $n > 0$, then M_n, the least iterable extender model with n Woodin cardinals does not know how to iterate itself (cf. Lemma 1.1). However, if $\delta_1 < \cdots < \delta_n$ are the Woodin cardinals of M_n, then inside M_n, $M_n||\delta_{i+1}$ is (ω, δ_{i+1})-iterable above δ_i for all $i < n$ (with the convention that $\delta_0 = 0$). In fact, more is true (cf. [8, Theorem 0.5]). This motivates Definition 0.1 below.

As a bit of terminology, if $\mathcal{M} = (J_\alpha[E]; \in, E, E_\alpha)$ is a premouse and $\beta \leq \mathcal{M} \cap \mathrm{OR}$, then $\mathcal{M}||\beta = (J_\beta[E \restriction \beta]; \in, E \restriction \beta, E_\beta)$ and $\mathcal{M}|\beta = (J_\beta[E \restriction \beta]; \in, E \restriction \beta, \emptyset)$. An iteration tree $\mathcal{T}$ is *above* α iff $\mathrm{crit}(E^\mathcal{T}_\beta) \geq \alpha$ for all $\beta + 1 < \mathrm{lh}(\mathcal{T})$; a premouse $\mathcal{M}$ is (ω, γ)-iterable above α iff there is an iteration strategy Σ which works for normal trees on $\mathcal{M}$ which are above α and have length $< \gamma$. We may think of $\mathcal{M}$ as being quasi-$(\omega, \omega_1, \gamma)$-iterable above α iff there is an iteration strategy Σ which works for countable stacks of normal iteration trees on $\mathcal{M}$ which are above α, have length $< \gamma$, and are such that at limit stages the $\mathcal{Q}$-structure identifies the right branch according to Σ. (Cf. [11, p. 1212] on the formal definition and also on the

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significance of "quasi-$(\omega, \omega_1, \gamma)$-iterable.") An ordinal δ is a *cutpoint* of $\mathcal{M}$ iff there is no $E_\nu^{\mathcal{M}} \neq \emptyset$ such that $\text{crit}(E_\nu^{\mathcal{M}}) \leq \delta \leq \nu$.[2] $\mathcal{M}$ is *tame* iff for all initial segments $\mathcal{M}||\xi$ of $\mathcal{M}$, if an ordinal δ is a Woodin cardinal in $\mathcal{M}||\xi$, then δ is a cutpoint of $\mathcal{M}||\xi$.

Definition 0.1 *Let $L[E]$ be an extender model. Suppose that (in V) $L[E]$ is quasi-$(\omega, \omega_1, \text{OR})$-iterable, as being witnessed by Σ. Let γ be either a cardinal of $L[E]$, or else $\gamma = \infty$. The ordinal $t < \gamma$ is called the* transition point of $L[E]$ below γ *iff t is least such that t is a cutpoint of $L[E]$ and $L[E] \models$ "$J_\gamma[E]$ is quasi-$(\omega, \omega_1, \gamma)$-iterable above t," as being witnessed by the restriction of Σ to the relevant trees.*

The following result will be shown in the first section of this paper.

Theorem 0.2 *Let $L[E]$ be a tame extender model. Suppose that (in V) $L[E]$ is quasi-$(\omega, \omega_1, \text{OR})$-iterable. Then for every $\gamma > \omega$ such that $J_\gamma[E] \models$ "there are only boundedly many Woodin cardinals," and either γ is a cardinal in $L[E]$ or else $\gamma = \infty$, the transition point of $L[E]$ below γ exists.*

The research leading to this paper was motivated by the question whether every iterable tame extender model thinks that there is a well-ordering of $\mathbb{R}$ which is ordinal definable from a real parameter. Notice that our Theorem 0.2 only shows that if $\gamma = \omega_1^{L[E]}$, then there is some $t < \gamma$ such that $J_\gamma[E]$ is γ-iterable (rather than $\gamma+1$-iterable) above t from the point of view of $L[E]$. The second author has shown in subsequent work that if $L[E]$ is assumed to be ω-small, then $L[E]$ indeed thinks that there is a well-ordering of $\mathbb{R}$ which is (in fact inside $L(\mathbb{R})$) definable from a real parameter (cf. [14]).

In the second section of this paper we apply Theorem 0.2 and show that $\diamondsuit^*_{\kappa,\lambda}$ holds in $L[E]$. It is known that $\diamondsuit^*_{\kappa,\lambda}$ holds in L for all $\omega < \kappa < \lambda$ (cf. [3, Theorem 35.21]).

Definition 0.3 *Let $\kappa \leq \lambda$ be cardinals. The principle $\diamondsuit^*_{\kappa,\lambda}$ denotes the following statement. There is a function $F \colon \mathcal{P}_\kappa(\lambda) \to V$ such that for every $X \in \mathcal{P}_\kappa(\lambda)$, $F(X)$ is a subset of $\mathcal{P}(X)$ of size at most* $\text{Card}(X)$*, and for all $A \subset \lambda$, there is a club $C \subset \mathcal{P}_\kappa(\lambda)$ such that for all $X \in C$, $X \cap A \in F(X)$.*

Theorem 0.4 *Let $L[E]$ be a tame extender model. Suppose that (in V) $L[E]$ is sufficiently iterable.[3] Then $\diamondsuit^*_{\kappa,\lambda}$ holds in $L[E]$ for all cardinals κ, λ of $L[E]$ such that $\omega_1^{L[E]} < \kappa < \lambda$.*

[2] Often, this is called a *strong* cutpoint.

[3] "Sufficient iterability" is a slight strengthening of being quasi-$(\omega, \omega_1, \text{OR})$-iterable; see Definition 2.3.

We do not know whether one may weaken $\omega_1^{L[E]} < \kappa$ to $\omega_1^{L[E]} \leq \kappa$ in the statement of Theorem 0.4.

Donder and Matet have shown (cf. [1], cf. also [9]) that if we replace "club" by "stationary" in Definition 0.3, then the resulting weaker principle already follows from $2^{<\kappa} < \lambda$.

We also show that (a weakened version of) $\diamondsuit^+_{\kappa,\lambda}$ holds in $L[E]$ for regular λ. Jensen has shown that $\diamondsuit^+_{\kappa,\lambda}$ holds in L for all $\omega < \kappa < \lambda$ such that λ is regular in L (cf. [2, §2, Theorem 2]). The principle $\diamondsuit^+_{\kappa,\lambda}$ implies the corresponding two cardinals version of Kurepa's Hypothesis (cf. [2, §1, Theorem 5]).

Definition 0.5 *Let $\kappa \leq \lambda$ be cardinals. The principle $\diamondsuit^{+,\text{ unctble.}}_{\kappa,\lambda}$ denotes the following statement. There is a function $F\colon \mathcal{P}_\kappa(\lambda) \to V$ such that for every $X \in \mathcal{P}_\kappa(\lambda)$, $F(X)$ is a subset of $\mathcal{P}(X)$ of size at most* Card(X)*, and for all $A \subset \lambda$, there is an unbounded set $D \subset \lambda$ such that for all uncountable $X \in \mathcal{P}_\kappa(\lambda)$, if $X \cap D$ is unbounded in* $\sup(X)$*, where* $\sup(X)$ *is a limit ordinal, then both $X \cap A \in F(X)$ and $X \cap D \in F(X)$.*

Jensen's original definition of $\diamondsuit^+_{\kappa,\lambda}$ results from the one just given by crossing out the word "uncountable," so that $\diamondsuit^+_{\kappa,\lambda}$ trivially implies $\diamondsuit^{+,\text{ unctble.}}_{\kappa,\lambda}$.

Theorem 0.6 *Let $L[E]$ be a tame extender model. Suppose that (in V) $L[E]$ is sufficiently iterable. Then $\diamondsuit^{+,\text{ unctble.}}_{\kappa,\lambda}$ holds in $L[E]$ for all cardinals κ, λ of $L[E]$ such that $\omega_1^{L[E]} < \kappa < \lambda$ and λ is regular in $L[E]$.*

We do not know if Jensen's original version of $\diamondsuit^+_{\kappa,\lambda}$ can be shown to hold in $L[E]$.

1 Self-iterability.

In this section we shall prove Theorem 0.2. Let us, though, first indicate why this is non-trivial. Lemma 1.1 is due to the second author, elaborating on H. Woodin's original proof in the case where $L[E]$ is assumed to be 1-small (so that, basically, $L[E] = M_1$, the least iterable inner model with one Woodin cardinal).

Lemma 1.1 *Let $L[E]$ be an extender model, and suppose that $L[E] \models$ "δ is the least Woodin cardinal." Assume that in $V^{\mathrm{Col}(\omega,\delta)}$, $L[E]$ is normally $(\omega, \delta^+ + 1)$-iterable. Let $\kappa < \delta$. Then $L[E] \models$ "I am not normally $(\omega, \delta^{+L[E]} + 1)$-iterable with respect to non-dropping iteration trees which only use extenders with critical points taken from the interval (κ, δ) and its images."*

PROOF. Suppose not, and let us work inside $L[E]$ to define an iterate W of $L[E]$. Let $\mathbb{P}$ denote Woodin's extender algebra corresponding to all extenders below δ (cf. [13, section 7.2]). Let $\bar{W}$ be the δ^{th} linear iterate of $L[E]$ which is obtained by hitting the least measure with critical point above κ (and its images) δ many times, and let $i\colon L[E] \to \bar{W}$ be the iteration map. Let W be a normal non-dropping iterate of $\bar{W}$ which arises by making $J_\delta[E]$ generic over the image of $\mathbb{P}$, and let $j\colon \bar{W} \to W$ be the iteration map. Set

$$\pi = j \circ i\colon L[E] \to W.$$

We will have that $\pi(\delta) < \delta^{+L[E]}$.

Let $G \in V$ be $\pi(\mathbb{P})$-generic over W such that $J_\delta[E] \in W[G]$. As $\mathbb{P}$ has the δ-c.c., $\pi(\delta)$ is a cardinal in $W[G]$. Therefore, if $\mathcal{Q} \lhd L[E]$ is least such that $\pi(\delta) \leq \mathcal{Q} \cap \mathrm{OR}$ and $\rho_\omega(\mathcal{Q}) = \delta$, then $\mathcal{Q} \notin W[G]$.

We claim that if H is $\mathrm{Col}(\omega, \pi(\delta))$-generic over $V \supset W[G]$, then $\mathcal{Q} \in W[G][H]$. This gives a contradiction, because then if H_0, H_1 are mutually $\mathrm{Col}(\omega, \pi(\delta))$-generic over V, $\mathcal{Q} \in W[G][H_0] \cap W[G][H_1]$, and hence $\mathcal{Q} \in W[G]$.

Let H be $\mathrm{Col}(\omega, \pi(\delta))$-generic over V. Inside $W[G][H]$, there is a tree T searching for a countable premouse $\mathcal{R}$ together with a fully elementary embedding

$$k\colon \mathcal{R} \to \pi(\mathcal{Q})$$

such that $\mathcal{R} \rhd J_\delta[E]$, δ is a cutpoint in $\mathcal{R}$, $\pi(\delta) \leq \mathcal{R} \cap \mathrm{OR}$, $k(\delta) = \pi(\delta)$ (and therefore also $\rho_\omega(\mathcal{R}) = \delta$), and if $\pi(\delta) \in \mathcal{Q}$, then $\pi(\delta) \in \mathcal{R}$ and $k(\pi(\delta)) = \pi(\pi(\delta))$. T is ill-founded in $V[H]$, due to the existence of $\mathcal{Q}$, $\pi \restriction \mathcal{Q} \in V$. Therefore, T is ill-founded in $W[G][H]$. Let $\mathcal{R} \in W[G][H] \subset V[H]$ be given by a branch through T. We have that $L[E]$ is normally $(\omega, \delta^+ + 1)$-iterable in $V[H]$ by our hypothesis. Hence $\pi(\mathcal{Q})$ is normally $(\omega, \delta^+ + 1)$-iterable above $\pi(\delta)$ inside $V[H]$, so that the existence of k guarantees that $\mathcal{R}$ is normally $(\omega, \delta^+ + 1)$-iterable above δ inside $V[H]$ as well. A standard coiteration argument performed inside $V[H]$ thus yields that in fact $\mathcal{R} = \mathcal{Q}$. Thus $\mathcal{Q} \in W[G][H]$.

$\square$ (Lemma 1.1)

The reader will have noticed that the above proof only uses that the Woodin cardinal δ be a cutpoint in $L[E]$ (rather than that δ is the least Woodin cardinal of $L[E]$). The following readily follows from Lemma 1.1.

Corollary 1.2 *Let $L[E]$ be an extender model, and suppose that $L[E] \models$ "κ is the least supremum of infinitely many Woodin cardinal." Assume that in $V^{\mathrm{Col}(\omega,\kappa)}$, $L[E]$ is normally (ω, κ)-iterable. Then the transition point of $L[E]$ below κ does not exist.*

One ingredient in the proof of Theorem 0.2 is the following version of Woodin's extender algebra (cf. [13, section 7.2] on its "classical" version).

Lemma 1.3 *Let κ be an infinite cardinal, and let $\mathcal{M}$ be a normally $(\omega, \kappa^+ + 1)$-iterable premouse of size κ such that for some $\delta < \mathcal{M} \cap \mathrm{OR}$, $\mathcal{M} \models$ "δ is a Woodin cardinal." Then there is a poset $\mathbb{P} \subset \mathcal{M}|\delta$ which is definable over $\mathcal{M}|\delta$ and such that $\mathcal{M} \models$ "$\mathbb{P}$ has the δ-c.c.," and for every $A \subset \kappa^+$ there is a normal non-dropping iteration tree $\mathcal{U}$ on $\mathcal{M}$ with last model $\mathcal{S} = \mathcal{M}^{\mathcal{U}}_\infty$ such that $A \cap \pi^{\mathcal{U}}_{0\infty}(\delta)$ is $\pi^{\mathcal{U}}_{0\infty}(\mathbb{P})$-generic over $\mathcal{S}$.*

Proof. $\mathbb{P}$ is like the usual extender algebra, except that for each $\alpha < \delta$ we have an atomic formula "$\check{\alpha} \in \underline{A}$." Notice that $B \models \varphi$ iff $B \cap \xi \models \varphi$ whenever ξ is above the supremum of all α such that the atomic formula "$\check{\alpha} \in \underline{A}$" shows up in φ. The rest is as in [13, section 7.2].

$\square$ (Lemma 1.3)

We shall write $\mathbb{P}^{\mathcal{M}|\delta}$ for the poset of Lemma 1.3.

We shall also need the following result on the local definability of $\Vdash$. (Cf. [12, Lemma 3.6].)

Lemma 1.4 *Let $\mathcal{M} = (M; A)$ be an amenable J-model, and let $\kappa \in \mathcal{M}$ be a cardinal of $\mathcal{M}$. Let $\mathbb{P}$ be a partial order which is Σ_ω-definable over $M||\kappa$. The relation*

$$\{(p, \varphi(\tau_1, \dots, \tau_n))\colon p \in \mathbb{P}, \varphi \text{ is a } \Sigma_0\text{−formula}, \tau_1, \dots, \tau_n \in M^{\mathbb{P}},$$

$$\text{and } p \Vdash^{\mathbb{P}}_{\mathcal{M}} \varphi(\tau_1, \dots, \tau_n)\}$$

is then Δ_1-definable over $\mathcal{M}$ (in the parameter $\mathbb{P}$). Therefore, the relation

$$\{(p, \exists x_1 \dots \exists x_m \varphi(x_1, \cdots x_m, \tau_1, \dots, \tau_n))\colon\ p \in \mathbb{P}, \varphi \text{ is a } \Sigma_0\text{−formula}, \tau_1, \dots, \tau_n \in M^{\mathbb{P}},$$

$$\text{and } \exists \sigma_1 \dots \exists \sigma_m p \Vdash^{\mathbb{P}}_{\mathcal{M}} \varphi(\sigma_1, \dots \sigma_m, \tau_1, \dots, \tau_n)\}$$

is Σ_1-definable over $\mathcal{M}$ (in the parameter $\mathbb{P}$).

Proof. Let us prove the first part. For $\alpha \leq M \cap \mathrm{OR}$, let us write

$$A^\alpha = \{(p, \sigma, \tau)\colon p \in \mathbb{P}, \sigma, \tau \in S^{\mathcal{M}}_\alpha, \text{ and } p \Vdash^{\mathbb{P}}_{\mathcal{M}} \sigma = \tau\}.$$

Here, $S^{\mathcal{M}}_\alpha$ is the α^{th} model of the S-hierarchy generating $\mathcal{M}$. It is not hard to prove the following simultaneously by induction, using the recursive characterization [4, Definition 3.3 (a)] of $p \Vdash^{\mathbb{P}}_{\mathcal{M}} \sigma = \tau$.

Claim 1. Let $\alpha \leq M \cap \mathrm{OR}$ be a limit ordinal.

(a) For all ordinals $\bar{\alpha} < \alpha$, $A^{\bar{\alpha}} \in \mathcal{M}||\max(\kappa, \alpha)$.

(b) If $\alpha \leq \kappa$, then A^α is Σ_ω-definable over $\mathcal{M}||\kappa$, and if $\alpha > \kappa$, then A^α is Δ_1-definable over $\mathcal{M}||\alpha$ (in the parameter $\mathbb{P}$).

Proof of Claim 1. Let β be a limit ordinal, and let (a) and (b) be shown for all limit ordinals $\alpha < \beta$.

Let us first show (a) for β. This is trivial if β is a limit of limit ordinals. Let $\beta = \lambda + \omega$, where λ is a limit ordinal. We show $A^{\lambda+n} \in \mathcal{M}||\max(\kappa, \beta)$ for all $n < \omega$ by induction on n. Well, $A^{\lambda} \in \mathcal{M}||\max(\kappa, \beta)$ follows from (b) of the inductive hypothesis. (Notice that if $\lambda < \kappa$, then A^{λ} is a bounded subset of $\mathcal{M}||\kappa$ which is definable over $\mathcal{M}||\kappa$, by (b) of the inductive hypothesis, and is hence an element of $\mathcal{M}||\kappa$, because $\kappa \in \mathcal{M}$ is a cardinal of $\mathcal{M}$.) If $A^{\lambda+n} \in \mathcal{M}||\max(\kappa, \beta)$, then

$$(p, \sigma, \tau) \in A^{\lambda+n+1} \text{ iff } \chi(p, \sigma, \tau, A^{\lambda+n}) \wedge \chi(p, \tau, \sigma, A^{\lambda+n}),$$

where (cf. [4, Definition 3.3 (a)]) $\chi(p, \sigma, \tau, A)$ is the following Σ_0-formula:

$$\forall(\pi_1, s_1) \in \sigma \forall q' \leq q \exists q \leq q'(q \leq s_1 \rightarrow \exists(\pi_2, s_s \in \tau(q \leq s_2 \wedge (q, \pi_1, \pi_2) \in A)).$$

Therefore, $A^{\lambda+n+1}$ may be obtained from $\mathbb{P} \times S^{\mathcal{M}}_{\lambda+n+1} \times S^{\mathcal{M}}_{\lambda+n+1} \in \mathcal{M}||\max(\kappa, \beta)$ via Σ_0 comprehension.

Let us now show (b) for β. We have that $(p, \sigma, \tau) \in A^{\beta}$ iff

$$\exists \alpha \exists A \subset \mathbb{P} \times S^{\mathcal{M}}_{\alpha} \times S^{\mathcal{M}}_{\alpha}((p, \sigma, \tau) \in A \wedge \forall(q, \sigma', \tau') \in \mathbb{P} \times S^{\mathcal{M}}_{\alpha} \times S^{\mathcal{M}}_{\alpha}$$
$$((q, \sigma', \tau') \in A \Leftrightarrow \chi(q, \sigma', \tau', A) \wedge \chi(q, \tau', \sigma', A))),$$

where χ is as above. This shows that if $\beta \leq \kappa$, then A^{β} is Σ_{ω}-definable over $\mathcal{M}||\kappa$, and if $\beta > \kappa$, then A^{β} is Σ_1-definable over $\mathcal{M}||\beta$ (in the parameter $\mathbb{P}$). It is easy to see that in the latter case A^{β} is also Π_1-definable over $\mathcal{M}||\beta$ (in the parameter $\mathbb{P}$). $\square$ (Claim 1)

Now let us write, for $\alpha \leq M \cap \mathrm{OR}$,

$$B^{\alpha} = \{(p, \varphi(\tau_1, \ldots, \tau_n)) \colon p \in \mathbb{P}, \varphi \text{ is } \Sigma_0, \tau_1, \ldots, \tau_n \in S^{\mathcal{M}}_{\alpha}, \text{ and } p \Vdash^{\mathbb{P}}_{\mathcal{M}} \varphi(\tau_1, \ldots, \tau_n)\}.$$

In order to prove the following we may use the characterizations of $p \Vdash^{\mathbb{P}}_{\mathcal{M}} \sigma \in \tau$, $p \Vdash^{\mathbb{P}}_{\mathcal{M}} \neg \varphi$, and $p \Vdash^{\mathbb{P}}_{\mathcal{M}} \varphi \wedge \psi$ given in [4, Definition 3.3 (b), (c), (d)] as well as the following one: $p \Vdash^{\mathbb{P}}_{\mathcal{M}} \exists x \in \tau \varphi(x, \tau_1, \ldots, \tau_n)$ iff

$$\{q \leq p \colon \exists(\sigma, q') \in \tau(q \leq q' \wedge q \Vdash^{\mathbb{P}}_{\mathcal{M}} \varphi(\sigma, \tau_1, \ldots, \tau_n)\}$$

is dense below p.

Claim 2. Let $\alpha \leq M \cap \mathrm{OR}$ be a limit ordinal.
(a) For all ordinals $\bar{\alpha} < \alpha$, $B^{\bar{\alpha}} \in \mathcal{M}||\max(\kappa, \alpha)$.

(b) If $\alpha \leq \kappa$, then B^α is Σ_ω-definable over $\mathcal{M}||\kappa$, and if $\alpha > \kappa$, then B^α is Δ_1-definable over $\mathcal{M}$ (in the parameter $\mathbb{P}$).

PROOF of Claim 2. This is shown in much the same way as Claim 1 above, only that χ has to be replaced by a different Σ_0-formula in order to deal with the recursive characterization of $p \Vdash^{\mathbb{P}}_{\mathcal{M}} \varphi(\tau_1, \ldots, \tau_n)$ for Σ_0-formulae φ which was stated above. $\Box$ (Claim 2)

The second part is now immediate. $\Box$ (Lemma 1.4)

Our basic idea for the proof of Theorem 0.2 is now as follows. Let γ be a cardinal of $L[E]$ or $\gamma = \infty$, and let $s < \gamma$ be the strict sup of the Woodin cardinals of $L[E]$ below γ. Let Σ be the unique iteration strategy for $L[E]||\gamma$ above s. Working in $L[E]$, we shall attempt to use the extender sequence $E \restriction \gamma$ to reconstruct the $\mathcal{Q}$-structures which determine Σ on trees of size $< \gamma$. This we do by comparing those $\mathcal{Q}$-structures with $L[E]||\gamma$, in the following way.

Suppose that $\mathcal{T} \in L[E]||\gamma$ is according to Σ, has limit length, and is above s. Let $b = \Sigma(\mathcal{T})$. We shall start "comparing" $\mathcal{Q}(b, \mathcal{T})$, regarded as a mouse over $\mathcal{M}(\mathcal{T})$,[4] with $L[E]||\gamma$, regarded as a mouse over $L[E]||t$. This involves comparing mice over different sets, moreover in $L[E]$ we do not yet know $\mathcal{Q}(b, \mathcal{T})$, indeed, we are trying to find it. Both problems are solved by executing in $L[E]$ a genericity iteration of $\mathcal{M}(\mathcal{T})$ (a structure which we do know now). At each limit stage λ of the construction of $\mathcal{U}$, we shall have a $\mathbb{P}^{\mathcal{M}(\mathcal{U} \restriction \lambda)}$-generic G such that

$$\mathcal{M}(\mathcal{U} \restriction \lambda)[G] \approx L[E]||\delta(\mathcal{U} \restriction \lambda),$$

where $\approx$ denotes a certain fine structure preserving intertranslatability. Notice that $\mathcal{U}$ moves not just $\mathcal{M}(\mathcal{T})$, but also its extension $\mathcal{Q}(b, \mathcal{T})$. Let $c = \Sigma(\mathcal{U} \restriction \lambda)$, where λ is a limit ordinal. In what we call the *$\mathcal{U}$-simple* case, we have

$$\mathcal{Q}(c, \mathcal{U} \restriction \lambda)[G] \approx L[E]||\xi,$$

where $\xi < \gamma$ is the height of $\mathcal{Q}(c, \mathcal{U} \restriction \lambda)$, and this intertranslatability extends the equivalence of $\mathcal{M}(\mathcal{U} \restriction \lambda)[G]$ with $L[E]||\delta(\mathcal{U} \restriction \lambda)$. Using G and $L[E]||\xi$, we can then invert the generic extension (via, as we shall call it, the *maximal $\mathcal{P}$-construction in $L[E]$ over $\mathcal{M}(\mathcal{U} \restriction \lambda)$*) so as to *find* $\mathcal{Q}(c, \mathcal{U} \restriction \lambda)$ inside $L[E]$. We then have c and i_c in $L[E]$. Since genericity iterations do not drop, i_c acts on $\mathcal{Q}(b, \mathcal{T})$. In what we shall call the *$\mathcal{U}$-terminal* case, we have that

$$i_c(\delta(\mathcal{T})) = \delta(\mathcal{U} \restriction \lambda),$$

[4] By our tameness assumption, $\delta(\mathcal{T})$ is a cutpoint of $\mathcal{Q}(b, \mathcal{T})$.

so that

$$i_c \colon \mathcal{Q}(b, \mathcal{T}) \to \mathcal{Q}(c, \mathcal{U} \restriction \lambda)$$

and we would set $\mathrm{lh}(\mathcal{U}) = \lambda + 1$. This enables us to find $\mathcal{Q}(b, \mathcal{T})$ inside $L[E]$. (A simple Σ^1_1 absoluteness argument shows $\mathcal{Q}(b, \mathcal{T}) \in L[E][h]$, for all generic collapses h of $\delta(\mathcal{T})$.) This in turn enables us to find $b = \Sigma(\mathcal{T})$ inside $L[E]$, as we set out to do.

If $\mathcal{U} \restriction \lambda$ is simple, but not terminal, then we can use c to continue building $\mathcal{U}$.

There are two ways this process can fail to find $\Sigma(\mathcal{T})$: either (a) some $\mathcal{U} \restriction \lambda$ fails to be simple, or (b) all $\mathcal{U} \restriction \lambda$ are simple but non-terminal. If $\mathcal{U} \restriction \lambda$ is not simple, then intuitively, $L[E]||\gamma$ regarded as a mouse over $L[E]||s$ is strictly stronger than $L[E]||\gamma$ regarded as a mouse over $L[E]||t$. For simplicity, let us assume that γ is a limit cardinal in $L[E]$, so that case (b) cannot occur. (Genericity iterations cannot go on that long.) We have seen then that if $L[E]||\gamma$ is not sufficiently strong to recover its iteration strategy above s by the method above, then there is a $t > s$ such that $L[E]||\gamma$ regarded as a mouse over $L[E]||s$ is strictly stronger than $L[E]||\gamma$ regarded as a mouse over $L[E]||t$. We then give up on trying to iterate $L[E]||\gamma$ above s, and move to iterating $L[E]||\gamma$ above t. Our strength order is well-founded, so after finitely many aborted attempts, we shall find the desired transition point t. (If γ is a successor cardinal, we can deal with case (b) in a way which produces descent in the same strength order.)

This completes our sketch. We now describe the method of inverting generic extensions, and the strength order on relativised mice, which we shall use.

Let $\mathcal{M}$ be a premouse with $\omega \cdot \gamma = \mathcal{M} \cap \mathrm{OR}$, and let $\delta \in \mathcal{M}$ be a cardinal of $\mathcal{M}$. Let us further assume that δ is a cutpoint of $\mathcal{M}$. Let $\bar{\mathcal{P}}$ be of height $\delta + \omega$ such that $\bar{\mathcal{P}}|\delta \subset \mathcal{M}|\delta$ is definable over $\mathcal{M}|\delta$. Let us suppose further that $\mathcal{M}|\delta$ is $\mathbb{P}^{\bar{\mathcal{P}}|\delta}$-generic over $\bar{\mathcal{P}}$, by which we mean that $\bar{\mathcal{P}} \models$ "δ is a Woodin cardinal," and $\bar{\mathcal{P}}[G] = \mathcal{M}|(\delta{+}\omega)$ for some generic filter G. We define a sequence $(\mathcal{P}_i \colon \delta + 1 \leq i \leq \bar{\gamma})$, some $\bar{\gamma} \leq \gamma$, as follows. We'll inductively maintain that $\mathcal{P}_i \models$ "δ is a Woodin cardinal," and $\mathcal{P}_i[G] = \mathcal{M}||(\omega \cdot i)$. (Notice that this holds for $i = \delta + \omega$ by our hypotheses.)

Set $\mathcal{P}_{\delta+1} = \bar{\mathcal{P}}$. At limit stages $\lambda > \delta$, we first take the "union" of the $\mathcal{P}_i$'s for $i < \lambda$, getting a model $\bar{\mathcal{P}}_\lambda$ such that $\bar{\mathcal{P}}_\lambda \models$ "δ is a Woodin cardinal," and $\bar{\mathcal{P}}_\lambda[G] = \mathcal{M}|(\omega \cdot \lambda)$. If $\mathcal{M}||(\omega \cdot \lambda)$ is active, then $\mathcal{P}_\lambda$ is the result of expanding $\bar{\mathcal{P}}_\lambda$ by the top extender $\dot{F}^{\mathcal{M}||(\omega \cdot \lambda)} \cap \bar{\mathcal{P}}_\lambda$; if $\mathcal{M}||(\omega \cdot \lambda)$ is passive, then we set $\mathcal{P}_\lambda = \bar{\mathcal{P}}_\lambda$. If $\mathcal{P}_i$ was constructed, then we stop the construction if $i = \gamma$, or if δ is not definably Woodin over $\mathcal{P}_i$.[5] Otherwise we let $\mathcal{P}_{i+\omega}$ be obtained from $\mathcal{P}_i$ by constructing one

[5]By this we mean here and in what follows that either $\rho_\omega(\mathcal{P}_i) < \delta$, or else for some $n < \omega$ there is an $r\Sigma^{\mathcal{P}_i}_{n+1}$-definable counterexample to the Woodinness of δ.

step further.

We set $\mathcal{P} = \mathcal{P}_{\bar\gamma}$. We shall write $\mathcal{P}(\mathcal{M}, \bar{\mathcal{P}}, \delta)$ for $\mathcal{P}$. The model $\mathcal{P}(\mathcal{M}, \bar{\mathcal{P}}, \delta)$ is called the *maximal $\mathcal{P}$-construction in $\mathcal{M}$ over $\bar{\mathcal{P}}$.*

A set of computations now proves the following. (The reader may find an expanded version of the argument which is to come in the proof of [12, Theorem 3.9].)

Lemma 1.5 *Let $\mathcal{M}$, δ, $\bar{\mathcal{P}}$, G, $(\mathcal{P}_i \colon \delta + 1 \leq i \leq \bar\gamma)$, $\mathcal{P} = \mathcal{P}_{\bar\gamma} = \mathcal{P}(\mathcal{M}, \bar{\mathcal{P}}, \delta)$ be as above. Let us in addition assume that all transitive collapses of countable substructures of $\mathcal{M}$ are quasi-$(\omega, \omega_1, \omega_1 + 1)$-iterable.*

Suppose that $i \leq \bar\gamma$, $\rho_\omega(\mathcal{M}||(\omega \cdot i)) \geq \delta$, and $\mathcal{M}||(\omega \cdot i)$ is sound. (By hypothesis, this will certainly be true if $i < \gamma$.) Then $\mathcal{P}_i$ is a premouse and $\mathcal{P}_i[G] = \mathcal{M}||(\omega \cdot i)$. Moreover, if δ is definably Woodin over $\mathcal{P}_i$, then $\rho_n(\mathcal{P}_i) = \rho_n(\mathcal{M}||(\omega \cdot i))$ for every $n < \omega$, and $\mathcal{P}_i$ is sound.

Suppose that $\rho_\omega(\mathcal{M}) < \delta$ and $\mathcal{M}$ is sound above δ. (But $\mathcal{M}$ need not to be fully sound.) Then δ is not definably Woodin over $\mathcal{P}$.

Proof. First part: This is shown by induction on (i, n), ordered lexicographically. Let us fix i.

Claim 1. $\mathcal{P}_i[G] = \mathcal{M}||(\omega \cdot i)$, and $\mathcal{P}_i$ is a premouse.

Proof of Claim 1. We use the fact that $\mathcal{P}_j[G] = \mathcal{M}||(\omega \cdot j)$ for all $j < i$. The more difficult case is the one where we assume $\mathcal{M}||(\omega \cdot i)$ to have a top extender, $\dot F^{\mathcal{M}||(\omega\cdot i)}$. But we may then recover $\dot F^{\mathcal{M}||(\omega\cdot i)}$ from $\dot F^{\mathcal{M}||(\omega\cdot i)} \cap (\mathcal{P}_i|(\omega \cdot i))$ in the usual way: For $a \in [\omega \cdot i]^{<\omega}$ and $X \in \mathcal{P}(\mathrm{crit}(\dot F^{\mathcal{M}||(\omega\cdot i)})) \cap \mathcal{M}||(\omega \cdot i)$, we have that $X \in (\dot F^{\mathcal{M}||(\omega\cdot i)})_a$ iff there is some $\bar X \in (\dot F^{\mathcal{M}||(\omega\cdot i)} \cap (\mathcal{P}_i|(\omega \cdot i)))_a$ with $X \supset \bar X$. Moreover,

$$\mathrm{ult}_0(\mathcal{P}_i[G]; \dot F^{\mathcal{M}||(\omega\cdot i)}) = \mathrm{ult}_0(\mathcal{P}_i; \dot F^{\mathcal{M}||(\omega\cdot i)} \cap (\mathcal{P}_i|(\omega \cdot i)))[G],$$

and $\dot F^{\mathcal{M}||(\omega\cdot i)}$ and $\dot F^{\mathcal{M}||(\omega\cdot i)} \cap (\mathcal{P}_i|(\omega \cdot i))$ have the same generators (cf. [12, Theorem 3.9, Claims 1 and 2]). It follows that $\mathcal{P}_i[G] = \mathcal{M}||(\omega \cdot i)$ and also that $\mathcal{P}_i$ is a premouse. $\square$ (Claim 1)

Let us now prove that $\rho_n(\mathcal{P}_i) = \rho_n(\mathcal{M}||(\omega \cdot i))$ and $\mathcal{P}_i$ is n-sound for every $n < \omega$, unless δ is not definably Woodin over $\mathcal{P}_i$. We thus may and shall assume that $\rho_\omega(\mathcal{P}_i) \geq \delta$. By the proof of the second part (cf. below), we also may and shall assume that $\rho_\omega(\mathcal{M}||(\omega \cdot i)) \geq \delta$.

The case $n > 1$ is not really different from the case $n = 1$: we'd have to use the appropriate reducts of $\mathcal{P}_i$ and $\mathcal{M}||(\omega \cdot i)$ rather than these structures themselves. Let us thus assume that $n = 1$. (Cf. [12, Theorem 3.9, Claims 6,7, and 8].)

Claim 2. $\rho_1(\mathcal{M}||(\omega\cdot i)) = \rho_1(\mathcal{P}_i)$.

PROOF of Claim 2. Let us first show that $\rho_1(\mathcal{P}_i) \le \rho_1(\mathcal{M}||(\omega\cdot i))$. Suppose that $A\cap\rho_1(\mathcal{M}||(\omega\cdot i)) \notin \mathcal{M}||(\omega\cdot i)$, where A is $\Sigma_1^{\mathcal{M}||(\omega\cdot i)}(\{p_1(\mathcal{M}||(\omega\cdot i))\})$. Say $\xi\in A$ iff $\mathcal{M}||(\omega\cdot i)\models\exists x\varphi(\xi,x,p_1(\mathcal{M}||(\omega\cdot i)))$, where φ is Σ_0. Consider

$$B = \{(q,\xi)\in\mathbb{P}^{\bar{\mathcal{P}}|\delta}\times(\omega\cdot i) \ :\ \exists\sigma\ q \Vdash^{\mathbb{P}^{\bar{\mathcal{P}}|\delta}}_{\mathcal{P}_i} \varphi(\sigma,\xi,p_1(\mathcal{M}||(\omega\cdot i)))\}.$$

We cannot have that $B\cap(\mathbb{P}^{\bar{P}|\delta}\times\rho_1(\mathcal{M}||(\omega\cdot i)))\in\mathcal{P}_i$, because otherwise $A\cap\rho_1(\mathcal{M}||(\omega\cdot i))\in\mathcal{P}_i[G]=\mathcal{M}||(\omega\cdot i)$. But B is $\Sigma_1^{\mathcal{P}_i}(\{p\})$ by the local definability of $\Vdash$ (cf. Lemma 1.4). Therefore, $\rho_1(\mathcal{P}_i)\le\rho_1(\mathcal{M}||(\omega\cdot i))$, as $\mathbb{P}^{\bar{P}|\delta}$ is coded by a subset of δ.

Let us now verify that $\rho_1(\mathcal{M}||(\omega\cdot i))\le\rho_1(\mathcal{P}_i)$. We may assume that $\rho_1(\mathcal{P}_i) < \mathcal{P}_i\cap\mathrm{OR}$, as otherwise there is nothing to prove. Let $\mathcal{P}^*$ be the 1-core of $\mathcal{P}_i$, i.e.,

$$\pi\colon\mathcal{P}^*\cong\mathrm{Hull}_1^{\mathcal{P}_i}(\rho_1(\mathcal{P}_i)\cup\{p_1(\mathcal{P}_i)\})\prec_{\Sigma_1}\mathcal{P}_i,$$

where $\mathcal{P}^*$ is transitive. Setting $\bar{p}=\pi^{-1}(p_1(\mathcal{P}_i))$, there is a partial $\Sigma_1^{\mathcal{P}^*}(\{\bar{p}\})$-definable surjection $f\colon\rho_1(\mathcal{P}_i)\to\mathcal{P}^*\cap\mathrm{OR}$. Say $f(\xi)=\eta$ iff $\mathcal{P}_i\models\varphi(\xi,\eta,\bar{p})$, where φ is Σ_1. We may define $B\subset\rho_1(\mathcal{P}_i)\times\rho_1(\mathcal{P}_i)$ by setting $(\xi,\xi')\in B$ iff $f(\xi)\le f(\xi')$. As $p_1(\mathcal{P}_i)$ is 1-universal,[6] $\mathcal{P}(\rho_1(\mathcal{P}_i))\cap\mathcal{P}^*=\mathcal{P}(\rho_1(\mathcal{P}_i))\cap\mathcal{P}_i$, i.e.,

$$\rho_1(\mathcal{P}_i)^{+\mathcal{P}^*}=\rho_1(\mathcal{P}_i)^{+\mathcal{P}_i}.$$

But we also have that

$$\rho_1(\mathcal{P}_i)^{+\mathcal{P}_i}=\rho_1(\mathcal{P}_i)^{+\mathcal{M}||(\omega\cdot i)}$$

by our inductive hypothesis, so that the order type of (the transitive collapse of) B is at least $\rho_1(\mathcal{P}_i)^{+\mathcal{M}||(\omega\cdot i)}$. Notice that B is $\Sigma_1^{\mathcal{M}||(\omega\cdot i)}(\{p_1(\mathcal{P}_i),\bar{\mathcal{P}}\})$, because $(\xi,\xi')\in B$ iff

$$\mathcal{M}||(\omega\cdot i)\models\exists\gamma\ \exists\eta\ \exists\eta'\ S_\gamma^{\mathcal{P}_i}\models\varphi(\xi,\eta,p_1(\mathcal{P}_i))\wedge\varphi(\xi',\eta',p_1(\mathcal{P}_i))\wedge\eta\le\eta')$$

and the S-hierarchy producing $\mathcal{P}_i$ is $\Sigma_1^{\mathcal{M}||(\omega\cdot i)}(\{\bar{\mathcal{P}}\})$.

We claim that $B\notin\mathcal{M}||(\omega\cdot i)$.

Well, if $\rho_1(\mathcal{P}_i)^{+\mathcal{M}||(\omega\cdot i)}<\mathcal{M}||(\omega\cdot i)\cap\mathrm{OR}$, then $\mathcal{M}||\rho_1(\mathcal{P}_i)^{+\mathcal{M}||(\omega\cdot i)}\models\mathsf{ZFC}^-$, and hence if $B\in\mathcal{M}||(\omega\cdot i)$, then the order type of B would have to be less than the height of $\mathcal{M}||(\omega\cdot i)||\rho_1(\mathcal{P}_i)^{+\mathcal{M}||(\omega\cdot i)}$, i.e., less than $\rho_1(\mathcal{P}_i)^{+\mathcal{M}||(\omega\cdot i)}$. This contradiction shows $B\notin\mathcal{M}||(\omega\cdot i)$ in this case.

[6] As $\mathcal{M}||(\omega\cdot i)$ is quasi-$(\omega,\omega_1,\omega_1+1)$-iterable, $\mathcal{P}_i$ is quasi-$(\omega,\omega_1,\omega_1+1)$-iterable as well, which implies that $p_1(\mathcal{P}_i)$ is 1-universal.

On the other hand, if $\rho_1(\mathcal{P}_i)^{+\mathcal{M}||(\omega\cdot i)} = \mathcal{M}||(\omega\cdot i)\cap\mathrm{OR}$, then in fact $\mathcal{P}^* = \mathcal{P}_i$ and π is the identity by the 1-universality of $p_1(\mathcal{P}_i)$, which also gives that $B \notin \mathcal{M}||(\omega\cdot i)$.

We have verified that B witnesses $\rho_1(\mathcal{M}||(\omega \cdot i) \leq \rho_1(\mathcal{P}_i)$. □ (Claim)

Claim 3. $p_1(\mathcal{M}||(\omega \cdot i)) = p_1(\mathcal{P}_i)$.

Proof of Claim 3. The above proof of $\rho_1(\mathcal{P}_i) \leq \rho_1(\mathcal{M}||(\omega \cdot i))$ shows that $p_1(\mathcal{P}_i) \leq^* p_1(\mathcal{M}||(\omega \cdot i))$. Conversely, the above proof of $\rho_1(\mathcal{M}||(\omega \cdot i)) \leq \rho_1(\mathcal{P}_i)$ is easily seen to show that $p_1(\mathcal{M}||(\omega \cdot i)) \leq^* p_1(\mathcal{P}_i)$. □ (Claim 3).

Claim 4. $\mathcal{P}_i$ is 1-sound.

Proof of Claim 4. Let $f \in \Sigma_1^{\mathcal{M}||(\omega\cdot i)}(\{p_1(\mathcal{M}||(\omega \cdot i))\})$ be a partial function from $\rho_1(\mathcal{M}||(\omega\cdot i))$ onto $\mathcal{M}||(\omega\cdot i)$. Let $f(\xi) = x$ iff $\mathcal{M}||(\omega\cdot i) \models \exists y\, \varphi(y, \xi, x, p_1(\mathcal{M}||(\omega\cdot i)))$, where φ is Σ_0. Then g is a partial function from $\mathbb{P}^{\bar{\mathcal{P}}|\delta} \times \rho_1(\mathcal{M}||(\omega\cdot i))$ onto $\mathcal{P}_i$, if we set $g(q, \xi) = x$ iff $\exists \sigma\ q \Vdash^{\mathcal{P}_i}_{\mathcal{P}_i} \varphi(\sigma, \xi, x, p_1(\mathcal{M}||(\omega \cdot i)))$. Again, g is $\Sigma_1^{\mathcal{P}_i}(\{p_1(\mathcal{M}||(\omega\cdot i))\})$ by the local definability of $\Vdash$ (cf. Lemma 1.4). Also, because $\mathbb{P}^{\bar{\mathcal{P}}|\delta}$ is coded by a subset of δ, we get a $\Sigma_1^{\mathcal{P}_i}(\{p_1(\mathcal{M}||(\omega \cdot i))\})$-definable partial function from $\rho_1(\mathcal{M}||(\omega \cdot i)) = \rho_1(\mathcal{P}_i)$ onto $\mathcal{P}_i$. Because $p_1(\mathcal{M}||(\omega \cdot i)) = p_1(\mathcal{P}_i)$, we are done. □ (Claim 4)

Second part: Suppose that δ is definably Woodin over $\mathcal{P}$. In particular, $\bar{\gamma} = \gamma$. Also, by the proof of the first part, $\rho_n(\mathcal{P}) = \rho_n(\mathcal{M})$ as long as $\rho_n(\mathcal{M}) \geq \delta$, and $\mathcal{P}$ is sound above δ. Moreover, $\rho_n(\mathcal{P}) \leq \delta$ for the least n with $\rho_n(\mathcal{M}) < \delta$. Therefore, as we assume δ to be definably Woodin over $\mathcal{P}$, $\rho_\omega(\mathcal{P}) = \delta$, and $\mathcal{P}$ is fully sound.

Let $\mathcal{P} + \omega$ be the premouse obtained from $\mathcal{P}$ by constructing one step further. We then have that δ is a Woodin cardinal in $\mathcal{P} + \omega$, and hence $\mathcal{M}$ is $\mathbb{P}^{\bar{P}|\delta}$-generic over $\mathcal{P} + \omega$, so that $\mathcal{M} \in (\mathcal{P} + \omega)[G]$. Now let $A \subset \delta$ be bounded such that A witnesses $\rho_\omega(\mathcal{M}) < \delta$. As $\mathcal{M} \in (\mathcal{P} + \omega)[G]$, we have that $A \in (\mathcal{P} + \omega)[G]$. Let $A = \tau^G$, where $\tau \in (\mathcal{P} + \omega)^{\mathbb{P}^{\bar{\mathcal{P}}}|\delta}$. Because $\mathbb{P}^{\bar{\mathcal{P}}}|\delta$ has the δ-c.c. inside $\mathcal{P} + \omega$, we may assume $\tau \in \bar{\mathcal{P}}|\delta$. But by $\rho_\omega(\mathcal{P}) = \delta$, $\mathcal{P}$ and $\mathcal{P} + \omega$ have the same bounded subsets of δ, so that $\tau \in \mathcal{P}$ and thus $A = \tau^G \in \mathcal{P}[G]$. However, $\mathcal{P}[G] \subset \mathcal{M}$. Contradiction!

□ (Lemma 1.5)

We shall use the "$+\omega$" notation from the proof of Lemma 1.5 more often. I.e., if $\mathcal{R}$ is a sound premouse, then by $\mathcal{R} + \omega$ we mean the premouse which is obtained by constructing over $\mathcal{R}$ one step further.

We now aim to define $\mathcal{P}(\mathcal{M}, \bar{\mathcal{P}}, \delta)$ also in situations where δ may not be a cardinal of $\mathcal{M}$ or in which δ may not be a cutpoint in $\mathcal{M}$.

Let $\mathcal{M}$ be a premouse with $\omega \cdot \gamma = \mathcal{M} \cap \mathrm{OR}$, and let $\delta \in \mathcal{M}$ (not necessarily a cardinal of $\mathcal{M}$). Let $\bar{\mathcal{P}}$ be of height $\delta + \omega$ such that $\bar{\mathcal{P}}|\delta \subset \mathcal{M}|\delta$ is definable over $\mathcal{M}|\delta$, and let us suppose further that $\mathcal{M}|\delta$ is $\mathbb{P}^{\bar{\mathcal{P}}|\delta}$-generic over $\bar{\mathcal{P}}$.

Let us first still suppose that δ is a cutpoint in $\mathcal{M}$. We then define $\mathcal{P}(\mathcal{M},\bar{\mathcal{P}},\delta)$ to be $\mathcal{P}(\mathcal{M}||\theta,\bar{\mathcal{P}},\delta)$, where $\theta \leq \gamma$ is largest such that δ is a cardinal in $\mathcal{M}||\theta$. Now suppose that δ is not a cutpoint in $\mathcal{M}$. Let $\alpha \leq \gamma$ be least such that $E^{\mathcal{M}}_{\alpha} \neq \emptyset$, $\alpha \geq \delta$, and $\mathrm{crit}(F) \leq \delta$. Set $F = E^{\mathcal{M}}_{\alpha}$, and let $\zeta \leq \gamma$ be largest such that $\mathrm{crit}(F)^{+\mathcal{M}||\alpha} = \mathrm{crit}(F)^{+\mathcal{M}||\zeta}$. We then define $\mathcal{P}(\mathcal{M},\bar{\mathcal{P}},\delta)$ to be $\mathcal{P}(\mathrm{ult}_n(\mathcal{M}||\zeta,F),\bar{\mathcal{P}},\delta)$, where $n<\omega$ is least such that $\rho_{n+1}(\mathcal{M}||\zeta) \leq \mathrm{crit}(F)$. Again, the model $\mathcal{P}(\mathcal{M},\bar{\mathcal{P}},\delta)$ is called the *maximal $\mathcal{P}$-construction in $\mathcal{M}$ over $\bar{\mathcal{P}}$.*

With the help of Lemma 1.5 we may now show the following. We let $<^*$ denote the prewellordering of of mice (cf. [15, p. 181 f.]).

Lemma 1.6 *Let $\mathcal{M}$, δ, $\bar{\mathcal{P}}$, and $\mathcal{P} = \mathcal{P}_{\bar{\gamma}} = \mathcal{P}(\mathcal{M},\bar{\mathcal{P}},\delta)$ be as above. (We now allow δ to be not a cardinal of $\mathcal{M}$ or to be not a cutpoint in $\mathcal{M}$). Let us suppose that $\mathcal{M}$ is tame and (ω,OR)-iterable, and that there is a tame and (ω,OR)-iterable $\mathcal{Q} \trianglerighteq \bar{\mathcal{P}}$ such that δ is not definably Woodin over $\mathcal{Q}$.*

Each one of the following two hypotheses imply that δ is not definably Woodin over $\mathcal{P}$.

(a) δ is a cutpoint in $\mathcal{M}$, and either δ is not a cardinal in $\mathcal{M}$ or else $\rho_\omega(\mathcal{M}) < \delta$.

(b) δ is not a cutpoint in $\mathcal{M}$, and if $\alpha \leq \gamma$ is least such that $F = E^{\mathcal{M}}_{\alpha} \neq \emptyset$, $\alpha \geq \delta$, and $\mathrm{crit}(F) \leq \delta$, and if $\zeta \leq \mathcal{M}\cap\mathrm{OR}$ is largest such that $\mathrm{crit}(F)^{+\mathcal{M}||\zeta} = \mathrm{crit}(F)^{+\mathcal{M}||\alpha}$, then $\rho_\omega(\mathcal{M}||\zeta) \leq \mathrm{crit}(F)$.

Proof. If (a) holds, then we may just apply Lemma 1.5. Let us now suppose that (b) holds.

Let $\mathcal{N} = \mathrm{ult}_n(\mathcal{M}||\zeta,F)$, where $n<\omega$ is least such that $\rho_{n+1}(\mathcal{M}||\zeta) \leq \mathrm{crit}(F)$, so that $\mathcal{P}(\mathcal{M},\bar{\mathcal{P}},\delta) = \mathcal{P}(\mathcal{N},\bar{\mathcal{P}},\delta)$. Let us write $\mathcal{P}$ for $\mathcal{P}(\mathcal{N},\bar{\mathcal{P}},\delta)$. If $\mathrm{crit}(F) < \delta$, then $\rho_{n+1}(\mathcal{N}) = \rho_{n+1}(\mathcal{M}||\zeta) \leq \mathrm{crit}(F) < \delta$, so that we may again just apply Lemma 1.5. Let us thus suppose that $\mathrm{crit}(F) = \delta$. Notice that we do not assume that $\rho_\omega(\mathcal{M}||\zeta) < \delta$.

In order to derive a contradiction, let us assume that δ is definably Woodin over $\mathcal{P}$. We must then have that $\mathcal{P}\cap\mathrm{OR} = \mathcal{N}\cap\mathrm{OR}$, and $\mathcal{N}|\delta = \mathcal{M}|\delta$ is $\mathbb{P}^{\mathcal{P}|\delta}$-generic over $\mathcal{P}$, i.e., there is some $\mathbb{P}^{\mathcal{P}|\delta}$-generic G such that $\mathcal{N} = \mathcal{P}[G]$. Let $\mathcal{Q} \trianglerighteq \bar{\mathcal{P}}$ be a minimal iterable premouse such that δ is not definably Woodin over $\mathcal{Q}$, i.e., $\mathcal{Q} \models$ "δ is a Woodin cardinal," but δ is not definably Woodin over $\mathcal{Q}$. By tameness, δ is a cutpoint of $\mathcal{Q}$. Therefore, the comparison of $\mathcal{P}$, $\mathcal{Q}$ is above $\delta+1$ on both sides. As δ is definably Woodin over $\mathcal{P}$, but not in $\mathcal{Q}$, we in fact easily get that $\mathcal{P} <^* \mathcal{Q}$.

On the other hand, G is also $\mathbb{P}^{\mathcal{P}|\delta}$-generic over $\mathcal{Q}$, and we may hence compare $\mathcal{M}||\zeta$ with $\mathcal{Q}[G]$. The comparison will again be above $\delta+1$ on the $\mathcal{Q}[G]$-side, so that no iterate of $\mathcal{Q}[G]$ arising in this comparison will have an extender with critical point δ on its sequence. But $\mathcal{M}||\zeta$ does have such an extender, which easily gives that – because $\rho_\omega(\mathcal{M}||\zeta) \leq \delta$ – the final iterate of $\mathcal{M}||\zeta$ will be non-sound. This implies that $\mathcal{Q}[G] \leq^* \mathcal{M}||\zeta$, and therefore $\mathcal{Q}[G] \leq^* \mathcal{N}$.

However, we may construe the iteration trees on $\mathcal{Q}[G]$ and $\mathcal{N}$, which witness the fact that $\mathcal{Q}[G] \leq^* \mathcal{N}$, as iteration trees on $\mathcal{Q}$ and $\mathcal{P}$, respectively. These trees then witness that $\mathcal{Q} \leq^* \mathcal{P}$. But we also showed that $\mathcal{P} <^* \mathcal{Q}$. Contradiction!

$\square$ (Lemma 1.6)

The reader will gladly verify that much weaker forms of iterability will be enough for guaranteeing the conclusion of Lemma 1.6.

We may also formulate the content of Lemma 1.6 in a negative way. In the situation of Lemma 1.6, δ may end up being definably Woodin over $\mathcal{P}$ only if either

(a) δ is a cardinal in $\mathcal{M}$, $\rho_\omega(\mathcal{M}) \geq \delta$, and δ is not the critical point of an extender on the sequence of $\mathcal{M}$, or else

(b) δ is not a cardinal in $\mathcal{M}$, and there is an extender $F = E_\alpha^{\mathcal{M}}$ with $\mathrm{crit}(F) < \mathrm{crit}(F)^{+\mathcal{M}||\alpha} = \mathrm{crit}(F)^{+\mathcal{M}} \leq \delta \leq \alpha$ and $\rho_\omega(\mathcal{M}) > \mathrm{crit}(F)$.

The following order will play a crucial role in the proof of Theorem 0.2.

Definition 1.7 *Let $\mathcal{M}$ and $\mathcal{N}$ be quasi-$(\omega, \omega_1, \mathrm{OR})$-iterable premice, and let $\xi \in \mathcal{M}$ and $\{\xi', \mathcal{M}\} \subset \mathcal{N}$. Let Σ be an iteration strategy for $\mathcal{M}$, and let Σ' be an iteration strategy for $\mathcal{N}$. We write $(\mathcal{N}, \xi', \Sigma') \triangleleft^{\mathrm{gen}} (\mathcal{M}, \xi, \Sigma)$ iff the following holds true. There is an iteration tree $\mathcal{V} \in \mathcal{N}$ on $\mathcal{M}$ according to Σ and above ξ with a last model, $\mathcal{M}_\infty^{\mathcal{V}}$, and there is some $\delta \in \mathcal{M}_\infty^{\mathcal{V}}$, $\delta < \xi'$, which is a Woodin cardinal in $\mathcal{M}_\infty^{\mathcal{V}}$ such that δ is a cutpoint of $\mathcal{M}_\infty^{\mathcal{V}}$ and*

(a) $\mathcal{M}_\infty^{\mathcal{V}}|\delta$ is definable over $\mathcal{N}|\delta$,

(b) $\mathcal{N}|\delta$ is $\mathbb{P}^{\mathcal{M}_\infty^{\mathcal{V}}|\delta}$-generic over $(\mathcal{M}_\infty^{\mathcal{V}}|\delta) + \omega$,

(c) $\mathcal{M}_\infty^{\mathcal{V}} \models$ "δ is a Woodin cardinal," but δ is not definably Woodin over $\mathcal{M}_\infty^{\mathcal{V}}$,

(d) $\mathcal{P}(\mathcal{N}, \mathcal{M}_\infty^{\mathcal{V}}|\delta + \omega, \delta) <^* \mathcal{M}_\infty^{\mathcal{V}}$, and

(e) Σ', restricted to iteration trees above ξ', is induced by Σ.

We construe initial segments of iterates of $\mathcal{M}$ as iterates of $\mathcal{M}$, too. A typical $\mathcal{M}_\infty^{\mathcal{V}}$ as above may be the $\mathcal{Q}$-structure for $\mathcal{M}(\mathcal{V} \restriction \mathrm{lh}(\mathcal{V}) - 1)$, where $\mathrm{lh}(\mathcal{V}) - 1$ is a limit ordinal, in which case (d) would say that the maximal $\mathcal{P}$-construction in $\mathcal{N}$ does not reach this $\mathcal{Q}$-structure.

A comment on (e) should be in order. Suppose that $(\mathcal{N}, \xi', \Sigma') \triangleleft^{\mathrm{gen}} (\mathcal{M}, \xi, \Sigma)$ as being witnessed by $\mathcal{V}$ and δ. By (c), (d), and Lemma 1.6, we then have that either

(a) δ is a cutpoint in $\mathcal{N}$, δ is a cardinal of $\mathcal{N}$, $\rho_\omega(\mathcal{N}) \geq \delta$, and $\mathcal{P}(\mathcal{N}, \mathcal{M}_\infty^{\mathcal{V}}|\delta + \omega, \delta) \cap \mathrm{OR} = \mathcal{N} \cap \mathrm{OR}$, or else

(b) δ is not a cutpoint in $\mathcal{N}$, and if $\alpha \leq \mathcal{N} \cap \mathrm{OR}$ is least such that $F = E_\alpha^{\mathcal{N}} \neq \emptyset$, $\alpha \geq \delta$, and $\mathrm{crit}(F) \leq \delta$, then $\mathrm{crit}(F)^{+\mathcal{N}||\alpha} = \mathrm{crit}(F)^{+\mathcal{N}}$, $\rho_\omega(\mathcal{N}) > \mathrm{crit}(F)$,

$\mathcal{P}(\mathcal{N}, \mathcal{M}_\infty^{\mathcal{V}}|\delta+\omega,\delta) = \mathcal{P}(\mathrm{ult}_0(\mathcal{N},F), \mathcal{M}_\infty^{\mathcal{V}}|\delta+\omega,\delta)$ and $\mathcal{P}(\mathcal{N}, \mathcal{M}_\infty^{\mathcal{V}}|\delta+\omega,\delta) \cap \mathrm{OR} = \mathrm{ult}_0(\mathcal{N},F) \cap \mathrm{OR}$.

Let us write $\mathcal{N}^* = \mathcal{N}$ if we are in case (a), and $\mathcal{N}^* = \mathrm{ult}_0(\mathcal{N},F)$ if we are in case (b).

By (d), there is an iteration tree $\mathcal{V}'$ on $\mathcal{M}_\infty^{\mathcal{V}}$ according to Σ with last model $\mathcal{M}_\infty^{\mathcal{V}^\frown\mathcal{V}'}$ and an embedding $\sigma\colon \mathcal{P}(\mathcal{N}, \mathcal{M}_\infty^{\mathcal{V}}|\delta+\omega,\delta) \to \mathcal{M}_\infty^{\mathcal{V}^\frown\mathcal{V}'}$. An iteration strategy for $\mathcal{M}_\infty^{\mathcal{V}^\frown\mathcal{V}'}$, and hence Σ, thus induces an iteration strategy for $\mathcal{P}(\mathcal{N}, \mathcal{M}_\infty^{\mathcal{V}}|\delta+\omega,\delta)$. But any iteration tree on $\mathcal{N}^*$ may be construed as an iteration tree on $\mathcal{P}(\mathcal{N}, \mathcal{M}_\infty^{\mathcal{V}}|\delta+\omega,\delta)$, so that Σ also induces an iteration strategy for $\mathcal{N}^*$. Thus Σ also induces an iteration strategy for $\mathcal{N}$. (e) expresses that the iteration strategy which is induced that way is exacly Σ'.

Lemma 1.8 *The relation $\lhd^{\mathrm{gen}}$ is well-founded.*

Proof. Suppose that $(\mathcal{N}_{n+1}, \xi_{n+1}, \Sigma_{n+1}) \lhd^{\mathrm{gen}} (\mathcal{N}_n, \xi_n, \Sigma_n)$ for every $n < \omega$. We aim to see that there is then a degenerate Σ_0-iteration of $\mathcal{N}_0$.

Well, by $(\mathcal{N}_{n+1}, \xi_{n+1}, \Sigma_{n+1}) \lhd^{\mathrm{gen}} (\mathcal{N}_n, \xi_n, \Sigma_n)$ there is an iteration tree $\mathcal{V}_n^* = \mathcal{V}_n{}^\frown\mathcal{V}_n'$ on $\mathcal{N}_n$ above ξ_n such that both $\mathcal{V}_n$ and $\mathcal{V}_n^*$ have last models, $\mathcal{M}_\infty^{\mathcal{V}_n}$ and $\mathcal{M}_\infty^{\mathcal{V}_n^*}$, respectively, there is some $\delta_n \in \mathcal{M}_\infty^{\mathcal{V}_n}$, and there is an iteration tree $\mathcal{U}_{n+1}$ on $\mathcal{N}_{n+1}$ with a last model, $\mathcal{M}_\infty^{\mathcal{U}_{n+1}}$, such that

(a) there is a drop on the main branch of $\mathcal{V}_n'$, and $\mathcal{V}_n'$ is above $\delta_n + 1$,

(b) there is no drop on the main branch of $\mathcal{U}_{n+1}$, and $\mathcal{U}_{n+1}$ is either above δ_n+1, or else there is some (least) α such that $E_\alpha^{\mathcal{N}_{n+1}} \neq \emptyset$, $\alpha \geq \delta_n$, $\mathrm{crit}(E_\alpha^{\mathcal{N}_{n+1}}) \leq \delta_n$, and $\mathcal{U}_{n+1}$ can be written as $E_\alpha^{\mathcal{N}_{n+1}}{}^\frown\bar{\mathcal{U}}_{n+1}$, where $\bar{\mathcal{U}}_{n+1}$ is above $\delta_n + 1$,

(c) $\mathcal{N}_{n+1}|\delta_n = \mathcal{M}_\infty^{\mathcal{U}_{n+1}}|\delta_n$ is $\mathbb{P}^{\mathcal{M}_\infty^{\mathcal{V}_n^*}|\delta_n} = \mathbb{P}^{\mathcal{M}_\infty^{\mathcal{V}_n}|\delta_n}$-generic over $\mathcal{M}_\infty^{\mathcal{V}_n^*}$, and in fact

(d) $\mathcal{M}_\infty^{\mathcal{U}_{n+1}} = \mathcal{M}_\infty^{\mathcal{V}_n^*}[\mathcal{N}_{n+1}|\delta(\mathcal{V}_n)]$.

Here, $\mathcal{V}_n$ is supposed to be such that $\mathcal{P}(\mathcal{N}_{n+1}, \mathcal{M}_\infty^{\mathcal{V}_n}|\delta_n+\omega,\delta_n) <^* \mathcal{M}_\infty^{\mathcal{V}_n}$. In particular, δ_n is definably Woodin over $\mathcal{P}(\mathcal{N}_{n+1}, \mathcal{M}_\infty^{\mathcal{V}_n}|\delta_n+\omega,\delta_n)$, call it $\mathcal{P}$. The reason for (b) here is then that by Lemma 1.6, either δ is not overlapped in $\mathcal{N}_{n+1}$, $\mathcal{P} \cap \mathrm{OR} = \mathcal{N}_{n+1} \cap \mathrm{OR}$, and the iteration tree $\mathcal{U}_{n+1}$ (which is non-dropping on its main branch) arising in the comparison of $\mathcal{P}$ with $\mathcal{M}_\infty^{\mathcal{V}_n}$ may be construed as an iteration tree on $\mathcal{N}_{n+1}$, or else δ is overlapped by a total extender $E_\alpha^{\mathcal{N}_{n+1}}$, $\mathcal{P} = \mathcal{P}(\mathrm{ult}_0(\mathcal{N}_{n+1}, E_\alpha^{\mathcal{N}_{n+1}}), \mathcal{M}_\infty^{\mathcal{V}_n}|\delta_n+\omega,\delta_n)$, and the iteration tree $\bar{\mathcal{U}}_{n+1}$ (which is non-dropping on its main branch) arising in the comparison of $\mathcal{P}$ with $\mathcal{M}_\infty^{\mathcal{V}_n}$ may be construed as an iteration tree on $\mathrm{ult}_k(\mathcal{N}_{n+1}, E_\alpha^{\mathcal{N}_{n+1}})$.

This situation is summarized in figure 1.

We may now complete the diagram by copying $\mathcal{V}_1^*$, $\mathcal{V}_2^*$, etc., down to the bottom line. Notice here that we may construe the iteration tree on $\mathcal{M}_\infty^{\mathcal{U}_1} = \mathcal{M}_\infty^{\mathcal{V}_0^*}[\mathcal{N}_1|\delta_0]$, which appears in this process and which will be above δ_0, as an iteration tree on

$$
\begin{array}{ccccccc}
 & & & & & \mathcal{V}_2^* & \\
 & & & & \mathcal{N}_2 & \cdots\cdots\cdots> & \\
 & & & & \mathcal{U}_2\downarrow & & \\
 & & & & \mathcal{M}_\infty^{\mathcal{U}_2} & & \\
 & & & & \| & & \\
 & & & & \mathcal{M}_\infty^{\mathcal{V}_1^*}[\mathcal{N}_2|\delta_1] & & \\
 & & & \mathcal{V}_1^* & \bigcup & & \\
 & & \mathcal{N}_1 & \cdots\cdots\cdots> & \mathcal{M}_\infty^{\mathcal{V}_1^*} & & \\
 & & \mathcal{U}_1\downarrow & & & & \\
 & & \mathcal{M}_\infty^{\mathcal{U}_1} & & & & \\
 & & \| & & & & \\
 & & \mathcal{M}_\infty^{\mathcal{V}_0^*}[\mathcal{N}_1|\delta_0] & & & & \\
 & \mathcal{V}_0^* & \bigcup & & & & \\
\mathcal{N}_0 & \cdots\cdots\cdots> & \mathcal{M}_\infty^{\mathcal{V}_0^*} & & & &
\end{array}
$$

Figure 1: An alleged ill-foundedness of $\lhd^{\rm gen}$.

$\mathcal{M}_\infty^{\mathcal{V}_0^*}$, etc. The iteration tree on $\mathcal{N}_0$ along the bottom line which is produced in this fashion will be according to Σ_0, because of (e) in Definition 1.7. But obviously said iteration tree will be degenerate. This is a contradiction!

$\square$ (Lemma 1.8)

Let us now start the proof of Theorem 0.2. Let us fix $L[E]$, a quasi-$(\omega, \omega_1, \mathrm{OR})$-iterable tame extender model. Let Σ denote the iteration strategy for $L[E]$. In what follows we shall write K for $L[E]$, with the idea that from the point of view of $L[E]$, the universe behaves somewhat like a core model. We shall first define, inside K, a (partial) iteration strategy Σ^K.

Let γ be either a cardinal of K, or else $\gamma = \infty$. We assume that $J_\gamma[E] = K||\gamma \models$ "there are only boundedly many Woodin cardinals." Set $s = \sup\{\delta < \gamma\colon \delta$ is Woodin in $K\}$, if there is a Woodin cardinal in K below γ; otherwise set $s = 0$. By hypothesis, $s < \gamma$. We first aim to find some $t < \gamma$, $t \geq s$, such that $K \models$ "$K||\gamma$ is γ-iterable above t." We shall then argue that there is in fact such a t which is a cutpoint in K.

Let $\mathcal{T} \in K$ be an iteration tree on $K||\gamma$ of limit length $< \gamma$ which is above s. As we aim to verify, inside K, quasi-$(\omega, \omega_1, \gamma)$-iterability of $K||\gamma$ above some t, we may and shall assume that if $\mathcal{T}$ is according to Σ, then in V, $\Sigma(\mathcal{T})$ is characterized as the unique cofinal branch b through $\mathcal{T}$ such that $\mathcal{M}(\mathcal{T}) \trianglelefteq \mathcal{Q}(\mathcal{T}) = \mathcal{Q}(b, \mathcal{T}) \trianglelefteq \mathcal{M}_b^{\mathcal{T}}$. Moreover, if K is able to see $\mathcal{Q}(\mathcal{T})$, then K can also identify b. Let us thus assume without loss of generality that $\delta(\mathcal{T})$ is Woodin in $\mathcal{M}(\mathcal{T}) + \omega$, because otherwise the identification of b is trivial.

We'll describe a tree $\mathcal{U} = \mathcal{U}(\mathcal{T})$ arising in the attempt to find $\mathcal{Q}(\mathcal{T})$. We'll actually use this procedure to define $\Sigma^K(\mathcal{T})$. $\mathcal{U}$ will be a normal non-dropping iteration tree on $\mathcal{M}(\mathcal{T}) + \omega$. If $\mathcal{T}$ is according to Σ, then we'll see that $\Sigma^K(\mathcal{T}) = \Sigma(\mathcal{T})$, unless $\Sigma(\mathcal{T}) \uparrow$. However, $\Sigma^K(\mathcal{T})$ may indeed remain undefined.

Set $s^* = \sup\{\delta < \delta(\mathcal{T}) \colon \delta \text{ is Woodin in } \mathcal{M}(\mathcal{T})\}$, if there is a Woodin cardinal in $\mathcal{M}(\mathcal{T})$; otherwise set $s^* = 0$. Notice that because K is tame, and as we assume $\delta(\mathcal{T})$ to be Woodin in $\mathcal{M}(\mathcal{T}) + \omega$, we'll have that $s^* < \delta(\mathcal{T})$. The tree $\mathcal{U}$ we are about to construct will be above s^*.

For a moment, let us consider any normal non-dropping iteration tree $\mathcal{U}$ on $\mathcal{M}(\mathcal{T}) + \omega$ which has a last model, $\mathcal{M}_\infty^{\mathcal{U}}$. Let

$$\pi_{0\infty}^{\mathcal{U}} \colon \mathcal{M}(\mathcal{T}) + \omega \to \mathcal{M}_\infty^{\mathcal{U}}$$

be the iteration map. Let us also suppose that $\mathcal{T}$ is according to Σ. Then the $\mathcal{Q}$-structure $\mathcal{Q}(\mathcal{T})$ exists, and we may construe $\mathcal{U}$ to act on all of $\mathcal{Q}(\mathcal{T})$ in the following way. Let $n < \omega$ be least such that either $\rho_{n+1}(\mathcal{Q}(\mathcal{T})) < \delta(\mathcal{T})$, or else there is an $r\Sigma_{n+1}^{\mathcal{Q}(\mathcal{T})}$-definable counterexample to the Woodinness of $\delta(\mathcal{T})$. There is then an $r\Sigma_{n+1}$-elementary map

$$\pi \colon \mathcal{Q}(\mathcal{T}) \to \mathcal{Q}^*$$

such that $\pi \supset \pi_{0\infty}^{\mathcal{U}}$, π is continuous at $\delta(\mathcal{T})$, and $\mathcal{Q}^* \trianglerighteq \mathcal{M}_\infty^{\mathcal{U}}$ is a premouse such that either $\rho_{n+1}(\mathcal{Q}^*) < \pi(\delta(\mathcal{T}))$, or else there is an $r\Sigma_{n+1}^{\mathcal{Q}^*}$-definable counterexample to the Woodinness of $\pi(\delta(\mathcal{T}))$. π is obtained by letting $\mathcal{U}$ act on $\mathcal{Q}(\mathcal{T})$, where we restrict the iteration maps to be just $r\Sigma_{n+1}$-elementary (even if we could afford more).

The construction of $\mathcal{U}$ will take place inside K and is as follows. The tree $\mathcal{U}$ attempts to make $K||\gamma$ generic over $\mathcal{M}(\mathcal{T})$ in the sense of Lemma 1.3. At limit steps λ of the construction of $\mathcal{U}$ we will use the maximal $\mathcal{P}$-construction in K over $\mathcal{M}(\mathcal{U} \restriction \lambda)$ to search for $\mathcal{Q}(\mathcal{U} \restriction \lambda)$. (Once this search is not successful, we stop the construction of $\mathcal{U}$, and we give up on trying to define $\Sigma^K(\mathcal{T})$.) $\mathcal{P}(K, \mathcal{M}(\mathcal{U} \restriction \lambda) + \omega, \delta(\mathcal{U} \restriction \lambda))$ will inherit the iterability-above-$\delta(\mathcal{U} \restriction \lambda)$ from K, so that indeed the maximal $\mathcal{P}$-construction in K over $\mathcal{M}(\mathcal{U} \restriction \lambda)$ is a good candidate for the $\mathcal{Q}$-structure for $\mathcal{U} \restriction \lambda$. If $\mathcal{M}_\zeta^{\mathcal{U}}$ is defined, then it may well be that $\mathcal{U} \restriction (\zeta + 1)$ is "maximal," i.e., $K|\pi_{0\zeta}^{\mathcal{U}}(\delta(\mathcal{T}))$ is $\mathbb{P}^{\mathcal{M}_\zeta^{\mathcal{U}}|\pi_{0\zeta}^{\mathcal{U}}(\delta(\mathcal{T}))}$-generic over $\mathcal{M}_\zeta^{\mathcal{U}}$. In this case, we would set $\zeta + 1 = \mathrm{lh}(\mathcal{U})$. If $\pi \supset \pi^{\mathcal{U}}_{0\zeta}$ is then as in the previous paragraph, where

$$\pi \colon \mathcal{Q}(\mathcal{T}) \to \mathcal{Q}^*,$$

then we might search for $\mathcal{Q}^*$ via the maximal $\mathcal{P}$-construction in K over $\mathcal{M}^{\mathcal{U}}_\zeta$, and we might then search for $\mathcal{Q}(\mathcal{T})$ via searching for an $r\Sigma_{n+1}$-elementary embedding from some $\bar{\mathcal{Q}}$ which end-extends $\mathcal{M}(\mathcal{T})$ into $\mathcal{Q}^*$. Notice that if $\mathcal{P}(K, \mathcal{M}^{\mathcal{U}}_\zeta, \pi^{\mathcal{U}}_{0\zeta}(\delta(\mathcal{T})))$ defines a counterexample to the Woodinness of $\pi^{\mathcal{U}}_{0\zeta}(\delta(\mathcal{T}))$, then $\mathcal{P}(K, \mathcal{M}^{\mathcal{U}}_\zeta, \pi^{\mathcal{U}}_{0\zeta}(\delta(\mathcal{T}))) = \mathcal{Q}^* \in K$, so that by absoluteness and uniqueness $\mathcal{Q}(\mathcal{T})$ (and hence $b = \Sigma(\mathcal{T})$) will be in some $K^{\mathrm{Col}(\omega,\epsilon)}$, and therefore in K as well.

The details of the construction of $\mathcal{U}$ are as follows. Suppose that $\mathcal{U} \restriction \zeta$ has been constructed. Let us assume first that ζ is a successor ordinal, $\zeta = \bar{\zeta} + 1$. Set

$$\mathbb{P} = \mathbb{P}^{\mathcal{M}^{\mathcal{U}}_{\bar{\zeta}} | \pi^{\mathcal{U}}_{0\zeta}(\delta(\mathcal{T}))}.$$

If every total extender from $\mathcal{M}^{\mathcal{U}}_{\bar{\zeta}}$ satisfies all the axioms correspondig to $\mathbb{P}$, then we stop the construction of $\mathcal{U}$, i.e., we set $\mathrm{lh}(\mathcal{U}) = \zeta$, $\mathcal{U} = \mathcal{U} \restriction \zeta$. In the latter case we will have that $K|\pi^{\mathcal{U}}_{0\zeta}(\delta(\mathcal{T}))$ is $\mathbb{P}$-generic over $\mathcal{M}^{\mathcal{U}}_{\bar{\zeta}}$. Set

$$\mathcal{P} = \mathcal{P}(K, \mathcal{M}^{\mathcal{U}}_\zeta || \pi^{\mathcal{U}}_{0\zeta}(\delta(\mathcal{T})) + \omega, \pi^{\mathcal{U}}_{0\zeta}(\delta(\mathcal{T}))).$$

If $\mathcal{P}$ is no $\mathcal{Q}$-structure for $\mathcal{M}^{\mathcal{U}}_\zeta || \pi^{\mathcal{U}}_{0\zeta}(\delta(\mathcal{T}))$, then $\Sigma^K(\mathcal{T}) \uparrow$. Otherwise we are in the $\mathcal{U}$-terminal case. Then let $\epsilon = \mathcal{P} \cap \mathrm{OR}$. In $K^{\mathrm{Col}(\omega,\epsilon)}$, there will be a unique pair $(\mathcal{Q}, \pi)$ such that $\pi \colon \mathcal{Q} \to \mathcal{P}$ is $r\Sigma_{n+1}$-elementary and $\mathcal{M}(\mathcal{T}) \trianglelefteq \mathcal{Q}$. But then $\mathcal{Q} = \mathcal{Q}(\mathcal{T}) \in K$, and hence $b = \Sigma(\mathcal{T}) \in K$ as well. We may then define $\Sigma^K(\mathcal{T})$ inside K as the unique cofinal hbranch through $\mathcal{T}$ which is obtained in this fashion.

Now suppose that not every total extender from $\mathcal{M}^{\mathcal{U}}_{\bar{\zeta}}$ satisfies all the axioms correspondig to $\mathbb{P}$. If $\bar{\zeta} = \gamma$, then we stop the construction of $\mathcal{U}$, i.e., we set $\mathrm{lh}(\mathcal{U}) = \gamma + 1$, $\mathcal{U} = \mathcal{U} \restriction \gamma + 1$. In this case, by decree, $\Sigma^K(\mathcal{T}) \uparrow$. Otherwise, i.e., if $\bar{\gamma} < \gamma$, we proceed as follows.

Set

$$\eta = \sup(\{\mathcal{Q}(\mathcal{U} \restriction \lambda) \cap \mathrm{OR} : \ \lambda \leq \bar{\zeta} \text{ is a limit ordinal } \}).$$

(Notice that we can compute this ordinal inside $L[E]$, and $\eta < \gamma$.) Let F be defined as follows. If $\bar{\zeta}$ is a limit ordinal, or if otherwise $\mathrm{lh}(E^{\mathcal{U}}_{\bar{\zeta}-1}) \geq \eta$, then we let F be the least total extender G from $\mathcal{M}^{\mathcal{U}}_{\bar{\zeta}}$ with $\mathrm{crit}(G) > s^*$ which does not satisfy the relevant axiom correspondig to $\mathbb{P}$. If $\bar{\zeta}$ is a successor ordinal and $\mathrm{lh}(E^{\mathcal{U}}_{\bar{\zeta}-1}) < \eta$, then we let F be the least total extender G from $\mathcal{M}^{\mathcal{U}}_{\bar{\zeta}}$ with $\mathrm{lh}(G) > \mathrm{lh}(E^{\mathcal{U}}_{\bar{\zeta}-1})$. We then define $\mathcal{U} \restriction \zeta + 1$ in the natural way, i.e., so that $\mathcal{U} \restriction \zeta + 1$ is a normal extension of $\mathcal{U} \restriction \zeta$, where $E^{\mathcal{U}}_{\bar{\zeta}} = F$.

Notice that if $\bar{\zeta}$ is a successor ordinal and $\mathrm{lh}(E^{\mathcal{U}}_{\bar{\zeta}-1}) < \eta$, then we may indeed apply an extender which does satisfy the relevant axiom corresponding to $\mathbb{P}$. To do so will produce *linear* segments of $\mathcal{U}$ which guarantee that at the next limit ordinal λ where finding the $\mathcal{Q}$-structure for $\mathcal{U} \upharpoonright \lambda$ is non-trivial (i.e., where $\delta(\mathcal{U} \upharpoonright \lambda)$ is a Woodin cardinal in $\mathcal{M}(\mathcal{U} \upharpoonright \lambda) + \omega$) we shall have that actually $\mathcal{U} \upharpoonright \lambda$ (and hence $\mathcal{M}(\mathcal{U} \upharpoonright \lambda)$) is definable over $L[E]||\delta(\mathcal{U} \upharpoonright \lambda)$, so that we may start the $\mathcal{P}$-construction over $\mathcal{M}(\mathcal{U} \upharpoonright \lambda)$ inside $L[E]$. Notice also that we only start these linear segments at successor ordinals, i.e., on a non-stationary set.

Now suppose that $\zeta \leq \gamma$ is a limit ordinal. If $\delta(\mathcal{U} \upharpoonright \zeta)$ is not a Woodin cardinal in $\mathcal{M}(\mathcal{U} \upharpoonright \zeta) + \omega$, then $\mathcal{Q}(\mathcal{U} \upharpoonright \zeta) = \mathcal{M}(\mathcal{U} \upharpoonright \zeta)$, and we define $\mathcal{U} \upharpoonright \zeta + 1$ as that extension of $\mathcal{U} \upharpoonright \zeta$ where $[0, \zeta)_{\mathcal{U}}$ is given by $\mathcal{M}(\mathcal{U} \upharpoonright \zeta)$. Notice that in this case $\pi^{\mathcal{U}}_{0\zeta}(\delta(\mathcal{T})) > \delta(\mathcal{U})$. Let us now finally assume that $\delta(\mathcal{U} \upharpoonright \zeta)$ is in fact a Woodin cardinal in $\mathcal{M}(\mathcal{U} \upharpoonright \zeta) + \omega$. Set

$$\mathbb{P} = \mathbb{P}^{\mathcal{M}(\mathcal{U} \upharpoonright \zeta)}.$$

By the construction of $\mathcal{U} \upharpoonright \zeta$, $K|\delta(\mathcal{U} \upharpoonright \zeta)$ is then $\mathbb{P}$-generic over $\mathcal{M}(\mathcal{U} \upharpoonright \zeta) + \omega$. Moreover, we must have that

$$\sup(\{\mathcal{Q}(\mathcal{U} \upharpoonright \lambda) \cap \mathrm{OR} : \lambda < \zeta \text{ is a limit ordinal }\}) < \delta(\mathcal{U} \upharpoonright \zeta)$$

by the construction of $\mathcal{U} \upharpoonright \zeta$, so that the entire construction of $\mathcal{U} \upharpoonright \zeta$ took place in $K|\delta(\mathcal{U} \upharpoonright \zeta)$ and hence $\mathcal{U} \upharpoonright \zeta$ is definable over $K|\delta(\mathcal{U} \upharpoonright \zeta)$. We may thus perform the maximal $\mathcal{P}$-construction in $L[E]$ over $\mathcal{M}(\mathcal{U} \upharpoonright \zeta)$.

If $\mathcal{P} = \mathcal{P}(K, \mathcal{M}(\mathcal{U} \upharpoonright \zeta) + \omega, \delta(\mathcal{U} \upharpoonright \zeta))$ is no $\mathcal{Q}$-structure for $\mathcal{U} \upharpoonright \zeta$, then we stop the construction of $\mathcal{U}$, i.e., we set $\mathrm{lh}(\mathcal{U}) = \zeta$, $\mathcal{U} = \mathcal{U} \upharpoonright \zeta$, and we also decide to have $\Sigma^K(\mathcal{T}) \uparrow$. Let us thus assume that $\mathcal{P}$ is indeed a $\mathcal{Q}$-structure for $\mathcal{U} \upharpoonright \zeta$. This is the $\mathcal{U}$-simple case. We will then define $\mathcal{U} \upharpoonright \zeta + 1$ as that extension of $\mathcal{U} \upharpoonright \zeta$ where $[0, \zeta)_{\mathcal{U}}$ is given by $\mathcal{P}$.

This finishes the definition of $\Sigma^K(\mathcal{T})$. Notice that the map $\mathcal{T} \mapsto \Sigma^K(\mathcal{T})$ is in K, and that if $\mathcal{T}$ is according to Σ and $\Sigma^K(\mathcal{T}) \downarrow$, then $\Sigma^K(\mathcal{T}) = \Sigma(\mathcal{T})$. In particular, if $\mathcal{T} \in K$ is according to Σ^K, then $\mathcal{T}$ is also according to Σ.

A moment of reflection shows that the tree $\mathcal{U}(\mathcal{T})$ arising in the attempt to find $\mathcal{Q}(\mathcal{T})$ cannot have length $\geq \delta(\mathcal{T})^{+K} + 1$, because otherwise the usual reflection argument would show that one of the extenders $E^{\mathcal{U}}_{\alpha}$ would satisfy all the relevant axioms after all. (Notice that by the rules for forming $\mathcal{U}$, we apply extenders which do satisfy the relevant axioms only on a non-stationary set, cf. above.) Therefore $\mathrm{lh}(\mathcal{U}(\mathcal{T})) \leq \gamma$.

Suppose now that there is no $\xi < \gamma$, $\xi > s$, such that Σ^K is total with respect to trees $\mathcal{T}$ on $K|\gamma$ which are above ξ, which are according to Σ, and which are such that for all limit ordinals $\lambda \leq \mathrm{lh}(\mathcal{T})$, $\Sigma(\mathcal{T} \upharpoonright \lambda)$ is the unique b such that $\mathcal{M}(\mathcal{T} \upharpoonright \lambda) \trianglelefteq \mathcal{Q}(\mathcal{M}(\mathcal{T} \upharpoonright \lambda)) \trianglelefteq \mathcal{M}^{\mathcal{T} \upharpoonright \lambda}_b$.

Let us first consider the case where γ is a successor cardinal of K, say $\gamma = \bar{\gamma}^{+K}$. We are then going to define a sequence $(\xi_n, \eta_n, \mathcal{T}_n, \mathcal{U}_n \colon n < \omega)$ such that for every $n < \omega$, $\bar{\gamma} < \xi_n < \eta_n$, $\mathcal{T}_n$ is a normal tree of limit length on $K||\eta_n$ above ξ_n according to Σ^K such that $\Sigma^K(\mathcal{T}_n)$ is not defined, $\mathcal{U}_n$ is the normal non-dropping tree on $\mathcal{M}(\mathcal{T}_n) + \omega$ arising in the attempt to find $\Sigma(\mathcal{T}_n)$, and

$$(K||\eta_{n+1}, \xi_{n+1}, \Sigma) \lhd^{\text{gen}} (K||\eta_n, \xi_n, \Sigma).$$

This last clause yields a contradiction.

Let ξ_n, η_n, $\mathcal{T}_n$, $\mathcal{U}_n$ be given. We may then pick $\xi_{n+1} > \eta_n$ such that $\mathcal{T}_n \in K|\xi_{n+1}$. Let $\eta_{n+1} > \xi_{n+1}$ be such that there is a normal tree of limit length, call it $\mathcal{T}_{n+1}$, on $K||\eta_{n+1}$ above ξ_{n+1} according to Σ^K such that $\Sigma^K(\mathcal{T}_{n+1})$ is not defined. We may and shall assume that $\mathcal{T}_{n+1}$ does not start with a drop; hence $K||\eta_{n+1}$ has a measurable cardinal, namely $\text{crit}(E_0^{\mathcal{T}_{n+1}}) \geq \xi_{n+1}$, and hence $\delta(\mathcal{T}_n)^{+K||\eta_{n+1}}$ exists. Set $\alpha = \delta(\mathcal{T}_n)^{+K||\eta_{n+1}}$. If $\text{lh}(\mathcal{U}_n) < \alpha$, then $\mathcal{P}(K||\eta_{n+1}, \mathcal{M}(\mathcal{U}_n) + \omega, \delta(\mathcal{U}_n)) \trianglelefteq \mathcal{P}(K, \mathcal{M}(\mathcal{U}_n) + \omega, \delta(\mathcal{U}_n)) <^* \mathcal{Q}(\mathcal{U}_n) \trianglelefteq \mathcal{M}^{\mathcal{U}_n}_{\Sigma(\mathcal{U}_n)}$, which shows that $(K||\eta_{n+1}, \xi_{n+1}, \Sigma) \lhd^{\text{gen}} (K||\eta_n, \xi_n, \Sigma)$.[7] Now suppose that $\text{lh}(\mathcal{U}_n) \geq \alpha$. By Lemma 1.5, for each $\lambda < \alpha$ we have that $\mathcal{P}(K|\alpha, \mathcal{M}(\mathcal{U}_n \upharpoonright \lambda) + \omega, \delta(\mathcal{U}_n \upharpoonright \lambda)) = \mathcal{P}(K|\xi_{n+1}, \mathcal{M}(\mathcal{U}_n \upharpoonright \lambda) + \omega, \delta(\mathcal{U}_n \upharpoonright \lambda))$. We therefore must have some $\lambda \leq \alpha$ such that $\mathcal{P}(K||\eta_{n+1}, \mathcal{M}(\mathcal{U}_n \upharpoonright \lambda) + \omega, \delta(\mathcal{U}_n \upharpoonright \lambda)) <^* \mathcal{Q}(\mathcal{U}_n \upharpoonright \lambda)$, because otherwise $\mathcal{U}_n \upharpoonright \alpha \subset K|\alpha$, $\mathcal{U}_n \upharpoonright \alpha$ is definable over $K|\alpha$, $\mathcal{U}_n \upharpoonright (\alpha + 1) \in K||\eta_{n+1}$, and the usual reflection argument would show that one of the extenders $E^{\mathcal{U}}_\beta$, $\beta < \alpha$, would satisfy all the relevant axioms after all. This again shows that $(K||\eta_{n+1}, \xi_{n+1}, \Sigma) \lhd^{\text{gen}} (K||\eta_n, \xi_n, \Sigma)$.

Let us now suppose that γ is a limit cardinal of K. We are then going to define an increasing sequence $(\xi_n \colon n < \omega)$ of ordinals below γ such that

$$(K||\gamma, \xi_{n+1}, \Sigma) \lhd^{\text{gen}} (K||\gamma, \xi_n, \Sigma)$$

for all $n < \omega$. This again gives a contradiction.

Set $\xi_0 = 0$. Suppose that ξ_n has been chosen. Let $\mathcal{T}$ on $K||\gamma$ be of length $< \gamma$ and according to Σ^K such that $\Sigma^K(\mathcal{T}) \uparrow$. Let $\mathcal{U}$ be the tree arising in the attempt to find $\Sigma(\mathcal{T})$. We know that $\text{lh}(\mathcal{U}) \leq \delta(\mathcal{T})^{+K} < \gamma$. We then either have that $\mathcal{P}(K||\gamma, \mathcal{M}(\mathcal{U}) + \omega, \delta(\mathcal{U})) <^* \mathcal{Q}(\mathcal{U})$, or else $\mathcal{P}(K||\gamma, \mathcal{M}(\mathcal{U}) + \omega, \delta(\mathcal{U})) = \mathcal{Q}(\mathcal{U})$, $K|\pi^{\mathcal{U}}_{0\infty}(\delta(\mathcal{T}))$ is $\mathbb{P}^{\mathcal{M}^{\mathcal{U}}_\infty|\pi^{\mathcal{U}}_{0\infty}(\delta(\mathcal{T}))}$-generic over $\mathcal{M}^{\mathcal{U}}_\infty$, but $\mathcal{P}(K||\gamma, \mathcal{M}^{\mathcal{U}}_\infty + \omega, \pi^{\mathcal{U}}_{0\infty}(\delta(\mathcal{T}))) <^* \mathcal{M}^{\mathcal{T}^\frown\mathcal{U}}_\infty$. In both cases, we get that $(K||\gamma, \xi, \Sigma) \lhd^{\text{gen}} (K||\gamma, \xi_n, \Sigma)$ whenever $\xi > \delta(\mathcal{T})^{+K}$. We may thus simply set $\xi_{n+1} = \delta(\mathcal{T})^{+K} + 1$.

[7] Notice that clause (e) in Definition 1.7 is given by the fact that Σ, restricted to iteration trees $\mathcal{T}$ such that for all limit ordinals $\lambda \leq \text{lh}(\mathcal{T})$, $\Sigma(\mathcal{T} \upharpoonright \lambda)$ is characterized as the unique cofinal branch b through $\mathcal{T} \upharpoonright \lambda$ such that $\mathcal{M}(\mathcal{T} \upharpoonright \lambda) \trianglelefteq \mathcal{Q}(\mathcal{T} \upharpoonright \lambda) \trianglelefteq \mathcal{M}^{\mathcal{T} \upharpoonright \lambda}_b$, is in fact *unique*.

We have shown that if γ is either a cardinal of K or else $\gamma = \infty$, and if $J_\gamma[E] \models$ "there are only boundedly many Woodin cardinals," then there is some $t < \gamma$ such that $K \models$ "$K|\gamma$ is γ-iterable above t." We now want to see that we may choose t as a cutpoint of K. Well, if there is no $\mu < \gamma$ such that μ is $< \gamma$-strong in K, then this is trivial. So let us assume that $\mu < \gamma$ is least such that μ is $< \gamma$-strong in K.

Let $\bar{t} < \mu$ be such that $K \models$ "$K|\mu$ is μ-iterable above $\bar{t}$." Notice that by the choice of μ, we may assume that $\bar{t}$ is a cutpoint in K. We claim that $K \models$ "$K|\gamma$ is γ-iterable above $\bar{t}$." To see this, let $\mathcal{T}$ be a tree on $K|\gamma$ of length $< \gamma$. We have $\mathcal{T} \in K|\nu$ for some $\nu < \gamma$ such that $E = E^K_\nu$ is an extender with $\mathrm{crit}(E) = \mu$. We may find $\Sigma^{\mathrm{ult}(K;E)}(\mathcal{T})$ in $\mathrm{ult}(K;E) \subset K$. We must have $\Sigma^{\mathrm{ult}(K;E)}(\mathcal{T}) = \Sigma(\mathcal{T})$. The fact that $K \models$ "$K|\gamma$ is γ-iterable above $\bar{t}$" is therefore witnessed by taking the "amalgamation" (i.e., formally, the union) of all $\Sigma^{\mathrm{ult}(K;E)}$, restricted to iteration trees $\mathcal{T} \in K|\nu$ above $\bar{t}$, where $E = E^K_\nu$ is an extender with $\mathrm{crit}(E) = \mu$.

This finishes the proof of Theorem 0.2.

$\square$ (Theorem 0.2)

Definition 1.9 *Let $L[E]$ be an extender model. Let γ be either a cardinal of $L[E]$, or else $\gamma = \infty$. We shall write $t(\gamma) = t^{L[E]}(\gamma)$ for the transition point of $L[E]$ below γ (if it exists; otherwise we let $t(\gamma) = t^{L[E]}(\gamma)\uparrow$).*

A consequence of Theorem 0.2 is the following.

Lemma 1.10 *Let $L[E]$ be tame and quasi-$(\omega, \omega_1, \mathrm{OR})$-iterable, and write $K = L[E]$. Let κ be a cardinal of K. Suppose that either κ is a limit cardinal of K, or else there is some μ below the predecessor of κ in K such that μ is $< \kappa$-strong in K. Let $t = t^K(\kappa)$. Then inside K, K is $(\omega, \omega_1, \kappa)$-iterable above t.*

Proof. Let $\mathcal{T} \in K$ with tree order T be an iteration tree on K above t of length $\bar{\kappa} < \kappa$. We may construe $\mathcal{T}$ as being a tree on $K||\eta$ for some η. Working inside K, we may let $\pi\colon \bar{K} \to K||\eta$ be such that $\bar{K}$ is transitive, $\mathrm{Card}^K(\bar{K}) = \mathrm{Card}^K(\bar{\kappa}) < \kappa$, and $T \in \bar{K}|\mu = K|\mu$, where $\mu > \mathrm{Card}^K(\bar{\kappa})$ is the critical point of π. We may and shall also assume that $t < \mu$. The tree order T induces an iteration tree $\bar{\mathcal{T}}$ on $\bar{K}$.

Let us now first assume that κ is a limit cardinal. Then inside K, $K||\bar{\kappa}^{+K}$ is $\bar{\kappa}^{+K}+1$ iterable above t. There is hence an iterate $\mathcal{M}^{\mathcal{U}}_\infty$ of $K||\bar{\kappa}^{+K}$ with $\mathrm{lh}(\mathcal{U}) < \bar{\kappa}^{+K}$ and some $\sigma\colon \bar{K} \to \mathcal{M}^{\mathcal{U}}_\infty$. Let us copy $\bar{\mathcal{T}}$ onto $\mathcal{M}^{\mathcal{U}}_\infty$, using σ, which gives $\sigma\bar{\mathcal{T}}$. Inside K, the iteration strategy for $K||\bar{\kappa}^{+K}$ gives us a maximal branch b through $\sigma\bar{\mathcal{T}}$. b is also a maximal branch through $\bar{\mathcal{T}}$ as well as through $\mathcal{T}$.

But we must now have that $b = \Sigma(\mathcal{T})$, because otherwise there would be two different cofinal branches through $\bar{\mathcal{T}} \restriction \sup(b)$ which are according to the true iteration strategy.

Now suppose that $\kappa = \lambda^{+K}$, and let $\mu < \lambda$ be least such that μ is $< \kappa$-strong in K. Let $\bar{t} < \mu$ be the transition point of K below μ. Let us pick $E = E^K_\nu$ with $\mathrm{crit}(E) = \mu$ and $\nu < \kappa$ such that $\bar{K}$, $\bar{\mathcal{T}} \in K|\nu$. Of course, $\bar{t} < t$ is the transition point of $\mathrm{ult}(K; E)$ below $i_E(\mu)$. But then the above argument yields $\Sigma(\mathcal{T})$ inside $\mathrm{ult}(K; E) \subset K$.

We have shown that we may define $\mathcal{T} \mapsto b = \Sigma(\mathcal{T})$ for trees $\mathcal{T}$ above t and of length $< \kappa$ inside K.

$\square$ (Lemma 1.10)

We do not know if Lemma 1.10 holds true if $\kappa = \lambda^{+K}$ and there is a cutpoint in K in the half open interval $[\lambda, \kappa)$.

Definition 1.11 *Let $\mathcal{M}$ be a premouse, and let $\delta \in \mathcal{M}$. Then $\mathcal{M}$ is called* countably iterable *above δ iff for all sufficiently elementary $\pi \colon \bar{\mathcal{M}} \to \mathcal{M}$ with $\delta \in \mathrm{ran}(\pi)$, where $\bar{\mathcal{M}}$ is countable and transitive, $\bar{\mathcal{M}}$ is quasi-$(\omega, \omega_1, \omega_1)$-iterable above $\pi^{-1}(\delta)$. We also call $\mathcal{M}$* countably $+1$ iterable *above δ iff for all sufficiently elementary $\pi \colon \bar{\mathcal{M}} \to \mathcal{M}$ with $\delta \in \mathrm{ran}(\pi)$, where $\bar{\mathcal{M}}$ is countable and transitive, $\bar{\mathcal{M}}$ is quasi-$(\omega, \omega_1, \omega_1 + 1)$-iterable above $\pi^{-1}(\delta)$.*

We immediately get the following.

Lemma 1.12 *Let $L[E]$ be tame and quasi-$(\omega, \omega_1, \mathrm{OR})$-iterable, and let us write $K = L[E]$. Then inside K, K is countably iterable above $t(\aleph_1^K)$.*

Let $\kappa \geq \aleph_2^K$ be a cardinal of $L[E]$ which is not a limit of Woodin cardinals of $L[E]$. Then inside K, K is countably $+1$ iterable above $t(\kappa)$.

Here is another helpful one.

Lemma 1.13 *Let $\mathcal{M}$, $\mathcal{N}$ be tame. Let $\mathcal{M}$ be countably iterable and such that no $\delta \in \mathcal{M}$ is definably Woodin over $\mathcal{M}$ Let $\mathcal{N}$ be $(\omega, \kappa^+ + 1)$-iterable, where*

$$\kappa = \max(\mathrm{Card}(\mathcal{M}), \mathrm{Card}(\mathcal{N})),$$

and no $\delta \in \mathcal{N}$ is definably Woodin over $\mathcal{N}$. Then $\mathcal{M}$, $\mathcal{N}$ can be successfully compared.

Proof. The $\mathcal{N}$-side of the comparison gives us the $\mathcal{Q}$-structures for the $\mathcal{M}$-side. The result then follows by the usual reflection argument.

$\square$ (Lemma 1.13)

Lemma 1.14 *Let $L[E]$ be tame and quasi-$(\omega, \omega_1, \mathrm{OR})$-iterable. Let η be a double successor cardinal of $L[E]$. Then $J_\eta[E]$ is universal with respect to premice $\mathcal{M} \in$*

$L[E]$[8] *of height at most (and including)* η *such that* $L[E] \models$ *"*$\mathcal{M}$ *is coiterable with* $J_\eta[E]$. *Moreover, the coiteration of* $\mathcal{M}$ *with* $K||\eta$ *lasts fewer than* η *steps.*

PROOF. For the purpose of this proof, let us call a premouse $\mathcal{N}$ "good" iff $\mathcal{N} \models$ "if there is a largest cardinal, λ, then $\mathrm{cf}(\lambda)$ is not measurable." In order to prove the Lemma, let us work in $L[E]$, which we denote by K.

As η is a double successor cardinal, $K||\eta$ is "good." Set $\eta = \rho^{++}$, where ρ is a cardinal. If $\mathcal{M} \cap \mathrm{OR} < \eta$, or if $\mathcal{M}$ is "good," then the arguments in [7, Section 3] easily give the desired result.

Let us thus suppose that $\mathcal{M} \cap \mathrm{OR} = \eta$, and that $\mathcal{M}$ is not "good." Let λ be the largest cardinal of $\mathcal{M}$. Let $\mathcal{T}, \mathcal{U}$ be the iteration trees on $K||\eta$, $\mathcal{M}$, respectively, arising in the coiteration of $K||\eta$, $\mathcal{M}$.

Let α be least such that $\mathcal{M}^\mathcal{T}_\alpha||\eta = \mathcal{M}^\mathcal{U}_\alpha||\eta$. We're done if $\mathcal{M}^\mathcal{U}_\alpha||\eta \cap \mathrm{OR} = \eta$. So let us assume that $\mathcal{M}^\mathcal{U}_\alpha||\eta \cap \mathrm{OR} > \eta$. Standard arguments (using that $K||\eta$ has a largest cardinal) show that we cannot have that η extenders have been applied along $[0,\alpha]_\mathcal{U}$. In particular, there is no drop along $[0,\alpha]_\mathcal{U}$, and $\pi^\mathcal{U}_{0\alpha}(\lambda)$ is the largest cardinal of $\mathcal{M}^\mathcal{U}_\alpha||\eta$. Also, $\pi^\mathcal{U}_{0\alpha}(\lambda)$ must be *singular* in $\mathcal{M}^\mathcal{U}_\alpha||\eta$. But $\mathcal{M}^\mathcal{T}_\alpha||\eta = \mathcal{M}^\mathcal{U}_\alpha||\eta$, and so $\mathcal{M}^\mathcal{T}_\alpha||\eta$ has a largest cardinal. Therefore, there is no drop along $[0,\alpha]_\mathcal{T}$, and $\pi^\mathcal{T}_{0\alpha}(\rho^+)$ is the largest cardinal of $\mathcal{M}^\mathcal{T}_\alpha||\eta$. But $\pi^\mathcal{T}_{0\alpha}(\rho^+)$ is not singular in $\mathcal{M}^\mathcal{T}_\alpha||\eta$. Contradiction!

It is now easy to see that $\mathrm{lh}(\mathcal{T}) = \mathrm{lh}(\mathcal{U}) < \eta$.

$\square$ (Lemma 1.14)

The following tells us that there is a perfect substitute for thick classes for "weasels" of the form $K||\eta$. The proof is easy.

Lemma 1.15 *Let* $L[E]$ *be tame and quasi-*$(\omega, \omega_1, \mathrm{OR})$*-iterable. Let* η *be a double successsor cardinal of* $L[E]$, *say* $\eta = \rho^{++L[E]}$. *Then the following holds true inside* $L[E]$, *where we write* $K = L[E]$.

(a) Let $C \subset \eta$ *be* ρ^+*-closed and unbounded in* η. *Then* $h^{K||\eta}(C) = K||\eta$.[9]

(b) If $\mathcal{T}$ *is an iteration tree on* $K||\eta$ *of successor length* $\alpha+1 < \eta$ *such that there is no drop along* $[0,\alpha]_\mathcal{T}$, *then* $\{\xi < \eta \colon \pi^\mathcal{T}_{0\alpha}(\xi) = \xi\}$ *is* ρ^+*-closed and unbounded in* η.

(c) More generally, if $F \in K||\eta$ *is a (short or long) extender on* $K||\eta$, *then* $\{\xi < \eta \colon i_F(\xi) = \xi\}$ *is* ρ^+*-closed and unbounded in* η.

[8] This form of universality means that $\mathcal{M} \leq^* K||\eta$ for all such $\mathcal{M}$.

[9] Here, h denotes the Σ_1-Skolem function.

2 The two cardinals $\diamondsuit$ principles in $L[E]$.

The results in this section need more than just quasi-$(\omega, \omega_1, \mathrm{OR})$-iterability, namely versions of iterability which are stated in [10, Definition 6.7 and Theorem 6.9] and which are heavily exploited in [5], cf. [5, Facts 3.16.1 and 3.19.1].

Definition 2.1 *Let $\mathcal{M}$ be a premouse, and let Σ be a (partial) iteration strategy for $\mathcal{M}$. A phalanx $((\mathcal{P}_\alpha \colon \alpha \leq \theta), (\lambda_\alpha \colon \alpha < \theta))$ is called* $(\Sigma, \mathcal{M})$-generated *iff for each $\alpha \leq \theta$ there is a normal iteration tree $\mathcal{T}_\alpha$ on $\mathcal{M}$ according to Σ and of successor length such that $\mathcal{P}_\alpha \trianglelefteq \mathcal{M}_\infty^{\mathcal{T}_\alpha}$ $\mathrm{lh}(E_\nu^{\mathcal{T}_\alpha}) \leq \lambda_\beta$ for all $\nu + 1 < \mathrm{lh}(\mathcal{T}_\alpha)$ and for all $\beta < \alpha$. We say that $((\mathcal{P}_\alpha \colon \alpha \leq \theta), (\lambda_\alpha \colon \alpha < \theta))$ is* $(\Sigma, \mathcal{M})$-generated by trees above γ *iff we also have that $\mathrm{crit}(E_\nu^{\mathcal{T}_\alpha}) \geq \gamma$ for all $\alpha \leq \theta$ and $\nu + 1 < \mathrm{lh}(\mathcal{T}_\alpha)$.*

Definition 2.2 *Let $\mathcal{M}$ be a premouse, and let Σ be a (partial) iteration strategy for $\mathcal{M}$. We say that $\mathcal{M}$ is $*, \kappa$-iterable (above γ) iff the following holds. If $\vec{\mathcal{P}} = ((\mathcal{P}_\alpha \colon \alpha \leq \theta), (\lambda_\alpha \colon \alpha < \theta))$ is $(\Sigma, \mathcal{M})$-generated (by trees above γ), then $\vec{\mathcal{P}}$ is normally (ω, κ)-iterable (above γ).*

The inner models which show up in real life are typically $*, \mathrm{OR}$-iterable (cf. for instance [10, Theorem 6.9]).

Definition 2.3 *Let $\mathcal{M}$ be a premouse. We say that $\mathcal{M}$ is* sufficiently iterable *iff $\mathcal{M}$ is quasi-$(\omega, \omega_1, \mathrm{OR})$-iterable as well as $*, \mathrm{OR}$-iterable.*

The arguments of the previous section (cf. in particular Lemma 1.10) yield the following.

Lemma 2.4 *Let $L[E]$ be a tame extender model which is sufficiently iterable, and write $K = L[E]$. Let κ be a cardinal of K. Suppose that either κ is a limit cardinal of K, or else there is some μ below the predecessor of κ in K such that μ is $< \kappa$-strong in K. Let $t = t^K(\kappa)$. Then inside K, K is $*, \kappa$-iterable above t.*

We are now going to prove Theorems 0.4 and 0.6. Let us again write K for $L[E]$. As a warm-up, we shall first prove Theorems 0.4 and 0.6 in the simplified case where $K \models$ "I am quasi-$(\omega, \omega_1, \mathrm{OR})$-iterable as well as $*, \mathrm{OR}$-iterable." We'll then prove the full set of results in a second round.

So let us first suppose that $K \models$ "I am quasi-$(\omega, \omega_1, \mathrm{OR})$-iterable as well as $*, \mathrm{OR}$-iterable," and let us work inside K. Let θ be an uncountable cardinal of K, and let

$$\pi \colon \bar{K} \to K||\theta$$

be an elementary embedding which is continuous at limit points of cofinality ω and such that $\bar{K}$ is transitive. Let $(\kappa_\alpha | \alpha < \gamma)$ be the monotone enumeration of the infinite cardinals of $\bar{K}$. We also write $\kappa_\gamma = \bar{K} \cap \mathrm{OR}$. We shall need the apparatus of [5] and [6]. In particular, we need to refer to the statements $(1)_\alpha$ through $(6)_\alpha$ of [5]. However, all we really have to care about is summarized in the statements of Facts 1 and 2 below, which may be used as black boxes.

For some $\alpha \leq \gamma$ we call π *good at* κ_α provided that for all $\beta < \alpha$, $(2)_\beta$ holds. We remind the reader that [5] shows that[10]

$$\forall \beta < \alpha (2)_\beta \Rightarrow (6)_\alpha \Rightarrow (5)_\alpha \Rightarrow (4)_\alpha \Rightarrow (1)_\alpha,$$

where $(1)_\alpha$ says that the coiteration of $\bar{K}$ with K doesn't use extenders with index $\leq \kappa_{\alpha+1}$ on the $\bar{K}$-side. It is also shown in [5] that

$$\forall \beta < \alpha (4)_\beta \Rightarrow (3)_\alpha.$$

Moreover, it is shown in [6] that for every uncountable cardinal η there is a stationary set of $X \in \mathcal{P}_{\eta^+}(K||\theta)$ such that if $X = \mathrm{ran}(\pi)$ with π (and everything else) as above then for all $\alpha < \gamma$,

$$(3)_\alpha \Rightarrow (2)_\alpha.$$

This discussion readily yields the following two statements.

Fact 1. ([6]) There is a stationary set of $X \in \mathcal{P}_{\eta^+}(K||\theta)$ such that $X \prec K||\theta$ and if $\pi \colon \bar{K} \cong X \prec K||\theta$, where $\bar{K}$ is transitive, then π is good at $\bar{K} \cap \mathrm{OR}$.

Fact 2. ([5]) Let $\pi \colon \bar{K} \to K||\theta$, where $\bar{K}$ is transitive. Let $(\kappa_\alpha | \alpha < \gamma)$ be the monotone enumeration of the infinite cardinals of $\bar{K}$, and set $\kappa_\gamma = \bar{K} \cap \mathrm{OR}$. Suppose $\alpha < \gamma$ to be such that π is good at κ_α. Then $\bar{K}||\kappa_{\alpha+1}$ doesn't move in the comparison with K.

We are now ready to start the proof of Theorem 0.4.

Let us fix κ, λ as in the statement of Theorem 0.4. Let $X \in \mathcal{P}_\kappa(\lambda)$ be uncountable. We aim to define $F(X)$. Let us pick some

$$\sigma \colon K' \to K||\lambda,$$

[10] If $V = K$, then K is a very soundness witness for all of its initial segments, so that as the W of [5] we may just use K. Also, again if $V = K$, the arguments of [5] and [6] do not require $\mathrm{OR}+1$ iterability − OR-iterability is enough. We'll need to consider variants of the statements $(1)_\alpha$ through $(6)_\alpha$ below.

where K' is transitive, which is good at $K' \cap \mathrm{OR}$ and such that $\mathrm{Card}(K') = \mathrm{Card}(X)$ and $X \subset \mathrm{ran}(\sigma)$. Such a σ exists by Fact 1. As $\kappa < \lambda$, we may and shall assume that $\sigma \neq \mathrm{id}$.

Set $\bar{\lambda} = K' \cap \mathrm{OR}$. Because σ is good at $K' \cap \mathrm{OR}$, there is a normal tree $\mathcal{T}$ on K with last model $\mathcal{M}^{\mathcal{T}}_{\infty}$ such that $\mathcal{M}^{\mathcal{T}}_{\infty} \trianglerighteq K'$. For each K'-cardinal $\mu \leq \bar{\lambda}$, there is a least $\alpha = \alpha(\mu) \leq \mathrm{lh}(\mathcal{T})$ such that $\mathcal{M}^{\mathcal{T}}_{\alpha} \trianglerighteq K'|\mu$;[11] let $\xi = \xi(\mu) \leq \mathcal{M}^{\mathcal{T}}_{\alpha(\mu)} \cap \mathrm{OR}$ be largest such that $K'|\mu$ and $\mathcal{M}^{\mathcal{T}}_{\alpha(\mu)}||\xi$ have the same bounded subsets of μ. We define

$$F(X) = \{X \cap \sigma''B \ : \ \exists \mu \leq \bar{\lambda} \ B \in \mathcal{P}(\mu) \cap \mathcal{M}^{\mathcal{T}}_{\alpha(\mu)}||\xi(\mu)\}.$$

Claim 1. For each $X \in \mathrm{dom}(F)$, $F(X)$ has size at most $\mathrm{Card}(X)$.

Proof of Claim 1. It suffices to prove that

$$\mathrm{Card}(\mu^{+\mathcal{M}^{\mathcal{T}}_{\alpha(\mu)}||\xi(\mu)}) \leq \max\{\mathrm{Card}(\mathrm{crit}(\sigma)), \mathrm{Card}(\mu)\}$$

for each uncountable $\mu \leq \bar{\lambda}$. Let us fix such a μ, and write $\alpha = \alpha(\mu)$, $\xi = \xi(\mu)$.

What we need to verify is trivial if $\mathcal{M}^{\mathcal{T}}_{\alpha}||\xi$ is a set sized mouse, because then $\rho_{\omega}(\mathcal{M}^{\mathcal{T}}_{\alpha}||\xi) \leq \mu$ and $\mathcal{M}^{\mathcal{T}}_{\alpha}||\xi$ is sound above μ. Let us thus assume that $\mathcal{M}^{\mathcal{T}}_{\alpha}$ is a weasel. As $\sigma \neq \mathrm{id}$, there is then some least $\delta \leq \mathrm{crit}(\sigma)$ such that $(\mathcal{P}(\delta) \cap K) \setminus K' \neq \emptyset$. (Otherwise $\mathrm{crit}(\sigma)$ ends up being a superstrong cardinal in K. Either $\mathrm{crit}(\sigma)$ is inaccessible in K' and $\delta = \mathrm{crit}(\sigma)$, or else δ is the predecessor of $\mathrm{crit}(\sigma)$ in K'.) We may assume that $\mu \geq \delta$, because otherwise $\mathrm{Card}(\mu^{+\mathcal{M}^{\mathcal{T}}_{\alpha(\mu)}||\xi(\mu)}) \leq \mathrm{Card}(\mathrm{crit}(\sigma))$.

We have that $\alpha > 0$. It is easy to verify that if $\mathrm{crit}(\pi^{\mathcal{T}}_{0\alpha}) \geq \delta$, then $[0,\alpha)_T \cap \mathcal{D}^{\mathcal{T}} \neq \emptyset$, i.e., $\mathcal{M}^{\mathcal{T}}_{\alpha}$ is a set sized mouse. Therefore, as we assume $\mathcal{M}^{\mathcal{T}}_{\alpha}$ to be a weasel, if $\gamma + 1$ is least in $(0, \alpha]_T$, then, setting $\eta = \mathrm{crit}(E^{\mathcal{T}}_{\gamma})$, we must have that $\eta < \delta$.

But we must also have that

$$\pi^{\mathcal{T}}_{0\alpha}(\eta) \geq \mu.$$

This is because if $\pi^{\mathcal{T}}_{0\alpha}(\eta) < \mu$, then the map

$$\sigma \circ \pi^{\mathcal{T}}_{0\alpha} \restriction K||\eta^{+}$$

is easily seen to witness that η ends up being superstrong in K.

But now, by the choice of α, the set of generators of $\pi^{\mathcal{T}}_{0\alpha}$ (construed as an extender) is a subset of μ, and hence $\mathcal{M}^{\mathcal{T}}_{\alpha}||\mu^{+\mathcal{M}^{\mathcal{T}}_{\alpha}}$ has size μ. This shows that $\mathrm{Card}(\mu^{+\mathcal{M}^{\mathcal{T}}_{\alpha(\mu)}||\xi(\mu)}) \leq \mathrm{Card}(\mu)$.

$\square$ (Claim 1)

[11] In particular, $\mathcal{M}^{\mathcal{T}}_{\alpha}$ is not supposed to have an extender with index μ.

Claim 2. For all $A \subset \lambda$, there is a club $C \subset \mathcal{P}_\kappa(\lambda)$ such that for all $X \in C$, $X \cap A \in F(X)$.

Proof of Claim 2. Let us fix $A \subset \lambda$. We define a club $C \subset \mathcal{P}_\kappa(\lambda)$ as follows. Set $X \in C$ iff there is some $\pi \colon \bar{K} \to K||\lambda^+$, where $\bar{K}$ is transitive, such that $A \in \operatorname{ran}(\pi)$ and $X = \operatorname{ran}(\pi) \cap \lambda$. We have to see that for all $X \in C$, $X \cap A \in F(X)$.

Let us fix $X \in C$. Let

$$\sigma \colon K' \to K||\lambda$$

be the good embedding which was used to define $F(X)$. Write $\bar{\lambda} = \pi^{-1}(\lambda)$. As $\operatorname{ran}(\sigma) \supset \operatorname{ran}(\pi) \cap K||\lambda$, we may define $k_0 \colon \bar{K}||\bar{\lambda} \to K'$ simply as

$$k_0 = \sigma^{-1} \circ \pi \restriction \bar{K}||\bar{\lambda}.$$

Set $\lambda' = \sup(k_0{}''\bar{\lambda}) \leq K' \cap \mathrm{OR} \leq K' \cap \mathrm{OR}$. We may also define the liftup

$$k \colon \bar{K} \to K^* = \operatorname{ult}(\bar{K}, k_0),$$

where K^* consists of all $[a, f]$ such that $a \in [\lambda']^{<\omega}$ and $f \in \bar{K}$, $f \colon [\nu]^{\operatorname{Card}(a)} \to \bar{K}$ (for some ν such that $k(\nu) > \max(a)$). We have that $k(\bar{\lambda}) \geq \lambda'$ (where $k(\bar{\lambda}) > \lambda'$ is possible).

Let

$$\tilde{\sigma} \colon K^* \to K||\lambda^+$$

be the "interpolation map" which is defined by

$$\tilde{\sigma}([a, f]) = \pi(f)(a),$$

for $a \in [\sup k''\bar{\lambda}]^{<\omega}$ and $f \in \bar{K}$, $f \colon [\nu]^{\operatorname{Card}(a)} \to \bar{K}$ (for some ν such that $k(\nu) > \max(a)$). In particular, we may and shall identify K^* with its transitive collapse, so that always $[a, f] = k(f)(a)$. Figure 2 reflects this entire situation.

K^* inherits the iterability from $K||\lambda^+$. Write $\bar{A} = \pi^{-1}(A)$. Because σ is good at $K' \cap \mathrm{OR}$, $\tilde{\sigma}$ is good at λ', and hence by Fact 2, K^* doesn't move in the comparison with K. This readily implies that

$$k(\bar{A}) \cap \lambda' \in \mathcal{M}^{\mathcal{T}}_{\alpha(\lambda')}.$$

Therefore, $X \cap \sigma''(k(\bar{A}) \cap \lambda') \in F(X)$.

However, for all $\xi \in X$, say $\xi = \pi(\bar{\xi})$, we get that $\xi \in \sigma''(k(\bar{A}) \cap \lambda')$ iff $\sigma(k_0(\bar{\xi})) \in \sigma''(k(\bar{A}) \cap \lambda')$ iff $k(\bar{\xi}) = k_0(\bar{\xi}) \in k(\bar{A})$ iff $\bar{\xi} \in \bar{A}$ iff $\xi \in A$. Hence $X \cap \sigma''(k(\bar{A}) \cap \lambda') = X \cap A$, and so $X \cap A \in F(X)$. $\square$ (Claim 2)

$\square$ (Theorem 0.4, simplified case)

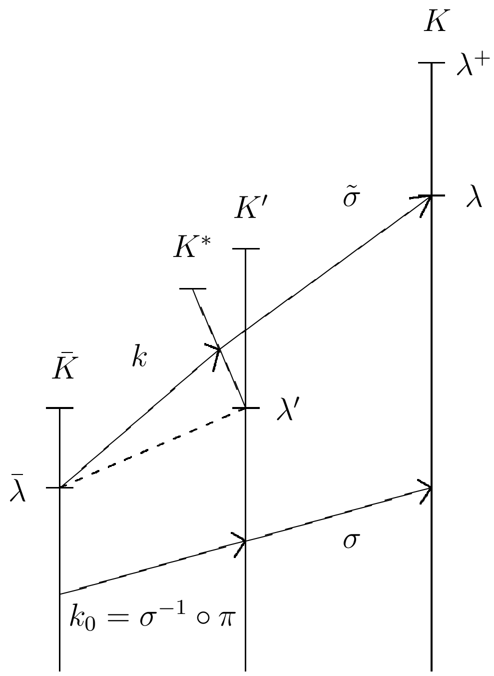

Figure 2: The definition of $F(X)$ works.

Let us now prove Theorem 0.6 in the simplified case where $K \models$ "I am quasi-$(\omega, \omega_1, \mathrm{OR})$-iterable as well as $*, \mathrm{OR}$-iterable."

Let us fix κ, λ as in the statement of Theorem 0.4. Let $X \in \mathcal{P}_\kappa(\lambda)$ be uncountable. We'll actually get by on an easier definition of $F(X)$ in this new situation (due to the fact that λ is now assumed to be regular). Much as before, we pick some

$$\sigma \colon K' \to K||\lambda,$$

where K' is transitive, which is good at $K' \cap \mathrm{OR}$ and such that $\mathrm{Card}(K') = \mathrm{Card}(X)$ and $X \cup \{X\} \subset \mathrm{ran}(\sigma)$. Again, such a σ exists by Fact 1 above, and we may and shall assume that $\sigma \neq \mathrm{id}$ (as $\kappa < \lambda$). This time, though, we also need to assume that σ is K-least with the properties as stated.

We now simply define

$$F(X) = \{X \cap \sigma(B) \colon B \in K'\}.$$

There is no analogue to Claim 1 above, as it is now trivial that $\mathrm{Card}(F(X)) \leq \mathrm{Card}(X)$. We are thus left with having to verify the following.

Claim. For all $A \subset \lambda$, there is an unbounded $D \subset \lambda$ such that for all $X \in \mathcal{P}_\kappa(\lambda)$, if $X \cap D$ is cofinal in $\sup(X)$, then $X \cap A \in F(X)$ as well as $X \cap D \in F(X)$.

PROOF of the Claim. Suppose not, and let $A \subset \lambda$ be a counterexample. There is then some $\epsilon < \lambda^+$ such that A is the ϵ^{th} element of K according to the canonical global well-ordering of K. The set A is then also definable over $K||\lambda^+$ from the parameters κ and ϵ.

Let us first define $D \subset \lambda$. Let $(X_i \colon i < \lambda)$ be a continuous chain of elementary substructures of $K||\lambda^+$ such that $(\kappa+1) \cup \{\epsilon\} \subset X_0$ and for all $i < \lambda$, $\mathrm{Card}(X_i) < \lambda$, $X_i \cap \lambda \in \lambda$, and $X_i \cup \{X_i\} \subset X_{i+1}$. We may and shall in fact require that each X_{i+1} is chosen least with these properties. Let $i < \lambda$. By the Condensation Lemma, X_i condenses to an initial segment of K, and we may write

$$\tau_i \colon K||\beta_i \cong X_i \prec K||\lambda^+.$$

Set $\alpha_i = X_i \cap \lambda = \tau_i^{-1}(\lambda)$; so α_i is the largest cardinal of $K||\beta_i$. Set $\epsilon_i = \tau^{-1}(\epsilon)$. Notice that $A \in \mathrm{ran}(\tau_i)$ for every $i < \lambda$, because A is definable over $K||\lambda^+$ from the parameters κ and ϵ; we then have $\tau_i^{-1}(A) = A \cap \alpha_i$ and $A \cap \alpha_i$ is definable over $K||\beta_i$ from the parameters κ and ϵ_i exactly as A is definable over $K||\lambda^+$ from the parameters κ and ϵ. Notice also that $\alpha_i < \beta_i < \alpha_{i+1}$ for all $i < \lambda$. For $i \leq j < \lambda$, we may define

$$\tau_{ij} \colon K||\beta_i \to K||\beta_j$$

by setting $\tau_{ij} = \tau_j^{-1} \circ \tau_i$; we have $\alpha_i = \mathrm{crit}(\tau_{ij})$, $\tau_{ij}(\alpha_i) = \alpha_j$, and $\tau_{ij}(A \cap \alpha_i) = A \cap \alpha_j$. It is clear that for every $i < \lambda$, the system

$$((K||\beta_k \colon k \leq i), (\tau_{kj} \colon k \leq j \leq i))$$

is definable over $K||\beta_i$ from the parameters κ and ϵ_i in much the same way as the entire system

$$((K||\beta_k \colon k < \lambda), (\tau_{kj} \colon k \leq j < \lambda))$$

was defined over $K||\lambda^+$ from the parameters κ and ϵ.

Let $\Gamma \colon \mathrm{OR} \times \mathrm{OR} \to \mathrm{OR}$ denote the Gödel pairing function. We now let

$$D = \{\Gamma(\epsilon_i, \beta_i) \ : \ i < \lambda\}.$$

D is obviously an unbounded subset of λ.

Let us now fix some $X \in \mathcal{P}_\kappa(\lambda)$ such that $X \cap D$ is cofinal in $\sup(X)$. We aim to verify that both $X \cap A \in F(X)$ as well as $X \cap D \in F(X)$, which will give a contradiction. Let

$$\sigma \colon K' \to K||\lambda$$

be the map which was used to define $F(X)$.

Let us write $\alpha = \sup(X)$. There is then some limit ordinal $\eta < \lambda$ such that $\alpha = \alpha_\eta$. To finish the proof of the Claim, it suffices to verify that $K||\beta_\eta \in \operatorname{ran}(\sigma)$ and $\epsilon_\eta \in \operatorname{ran}(\sigma)$. This is because if this holds true, then $A \cap \alpha \in \operatorname{ran}(\sigma)$ as well as

$$((K||\beta_i \colon i \leq \eta), (\tau_{ij} \colon i \leq j \leq \eta)) \in \operatorname{ran}(\sigma).$$

But then also $D \cap \alpha \in \operatorname{ran}(\sigma)$. However, if both $A \cap \alpha \in \operatorname{ran}(\sigma)$ as well as $D \cap \alpha \in \operatorname{ran}(\sigma)$, then both $X \cap A \in F(X)$ as well as $X \cap D \in F(X)$.

Let us therefore verify that $K||\beta_\eta \in \operatorname{ran}(\sigma)$ and $\epsilon_\eta \in \operatorname{ran}(\sigma)$.

We have that

$$(K||\beta_\eta, (\tau_{i\eta} \colon i < \eta))$$

is the direct limit of the system

$$((K||\beta_i \colon i < \eta), (\tau_{ij} \colon i \leq j < \eta)).$$

Cofinally many points of this system are elements of $\operatorname{ran}(\sigma)$. This is just because by hypothesis, $\{\Gamma(\epsilon_i, \beta_i) \colon \beta_i \in X\}$ is cofinal in α. But if $\Gamma(\epsilon_i, \beta_i) \in X$, then $K||\beta_i \in \operatorname{ran}(\sigma)$ as well as $\epsilon_i \in \operatorname{ran}(\sigma)$, and then $i \in \eta \cap \operatorname{ran}(\sigma)$ and

$$((K||\beta_k \colon k \leq i), (\tau_{kj} \colon k \leq j \leq i)) \in \operatorname{ran}(\sigma),$$

as this system is definable over $K||\beta_i$ from the parameters κ and ϵ_i. For $\Gamma(\epsilon_k, \beta_k)$, $\Gamma(\epsilon_i, \beta_i)$ both in X and $k \leq i$, let us write

$$\bar{K}_k = \sigma^{-1}(K||\beta_k) \text{ and } \bar{\tau}_{ki} = \tau^{-1}(\tau_{ki}).$$

Setting $\bar{\alpha}_i = \sigma^{-1}(\alpha_i)$ and $\bar{\epsilon}_i = \sigma^{-1}(\epsilon_i)$ for $\beta_i \in X$, we have that if $\Gamma(\epsilon_k, \beta_k)$ and $\Gamma(\epsilon_i, \beta_i)$ are both in X and $k \leq i$, then $\bar{\alpha}_k = \operatorname{crit}(\bar{\tau}_{ki})$, $\bar{\tau}_{ki}(\bar{\alpha}_k) = \bar{\alpha}_i$, and $\bar{\tau}_{ki}(\bar{\epsilon}_k) = \bar{\epsilon}_i$.

Let

$$(K^*, (\bar{\tau}_i \colon \Gamma(\epsilon_i, \beta_i) \in X))$$

denote the direct limit of the system

$$((\bar{K}_k \colon \beta_k \in X), (\bar{\tau}_{kj} \colon \Gamma(\epsilon_k, \beta_k) \in X \wedge \Gamma(\epsilon_j, \beta_j) \in X \wedge k \leq j)).$$

Let $\bar{\alpha} = \sigma^{-1}(\alpha)$. As $\alpha = \sup\{\alpha_i \colon \beta_i \in X\}$, $\bar{\alpha} = \sup\{\bar{\alpha}_i \colon \beta_i \in X\}$. Let us also write $\bar{\epsilon} = \bar{\tau}_i(\sigma^{-1}(\epsilon_i))$ (which is independent from i with $\Gamma(\epsilon_i, \beta_i) \in X$).

There is a natural embedding

$$\pi \colon K^* \to K||\beta_\eta$$

defined as follows. If $x \in K^*$, then pick $\beta_i \in X$ such that $x \in \mathrm{ran}(\bar{\tau}_i)$ and set $\pi(x) = \tau_{i\eta}(\sigma(\bar{\tau}_i^{-1}(x)))$. It is easy to see that π is well-defined. Notice that $\pi(\bar{\alpha}) = \tau_{i\eta}(\sigma(\bar{\tau}_i^{-1}(\bar{\alpha}))) = \tau_{i\eta}(\sigma(\bar{\alpha}_i)) = \tau_{i\eta}(\alpha_i) = \alpha$ (for any $i < \eta$ with $\beta_i \in X$).

Let us write $Z = \mathcal{P}(\bar{\alpha}) \cap K^* \cap K'$.

Subclaim 1. $\pi \restriction Z = \sigma \restriction Z$.

Proof of Subclaim 1. Let $\xi < \bar{\alpha}$, and let $U \in \mathcal{P}(\xi) \cap Z$. Let $\xi \in \mathrm{ran}(\bar{\tau}_i)$, i.e., $\xi < \bar{\alpha}_i$. Then $U \in \mathrm{ran}(\bar{\tau}_i)$ and $\pi(U) = \tau_{i\eta}(\sigma(\bar{\tau}_i^{-1}(U))) = \tau_{i\eta}(\sigma(U)) = \sigma(U)$, as $\sigma(\xi) < \alpha_i$.

Now let $U \in Z$. Then

$$\pi(U) = \bigcup_{\beta_i \in X} \pi(U \cap \bar{\alpha}_i) = \bigcup_{\beta_i \in X} \sigma(U \cap \bar{\alpha}_i) = \sigma(U).$$

This shows Subclaim 1. $\square$ (Subclaim 1)

Because σ is good at $K' \cap \mathrm{OR}$, Subclaim 1 in particular yields that π is good at $\bar{\alpha}$. (This only needs $\pi \restriction \bar{\alpha} = \sigma \restriction \bar{\alpha}$.) Let $\mathcal{T}$ be the normal tree on K which arises in the comparison of K with $K'||\bar{\alpha} = K^*||\bar{\alpha}$. As σ is good at $\bar{\alpha}$, $K'||\bar{\alpha}^{+K'} \trianglelefteq \mathcal{M}_\infty^\mathcal{T}$. As π is good at $\bar{\alpha}$, $K^* \trianglelefteq \mathcal{M}_\infty^\mathcal{T}$. Therefore, $K'||\bar{\alpha}^{+K'} \trianglelefteq K^*$ or $K^* \trianglelefteq K'||\bar{\alpha}^{+K'}$.

Subclaim 2. $K^* \triangleleft K'||\bar{\alpha}^{+K'}$.

Proof of Subclaim 2. Because $X \cap D$ is cofinal in α, $\mathrm{cf}(\alpha) = \mathrm{cf}(X \cap D) \leq \mathrm{Card}(X) < \kappa$. On the other hand, $\alpha > \kappa$, as $\kappa + 1 \subset X_0$. Therefore α is singular (in $K||\lambda$), so that $\bar{\alpha}$ is singular in K', and hence $\bar{\alpha}$ is singular in $K'||\bar{\alpha}^{+K'}$. On the other hand, λ is regular (in $K||\lambda^+$), and hence $\alpha = (\tau_\eta \circ \pi)^{-1}(\lambda)$ is regular in K^*. This yields that $K^* \triangleleft K'||\bar{\alpha}^{+K'}$. $\square$ (Subclaim 2)

By Subclaim 2, $K^* \in \mathrm{dom}(\sigma)$. But now

$$(K^*, (\bar{\tau}_i \colon \Gamma(\epsilon_i, \beta_i) \in X))$$

is the direct limit of the system

$$((\bar{K}_k \colon \Gamma(\epsilon_k, \beta_k) \in X), (\bar{\tau}_{kj} \colon \Gamma(\epsilon_k, \beta_k) \in X \wedge \Gamma(\epsilon_j, \beta_j) \in X \wedge k \leq j)),$$

so that $\sigma(K^*)$ is the model gotten by forming the direct limit of the system

$$((K||\beta_k \colon \Gamma(\epsilon_k, \beta_k) \in X), (\tau_{kj} \colon \Gamma(\epsilon_k, \beta_k) \in X \wedge \Gamma(\epsilon_j, \beta_j) \in X \wedge k \leq j)).$$

But the latter system gives the model $K||\beta_\eta$, so that in fact $K||\beta_\eta = \sigma(K^*)$. This shows that $K||\beta_\eta \in \mathrm{ran}(\sigma)$.

In order to verify $\epsilon_\eta \in \mathrm{ran}(\sigma)$, notice that $\sigma(\bar\epsilon) = \pi(\bar\epsilon)$ by Subclaim 2, and hence $\sigma(\bar\epsilon) = \pi(\bar\epsilon) = \tau_{i\eta}(\sigma(\bar\tau_i^{-1}(\bar\epsilon))) = \tau_{i\eta}(\epsilon_i) = \epsilon_\eta$, independently from i such that $\Gamma(\epsilon_i, \beta_i) \in X$. $\square$ (Claim)

$\square$ (Theorem 0.6, simplified case)

Let us now drop the hypothesis that $K \models$ "I am quasi-$(\omega, \omega_1, \mathrm{OR})$-iterable as well as $*, \mathrm{OR}$-iterable." We key idea for showing that $\diamondsuit^*_{\kappa,\lambda}$ and $\diamondsuit^{+,\text{ unctble.}}_{\kappa,\lambda}$ both hold in K remains of course the same. Let $\theta > \lambda$. An inspection of the argument we have given above shows that in order to verify $\diamondsuit^*_{\kappa,\lambda}$ in $L[E]$ (where $\omega_1^{L[E]} < \kappa < \lambda$) or in order to verify $\diamondsuit^{+,\text{ unctble.}}_{\kappa,\lambda}$ in $L[E]$ (where $\omega_1^{L[E]} < \kappa < \lambda$ and λ is regular in $L[E]$) we need to see that the following "amalgamation" of Facts 1 and 2 holds true.

Lemma 2.5 *Let $L[E]$ be a tame extender model, and suppose that (in V) $L[E]$ is sufficiently iterable. Let $\kappa < \lambda < \lambda^+ < \theta$ be uncountable cardinals of $L[E]$. Then inside $L[E]$ the following statement holds true. There is a stationary set $X \in \mathcal{P}_\kappa(J_\theta)[E]$ such that if*

$$\pi\colon J_{\bar\theta}[\bar E] \cong X \prec J_\theta[E],$$

then $J_{\pi^{-1}(\lambda^+)}[\bar E]$ is coiterable with $L[E]$ and in fact $J_{\pi^{-1}(\lambda^+)}[\bar E]$ does not move in the comparison with $L[E]$. Moreover, if λ' is a cardinal of $J_{\pi^{-1}(\lambda^+)}[\bar E]$, and if $\mathcal{N} \triangleright J_{\lambda'}[\bar E]$ with $\dot F^{\mathcal{N}} = \emptyset$ is such that λ' is the largest cardinal of $\mathcal{N}$ and there is an embedding $k\colon \mathcal{N} \to J_{\lambda^+}[E]$ with $k \restriction \lambda' = \pi \restriction \lambda'$, then $\mathcal{N}$ is coiterable with $L[E]$ and in fact $\mathcal{N}$ does not move in the comparison with $L[E]$.

Proof. Obviously, the "moreover" part of Lemma 2.5 implies the fact about $J_{\pi^{-1}(\lambda^+)}[\bar E]$. We shall, however, first prove the first part of Lemma 2.5. We'll then indicate how to prove the "moreover" part by a simple variant.

We again write $K = L[E]$. Let us work in K. Let

$$\pi\colon \bar K \to K||\theta$$

be given, where we pick π by a method as in [6]. We shall say more about how to pick π in a minute. If κ is countably closed, which by GCH is equivalent to the fact that κ is not the successor of a cardinal of cofinality ω, then we might just pick π in such a way that ${}^\omega\bar K \subset \bar K$ (cf. [5]). However, if κ is not countably closed, then a little bit more work is necessary (cf. [6]). We aim to see that π satisfies the first part in the statement of Lemma 2.5, and we shall then verify that π actually also satisfies the "moreover" part.

Let us write η for the cardinality of X, so that $\bar{K} \cap \mathrm{OR} < \eta^+$. We may and shall assume that $\pi \restriction (\eta + 1) = \mathrm{id}$. We'll then also have that $\pi \restriction (t(\kappa) + 1) = \mathrm{id}$.

Let us consider the comparison of K with $\bar{K}$. As K is η^+-iterable above $t(\eta^+)$ by Lemma 1.10, the first η^+ steps of this comparison are no problem, at least as long as $\bar{K}$ does not move.[12] We aim to see that $\bar{K}||\pi^{-1}(\lambda^+)$ does not move in this comparison (which then also shows that the comparison lasts fewer than η^+ steps).

Let $(\kappa_\alpha | \alpha \leq \gamma)$ enumerate the infinite cardinals of $\bar{K}||\pi^{-1}(\lambda^+)$, and set $\lambda_\alpha = \kappa_\alpha^{+\bar{K}}$ for $\alpha \leq \gamma$. We also set $\Lambda_\alpha = \sup(\pi''\lambda_\alpha)$ for $\alpha \leq \gamma$. Now let us suppose towards contradiction that $\bar{K}||\pi^{-1}(\lambda^+)$ does move in the comparison with K. Let $F = E_\nu^{\bar{K}}$ be the first extender used on the $\bar{K}$-side. Let κ_{α^*} be the $\bar{K}$-cardinality of ν, and let, for $\alpha \leq \alpha^*$, $\mathcal{P}_\alpha$, $\mathcal{Q}_\alpha$, $\mathcal{R}_\alpha$, and $\mathcal{S}_\alpha$ be defined exactly as in [5]. Let us remind the reader how these objects are defined.

Let $\mathcal{T}$ denote the iteration tree on K which arises in the comparison of K with $\bar{K}|\nu$ (in which, by the choice of ν, $\bar{K}|\nu$ does not move). For $\alpha \leq \alpha^*$, let $\xi(\alpha)$ be the least ξ such that $\bar{K}||\kappa_\alpha = \mathcal{M}_\xi^\mathcal{T}||\kappa_\alpha$. It is easy to see that we must then have $\mathcal{M}_\xi^\mathcal{T}|\lambda_\alpha = \bar{K}||\lambda_\alpha$ (but we may have that $\mathcal{M}_\xi^\mathcal{T}$ has an extender with index λ_α) or else $\alpha = \alpha^*$. Let $\mathcal{P}_\alpha$ be the longest initial segment of $\mathcal{M}_{\xi(\alpha)}^\mathcal{T}$ which has the same subsets of κ_α as $\bar{K}$ has. We may then let $\mathcal{R}_\alpha = \mathrm{ult}_n(\mathcal{P}_\alpha; E_{\pi \restriction \lambda_\alpha})$, where $n = 0$ if $\mathcal{P}_\alpha$ is a weasel and n is least with $\rho_{n+1}(\mathcal{P}_\alpha) \leq \kappa_\alpha < \rho_n(\mathcal{P}_\alpha)$ if $\mathcal{P}_\alpha$ is set-sized. We recursively define $\mathcal{S}_\alpha$ and $\mathcal{Q}_\alpha$ as follows. If $\mathcal{R}_\alpha$ is a premouse (rather than just a proto-mouse), then we set $\mathcal{S}_\alpha = \mathcal{R}_\alpha$; otherwise let $\pi(\kappa_{\zeta(\alpha)}) = \mathrm{crit}(\dot{F}^{\mathcal{R}_\alpha})$, let $\delta(\alpha)$ be the largest δ such that $\mathcal{P}(\pi(\kappa_{\zeta(\alpha)})) \cap \mathcal{S}_{\zeta(\alpha)}||\delta \subset K||\Lambda_{\zeta(\alpha)}$, and let $\mathcal{S}_\alpha = \mathrm{ult}_k(\mathcal{S}_{\zeta(\alpha)}||\delta(\alpha); \dot{F}^{\mathcal{R}_\alpha})$, where $k = 0$ if $\mathcal{S}_{\zeta(\alpha)}$ is a weasel and k is least with $\rho_{k+1}(\mathcal{S}_{\zeta(\alpha)}||\delta(\alpha)) \leq \pi(\kappa_{\zeta(\alpha)}) < \rho_k(\mathcal{S}_{\zeta(\alpha)}||\delta(\alpha))$ if $\mathcal{S}_{\zeta(\alpha)}||\delta(\alpha)$ is set-sized. Finally, we let $\mathcal{Q}_\alpha = \mathcal{P}_\alpha$ if $\mathcal{S}_\alpha = \mathcal{R}_\alpha$, and we let $\mathcal{Q}_\alpha = \mathrm{ult}_n(\mathcal{Q}_{\zeta(\alpha)}; \dot{F}^{\mathcal{P}_\alpha})$ otherwise, where $n = 0$ if $\mathcal{Q}_{\zeta(\alpha)}$ is a weasel, and n is least with $\rho_{n+1}(\mathcal{Q}_{\zeta(\alpha)}) \leq \kappa_{\zeta(\alpha)} < \rho_n(\mathcal{Q}_{\zeta(\alpha)})$ if $\mathcal{Q}_{\zeta(\alpha)}$ is set-sized.

Let us now consider the following statements, for $\alpha \leq \alpha^*$ (cf. [5] and [6]).

$(2)_\alpha^*$ $((K, \mathcal{S}_\alpha), \pi(\kappa_\alpha))$ is countably iterable above $t(\eta^+)$.

$(3)_\alpha^*$ $((\bar{K}, \mathcal{Q}_\alpha), \kappa_\alpha)$ is countably iterable above $t(\eta^+)$.

$(4)_\alpha^*$ $(((\mathcal{P}_\beta \colon \beta < \alpha)^\frown \bar{K}), (\lambda_\beta \colon \beta < \alpha))$ is countably iterable above $t(\eta^+)$.

$(5)_\alpha^*$ $(((\mathcal{R}_\beta \colon \beta < \alpha)^\frown K), (\Lambda_\alpha \colon \beta < \alpha))$ is countably iterable above $t(\eta^+)$ with respect to special iteration trees.

[12]If $\bar{K}$ were to move, then we might face a problem as $\bar{K}$ may have Woodin cardinals between $t(\eta^+)$ and $\pi^{-1}(\lambda^+)$.

$(6)^*_\alpha$ $(((\mathcal{S}_\beta \colon \beta < \alpha)^\frown K), (\Lambda_\alpha \colon \beta < \alpha))$ is countably iterable above $t(\eta^+)$ with respect to special iteration trees.

As in [5] and [6], we aim to prove that

$$\forall \beta < \alpha (2)_\beta \Rightarrow (6)_\alpha \Rightarrow (5)_\alpha \Rightarrow (4)_\alpha, \text{ and}$$

$$\forall \beta < \alpha (4)_\beta \Rightarrow (3)_\alpha.$$

We also want to have that

$$(3)_\alpha \Rightarrow (2)_\alpha.$$

We are now in a position to say exactly how we may and shall assume π to have been chosen, namely in such a way that for every witness to $\neg(2)_\alpha$ there is some (possibly different) witness Φ to $\neg(2)_\alpha$ (i.e., Φ is a countable phalanx which can be embedded into $((K, \mathcal{S}_\alpha), \pi(\kappa_\alpha))$ and which is not countably iterable above the preimage of $t(\pi(\lambda_\alpha)))$ such that Φ can be reembedded into $((\bar{K}, \mathcal{Q}_\alpha), \kappa_\alpha)$ to show that $\neg(3)_\alpha$ holds; cf. [6] on further details). In other words, we may and shall assume π to be such that $(3)_\alpha \Rightarrow (2)_\alpha$.

The proofs of $(6)_\alpha \Rightarrow (5)_\alpha$ and $(5)_\alpha \Rightarrow (4)_\alpha$ are exactly as in [5, Lemmas 3.17 and 3.18]. Therefore, the following remains to be shown.

Claim 1. $\forall \beta < \alpha (2)_\beta \Rightarrow (6)_\alpha$.

Claim 2. $\forall \beta < \alpha (4)_\beta \Rightarrow (3)_\alpha$.

Claim 3. If $(4)_{\alpha^*}$ holds, then $F = E^{\bar{K}}_\nu$ is not used in the comparison of K with $\bar{K}$.

As we are assuming F to be the first extender used in the comparison of K with $\bar{K}$, this will give a contradiction.

Proof of Claim 1. The idea for the proof of this Claim is as in [5, Lemma 3.19]. As there, it suffices to see that for all $\beta < \alpha$ the following holds true. If $\mathcal{S}_\beta$ is set-sized, then $\mathcal{S}_\beta \lhd K$, and if $\mathcal{S}_\beta$ is a weasel, then $E^K_{\Lambda_\beta} \neq \emptyset$ is a total extender on K and $\mathcal{S}_\beta = \mathrm{ult}(K; E^K_{\Lambda_\beta})$. This is enough as we may then argue as in the proof of [5, Lemma 3.19]. Notice that this is the first place where we need that inside K, K is $*, \eta^+$-iterable, which is given by Lemma 2.4.

Let us fix $\beta < \alpha$, and let us suppose $(2)^*_\beta$ to be true. Let us first assume that $\mathcal{S}_\beta$ is set-sized. Then we want to see that $\mathcal{S}_\beta \lhd K$.

Case 1. There is no $\mu < \pi(\kappa_\beta)$ such that μ is $< \pi(\lambda_\beta)$-strong in K.

Notice that we must have $t(\pi(\lambda_\beta)) \in \mathrm{ran}(\pi)$, so that $t(\pi(\lambda_\beta)) < \Lambda_\beta$. Our case hypothesis is easily seen to give that Λ_β is a cutpoint of K. We thus have that $K||\pi(\lambda_\beta)$ is $\pi(\lambda_\beta)$-iterable above Λ_β.

The case hypothesis also implies that if $E_\xi^{\mathcal{S}_\beta} \neq \emptyset$, where $\xi < \mathcal{S}_\beta \cap \mathrm{OR}$, then $\mathrm{crit}(E_\xi^{\mathcal{S}_\beta}) > \Lambda_\beta$. Therefore, we either have that Λ_β is also a cutpoint of $\mathcal{S}_\beta$, or else $\dot{F}^{\mathcal{S}_\beta} \neq \emptyset$ and $\mathrm{crit}(\dot{F}^{\mathcal{S}_\beta}) = \pi(\kappa_\beta)$. We have that $\rho_\omega(\mathcal{S}_\beta) \leq \pi(\kappa_\beta)$.

Let us take a countable hull of a rank initial segment of V (i.e, K) which contains $K||\pi(\lambda_\beta)$ as well as $\mathcal{S}_\beta$. Let c denote the collapsing map. The comparison of $c(K||\lambda_\beta)$ with $c(\mathcal{S}_\beta)$ is above $c(\Lambda_\beta)$ on the $c(K||\lambda_\beta)$-side and above $c(\Lambda_\beta)$ or at least above $c(\pi(\kappa_\beta))$ on the $c(\mathcal{S}_\beta)$-side. By how π was chosen, we may thus successfully compare $c(K||\lambda_\beta)$ with $c(\mathcal{S}_\beta)$ and deduce that $c(\mathcal{S}_\beta) \triangleleft c(K||\pi(\lambda_\beta))$. Therefore $\mathcal{S}_\beta \triangleleft K||\pi(\lambda_\beta)$.

Case 2. There is some $\mu < \pi(\kappa_\beta)$ such that μ is $< \pi(\lambda_\beta)$-strong in K.

Let μ be the least such. Let us pick $E = E_\xi^K$ such that $\mathcal{S}_\beta \in K|\xi$ and $\mathrm{crit}(E) = \mu$. Notice that $t(\kappa) < \mu$. We may then argue inside $\mathrm{ult}(K;E)$ to deduce that in fact $\mathcal{S}_\beta$ is an initial segment of $\mathrm{ult}(K;E)$, and therefore of K.

By how π was chosen, $((K, \mathcal{S}_\beta), \Lambda_\beta)$ is countably iterable above $t(\eta^+)$. This implies that $((\mathrm{ult}(K;E), \mathcal{S}_\beta), \Lambda_\beta)$ is countably iterable above $t(\eta^+)$ inside K as well as $\mathrm{ult}(K;E)$. Moreover, $\mathrm{ult}(K;E)||i_E(\mu)$ is $i_E(\mu)+1$ iterable above $t(\eta^+)$ inside $\mathrm{ult}(K;E)$. We may therefore use Lemma 1.13 to deduce that we may successfully compare $\mathrm{ult}(K;E)||i_E(\mu)$ with $((\mathrm{ult}(K;E)||i_E(\mu), \mathcal{S}_\beta), \Lambda_\beta)$. Standard arguments then yield that $\mathcal{S}_\beta \triangleleft \mathrm{ult}(K;E)||i_E(\mu)$, i.e., $\mathcal{S}_\beta \triangleleft K$.

Now suppose that $\mathcal{S}_\beta$ is class-sized. We aim to see that $E_{\Lambda_\beta}^K \neq \emptyset$, where $\mathrm{crit}(E_{\Lambda_\beta}^K)$ is also the critical point of the first extender used along the main branch going from K to $\mathcal{P}_\beta$. We also want to see that $\mathcal{S}_\beta = \mathrm{ult}(K; E_{\Lambda_\beta}^K)$.

The argument is an amalgamation of the argument for the case where $\mathcal{S}_\beta$ is set-sized and the argument from [5, Lemma 3.19] for the case where $\mathcal{S}_\beta$ is a weasel.. We omit further detail.

$\square$ (Claim 1)

Proof of Claim 2. This proof is basically as the proof of [5, Lemma 3.16], modulo arguments as in the proof of Claim 1 and of Claim 3. We also need to cite Lemma 2.4 here. We omit further detail.

$\square$ (Claim 2)

Proof of Claim 3. We need to split the proof into three cases.

Case 1. κ is a limit cardinal.

In this case, $\eta^+ < \kappa$, and we may use $K||\eta^{++}$ as our "core model." I.e., we apply the techniques of Lemma 1.14 and Lemma 1.15. Let us consider

$$\vec{\mathcal{P}} = ((\mathcal{P}_\alpha||\eta^{++}\colon \alpha \leq \alpha^*), (\lambda_\alpha\colon \alpha < \alpha^*))$$

and

$$\vec{\mathcal{P}}' = ((\mathcal{P}_\alpha||\eta^{++}\colon \alpha < \alpha^*)^\frown \bar{K}, (\lambda_\alpha\colon \alpha < \alpha^*)).$$

By $(4)_{\alpha^*}$, $\vec{\mathcal{P}}'$ is countably iterable above $t(\eta^+)$. By we may construe the $\vec{\mathcal{P}}$-side of the comparison of $\vec{\mathcal{P}}$ with $\vec{\mathcal{P}}'$ as an iteration of $K||\eta^{++}$ which is above $t(\eta^+)$. We have that $K||\eta^{++}$ is $\eta^{++}+1$ iterable above $t(\eta^+)$, and hence by Lemma 1.14 the comparison of $\vec{\mathcal{P}}$ with $\vec{\mathcal{P}}'$ will successfully terminate after fewer than η^{++} steps. Standard arguments, using Lemma 1.14 and Lemma 1.15, then yield that in fact F is on the sequence of $\mathcal{P}_{\alpha^*}$. This gives a contradiction!

Case 2a. κ is a successor cardinal, say $\kappa = \epsilon^+$, where ϵ is a cardinal, and there is no $\mu < \epsilon$ which is $< \kappa$ strong.

In this case, there is some cutpoint $c \geq t(\eta^+)$, $\epsilon \leq c < \mathrm{crit}(\pi) = \mathrm{ran}(\pi) \cap \kappa < \kappa$. Let $\xi > c$ be least such that $(\mathcal{P}(c) \cap K||(x+\omega)) \setminus \bar{K} \neq \emptyset$. ($\xi$ is well-defined, as $(\mathcal{P}(\mathrm{crit}(\pi)) \cap K) \setminus \bar{K} \neq \emptyset$.) We may then actually construe the coiteration of K with $\bar{K}$ as the coiteration of $K||\xi$ with $\bar{K}$, which will be above c on both sides.

In particular, none of the models $\mathcal{P}_\alpha$ for $\alpha \leq \alpha^*$ with $\kappa_\alpha \geq \mathrm{crit}(\pi)$ will be a weasel. Let us take a countable hull of some rank initial segment of V containing all sets of current interest. Let k denote the collapsing map. The coiteration of

$$((k(\mathcal{P}_\alpha)|\alpha \in ((\alpha^*+1) \cap \mathrm{ran}(k))), (\lambda_\alpha|\alpha \in (\alpha^*+1) \cap \mathrm{ran}(k)))$$

with

$$((k(\mathcal{P}_\alpha)|\alpha \in ((\alpha^*) \cap \mathrm{ran}(k))^\frown k(\bar{K})), (\lambda_\alpha|\alpha \in (\alpha^*+1) \cap \mathrm{ran}(k)))$$

doesn't involve k-images of weasels. These two phalanxes may therefore successufully be coitereated, and we may conclude that $k(\bar{K}||{\kappa_\alpha}^{+\bar{K}}) \lhd k(\mathcal{M}_{\alpha^*})$, and hence $\bar{K}||{\kappa_\alpha}^{+\bar{K}} \lhd \mathcal{M}_{\alpha^*}$. Contradiction!

Case 2b. κ is a successor cardinal, say $\kappa = \epsilon^+$, where ϵ is a cardinal, and there is some $\mu < \epsilon$ which is $< \kappa$ strong.

Let $\mu < \epsilon$ be least such that μ is $< \kappa$ strong. We have that $t(\eta^+) < \mu$. We may let $G = E^K_\xi \neq \emptyset$ be such that $\mathrm{crit}(G) = \mu$ and $\bar{K} \in K||\xi$. We'll then take $\mathrm{ult}(K;G)$ as our "core model."

More precisely, let

$$\vec{\mathcal{P}} = ((\mathcal{P}_\alpha^\dagger||i_G(\mu)|\alpha \leq \alpha^*), (\lambda_\alpha|\alpha < \alpha^*))$$

and

$$\vec{\mathcal{P}}' = ((\mathcal{P}_\alpha^\dagger||i_G(\mu)|\alpha < \alpha^*)^\frown \bar{K}, (\lambda_\alpha|\alpha < \alpha^*)),$$

where $\mathcal{P}_\alpha^\dagger$ is obtained by letting the iteration tree (including the final cut) which produces $\mathcal{P}_\alpha$ from K act on $\mathrm{ult}(K;G)$. (If $\mathcal{P}_\alpha$ is set-sized, then $\mathcal{P}_\alpha^\dagger = \mathcal{P}_\alpha$, but if $\mathcal{P}_\alpha$ is a weasel, then $\mathcal{P}_\alpha^\dagger \neq \mathcal{P}_\alpha$.)

By $(4)_{\alpha^*}$, $\vec{\mathcal{P}}'$ is countably iterable, and hence $\vec{\mathcal{P}}'$ is also countably iterable in $\mathrm{ult}(K;G)$. We may hence successfully coiterate $\vec{\mathcal{P}}$ with $\vec{\mathcal{P}}'$ inside $\mathrm{ult}(K;G)$. Standard arguments then yield that in fact F is on the sequence of $\mathcal{M}_{\alpha^*}^{\mathcal{T}}$. Contradiction! $\square$ (Claim 3)

The reader will now luckily verify that our argument actually also shows the "moreover" part of Theorem 2.5. The point is that if π is chosen as above and $k\colon \mathcal{N} \to K||\lambda^+$, where $k \restriction \lambda' = \pi \restriction \lambda'$, then we may just replace $\bar{K}$ by $\mathcal{N}$ in $(3)_\alpha^*$ and $(4)_\alpha^*$ and show the implications for the new versions of $(2)_\alpha^*$ through $(6)_\alpha^*$ exactly as before for the old versions. We may then finally deduce that $\mathcal{N}$ is not moved in the comparison with K as in the proof of Claim 3 above.

$\square$ (Lemma 2.5 and Theorems 0.4 and 0.6)

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