LINKING DESCRIPTIVE SET THEORY TO SYMBOLIC DYNAMICS

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General notation

This talk, applies set-theoretic ideas to a problem of analysis, and therefore our notation will draw on that of two mathematical traditions. Thus we usually denote the set $\{0, 1, 2, \ldots\}$ of *natural numbers* by ω, though occasionally by $\mathbb{N}$; this visual distinction allows us to write ω^n for the ordinal power and $\mathbb{N}^n$ for the set of n-tuples of natural numbers.

$\mathbb{N}^+$ is the set $\{1, 2, 3, \ldots\}$ of *positive integers*: in Definition 4·3 the difference between $\mathbb{N}$ and $\mathbb{N}^+$ is important.

On a space, such as Baire space, comprising all sequences of length ω of members of some set, we define the shift function $\mathfrak{s}$ thus:

$$\mathfrak{s}(\zeta)(n) = \zeta(n+1) \text{ for } n \geqslant 0.$$

Here we return to normal set-theoretic convention by considering the domain of such sequences to be $\omega = \{0, 1, 2, \ldots\}$.

We write $\circledcirc$ for the empty sequence: technically of course it is the same as the empty set, which we write as $\varnothing$; and also the same as the number zero, which we write as 0, since set-theorists customarily identify each natural number n with the set $\{0, 1, \ldots n-1\}$.

We denote by ${}^{<\omega}X$ the set of finite sequences of points in the set X, including the empty sequence.

When s is a finite sequence, we write $\ell h(s)$ for its length, so that $s = \langle s(0), s(1), \ldots s(\ell h(s)-1)\rangle$. We also write $\ell(s)$ for its last element, $s(\ell h(s)-1)$. Concatenation is denoted by $\frown$, so $\ell h(s^\frown\langle p\rangle) = \ell h(s)+1$.

For s a finite sequence and ζ an infinite sequence, we write $s \sqsubseteq \zeta$ to mean $\exists k \forall n < \ell h(s)\, s(n) = \zeta(k+n)$. Similarly if t is a finite sequence we write $s \sqsubseteq t$ to mean

$$\exists k \left(\ell h(t) \geqslant k + \ell h(s) \;\&\; \forall n < \ell h(s)\, s(n) = t(k+n)\right).$$

The original problem

Let $\mathcal{X}$ be a Polish space and let $f : \mathcal{X} \longrightarrow \mathcal{X}$ be a continuous function. For $k \in \omega$ we write f^k for the k^{th} iterate of f, so that for each $x \in \mathcal{X}$, $f^0(x) = x$ and $f^{k+1}(x) = f(f^k(x))$. For x and y in $\mathcal{X}$ we define

$$x \curvearrowright_f y \iff_{\text{df}} \exists \text{ strictly increasing } \alpha : \omega \to \omega \text{ with } \lim_{n\to\infty} f^{\alpha(n)}(x) = y.$$

We write $\omega_f(x)$ for the set $\{y \mid x \curvearrowright_f y\}$ of the accumulation points of the forward orbit of x under f, including the periodic points. When f is fixed in a discussion, we write $x \curvearrowright y$ for $x \curvearrowright_f y$, and we sometimes write $y \curvearrowleft x$ for $x \curvearrowright y$. We read $x \curvearrowright y$ as "x *attacks* y".

Proposition *(i)* $\omega_f(x)$ *is a closed subset of* $\mathcal{X}$*.*

(ii) if $x \curvearrowright y$ *and* $y \curvearrowright z$ *then* $x \curvearrowright z$*.*

We define an operator Γ_f on subsets of $\mathcal{X}$ by

$$\Gamma_f(X) = \bigcup\{\omega_f(x) \mid x \in X\}.$$

Then starting from a given point $a \in \mathcal{X}$, we define a transfinite sequence:

$$\begin{aligned} A^0(a,f) &= \omega_f(a) \\ A^{\beta+1}(a,f) &= \Gamma_f(A^\beta(a,f)) \\ A^\lambda(a,f) &= \bigcap_{\nu<\lambda} A^\nu(a,f) \qquad \text{for } \lambda \text{ a limit ordinal} \end{aligned}$$

By part (ii) of the Proposition, $A^0(a,f) \supseteq A^1(a,f) \supseteq A^2(a,f) \dots$; and indeed for all ordinals $\alpha < \beta$, $A^\alpha(a,f) \supseteq A^\beta(a,f)$.

DEFINITION The *escape set* or *boundary* is the union over all ordinals β of the set of those points in $\omega_f(a)$ eliminated at stage β of the iteration:

$$E(a,f) \ =_{\mathrm{df}} \ \bigcup_{\beta}\big(A^{\beta}(a,f) \smallsetminus A^{\beta+1}(a,f)\big).$$

Here $X \smallsetminus Y$ is the set-theoretic difference $\{x \mid x \in X \text{ and } x \notin Y\}$.

DEFINITION For $x \in E(a,f)$, we write $\beta(x,a,f)$ for the unique β with $x \in A^{\beta}(a,f) \smallsetminus A^{\beta+1}(a,f)$.

Thus if we make the following

DEFINITION $\theta(a, f) =_{\mathrm{df}}$ the least ordinal θ with $A^{\theta}(a, f) = A^{\theta+1}(a, f)$,

we know that $\theta(a, f)$ is always well defined and at most ω_1. Further for all $\delta \geqslant \theta$, $A^{\delta}(a, f) = A^{\theta}(a, f)$.

DEFINITION We write $A(a, f)$ for this final set $A^{\theta(a,f)}(a, f)$. We call $A(a, f)$ the *abode*, and the ordinal $\theta(a, f)$ the *score* of the point a under f.

Thus $E(a, f) = \omega_f(a) \setminus A(a, f)$. We say that points in $A(a, f)$ *abide*, and points in $E(a, f)$ *escape*.

The question raised in 1993 was to investigate the possible behaviour of the function $\theta(a, f)$: what are its possible values ?

References

[**1**] Ll. Alsedà, M. Chas and J. Smítal. On the structure of the ω-limit sets for continuous maps of the interval. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **9** (1999), no. 9, 1719–1729. MR 2000**i**:37047

[**2a**] A. R. D. Mathias. Delays, recurrence and ordinals. *Proc. London Math. Soc.* (3) **82** (2001) 257–298.

[**2b**] A. R. D. Mathias, Recurrent points and hyperarithmetic sets, in *Set Theory, Techniques and Applications*, Curaçao 1995 and Barcelona 1996 conferences, edited by C. A. Di Prisco, Jean A. Larson, Joan Bagaria and A. R. D. Mathias, Kluwer Academic Publishers, Dordrecht, Boston, London, 1998, 157–174.

Linking escape to well-foundedness

The slides of this section are taken from §2 of *Delays*.

We introduce the trees we shall use to calculate $\beta(b)$ for $b \in E(a, f)$. We shall define for our fixed a and for each $b \in \mathcal{X}$ a tree T^a_b of finite sequences and show using DC that $b \in A(a, f) \iff T^a_b$ is ill-founded.

2·0 Definition For $b \in \mathcal{X}$, set

$$T^a_b \ =_{\rm df} \ \big\{ s \in {}^{<\omega}\mathcal{X} \ \big| \ \ell h(s) > 0 \Longrightarrow \big(s(0) = b \ \& \\ \forall i :< \ell h(s) \ (a \curvearrowright s(i)) \ \& \\ \forall i :< \ell h(s) - 1 \ (a_{i+1} \curvearrowright a_i) \big) \big\}.$$

Note that if $t \succ s \in T^a_b$, then $t \in T^a_b$, so that T^a_b is closed under shortening. Our definition is of most interest when $b \in \omega_f(a)$, since

$$b \notin \omega_f(a) \iff T^a_b = \{\circledcirc\}.$$

2·1 LEMMA (DC) *$b \in A(a,f) \iff \exists$ an infinite sequence $\langle x_i | i < \omega\rangle$ such that $\forall i_{\in\omega}\, a \curvearrowright x_i$ and*

$$b = x_0 \curvearrowleft x_1 \curvearrowleft x_2 \curvearrowleft \dots \quad .$$

Proof : given such a sequence, one checks easily by induction on ξ that each of its members is in $A^\xi(a,f)$, hence is in $A(a,f)$; in particular $b = x_0$ is in $A(a,f)$. If no such sequence exists for a given b, then by DC the tree T^a_b will be well-founded under $\prec$, and hence we may define a rank function $\varrho = \varrho^a_b$ mapping T^a_b to the ordinals by

$$\varrho^a_b(s) = \sup\{\varrho^a_b(s^\frown\langle r\rangle) + 1 \mid r \in \mathcal{X} \ \&\ s^\frown\langle r\rangle \in T^a_b\}.$$

and show by induction on ξ that $\varrho^a_b(s) = \xi \implies \ell(s) \notin A^{\xi+1}(a,f)$: hence $b \notin A^{\varrho^a_b(\langle b\rangle)+1}(a,f)$.

2·2 COROLLARY (DC) *For* $b \in \omega_f(a)$, $b \in E(a, f) \iff T^a_b$ *is well-founded.*

2·3 PROPOSITION *For each* $b \in E(a, f)$, $\varrho^a_b(\langle b \rangle) < \omega_1$.

2·4 COROLLARY $\theta \leqslant \omega_1$

Proof : Each b in $E(a, f)$ leaves the A-sequence at stage $\varrho^a_b(\langle b \rangle) + 1$, which is countable. Hence by stage ω_1 all those points that are to escape have already done so. ⊣ (2·4)

Background reading

[**3**] Y. N. MOSCHOVAKIS. *Descriptive set theory.* (North Holland, 1980).

[**4**] A. S. KECHRIS. *Classical descriptive set theory.* Graduate Texts in Mathematics 156, (Springer, 1995).

Points at the end of a path

PROPOSITION *Let f be a continuous map of a Polish space $\mathcal{X}$ into itself, and suppose that we have an infinite sequence of points b_i, with $b_0 \curvearrowleft_f b_1 \curvearrowleft_f b_2 \ldots \curvearrowleft_f b$. Then we can choose integers n_i, (increasing if we wish), such that putting $y_i = f^{n_i}(b_i)$, the y_i form a Cauchy sequence converging to a point y with $b \curvearrowright_f y \curvearrowright_f y \curvearrowright_f b_i$ for each i.*

Proof : in these circumstances $f^n(b_j) \curvearrowright b_i$ for $j > i$ and arbitrary n. ⊣

That lends interest to the following definition:

DEFINITION Let $b_0 \curvearrowleft_f b_1 \curvearrowleft_f b_2 \ldots$ be an infinite path descending in the relation $\curvearrowright_f$. We say that a point y *lies at the end of the path* if it satisfies two conditions:

(i) there are numbers n_i such that $y = \lim_{i\to\infty} f^{n_i}(b_i)$;

(ii) for each i, $y \curvearrowright_f b_i$.

PROPOSITION *If both y and z are at the end of the same path, then $y \curvearrowright_f z \curvearrowright_f y$; in particular all points at the end of a given path are recurrent and attack each other.*

Proof : True because z attacks each b_i, hence attacks each $f^{n_i}(b_i)$; hence attacks y; and the situation is symmetric. $\dashv$

REMARK When, as here, $\mathcal{X} = \mathcal{Y}$ and $f = \mathfrak{s}$, the first condition will follow if one proves that to each ℓ there is a large i and an n_i with $y \restriction \ell \sqsubseteq \mathfrak{s}^{n_i}(b_i)$.

Building points of large countable score

Let $\gamma \in {}^{\omega}X$ for some set X. Let $(n_i)_i \in {}^{\omega}\omega$ be strictly increasing. Let $z_i = \gamma \restriction n_i$.

Let $Y = X \cup \{\mathsf{m}_i \mid i \in \omega\}$ where the markers m_i are distinct from each other and from all members of X.

Define $\beta \in {}^{\omega}Y$ by

$$\beta \ =_{\mathrm{df}} \ z_0 {}^\frown \langle \mathsf{m}_0 \rangle {}^\frown z_1 {}^\frown \langle \mathsf{m}_1 \rangle {}^\frown \ldots$$

Then $\beta \curvearrowright_{\mathfrak{s}} \gamma$ but $\gamma \not\curvearrowright_{\mathfrak{s}} \beta$.

For a detailed account see §4 of *Delays*;
for more general embeddings, see Delhommé [**6b**].

Preparations for a point of uncountable score

The slides of this section are taken from §3 of the paper

[**2c**] A. R. D. Mathias, Analytic sets under attack, *Math. Proc. Cambridge Phil. Soc.* **138** (2005) 465–485.

Finite trees and paths

We write $\ell h(u)$ for the length of a finite sequence u.

3·0 DEFINITION $\mathcal{F} =_{\mathrm{df}} \{u \mid u$ a non-empty finite sequence

$$\big(u(1), u(2), \ldots, u(\ell h(u))\big)$$

of natural numbers $u(i)$ with $0 \leqslant u(i) < i$ for $1 \leqslant i \leqslant \ell h(u)\}$.

3·1 REMARK Contrary to habitual practice among set theorists, the terms of u are indexed by $1, \ldots, \ell h(u)$ rather than $0, \ldots, \ell h(u) - 1$.

For $1 \leqslant k \leqslant \ell h(u)$ we write $u_{\leqslant k}$ for the sequence $\big(u(1), \ldots, u(k)\big)$; that will be an element of $\mathcal{F}$.

3·3 DEFINITION If $u = \big(u(1), u(2), \ldots, u(\ell h(u))\big) \in \mathcal{F}$, a *positive u-sequence* is a non-empty finite sequence $s = (p_1, \ldots, p_\ell)$ with $1 \leqslant p_1 < p_2 < \cdots < p_\ell \leqslant \ell h(u)$, so that $\ell = \ell h(s)$ and $p_\ell = \max s$; we further require that $u(p_1) = 0$, and for $1 \leqslant i < \ell h(s)$, $u(p_{i+1}) = p_i$.

The *u-sequences* are the positive u-sequences and the empty sequence, which we write as $\circledcirc$.

As above, we write $s_{\leqslant k}$ for the sequence $(p_1, \ldots, p_k)$, where $1 \leqslant k \leqslant \ell h(s)$; that too will be a positive u-sequence. Further, we interpret $s_{\leqslant 0}$ as the empty sequence, $\circledcirc$.

3·4 EXAMPLE If u is the sequence (0,0,2,1,0), the u-sequences are $\circledcirc$, (1), (2), (5), (1,4), and (2,3).

3·5 We shall build our point in a space of infinite sequences of *symbols*, of which there will be three kinds, *recorders*, *predictors* and *markers*. Certain symbols will contain information that is either an element u of $\mathcal{F}$—such symbols will be called *recorders*, because they contain information about the recent past of the infinite sequence of symbols under consideration—or else a pair of finite sequences s, u where $u \in \mathcal{F}$ and s is a positive u-sequence—such symbols will be called *predictors* because they contain information about the near future of that infinite sequence. Nothing is required of the third kind of symbol, the *markers*, save that there be a countable infinity of them and that they be all distinct from each other and from all recorders and predictors.

It is extremely important that, from the point of view of the shift function that we shall apply, each symbol is a single object; and, to give visual emphasis to that point, we shall use square brackets [,] to encase each individual symbol, whereas we shall use pointed brackets $\langle$, $\rangle$, to encase finite or infinite sequences of symbols.

We shall associate to each recorder and each predictor two natural numbers, its *weight* and its *height*.

3·6 DEFINITION A *recorder* is an object $[u]$ where u is in $\mathcal{F}$. Its *weight* is 0 and its *height* is the length $\ell h(u)$ of u as a member of $\mathcal{F}$.

3·7 DEFINITION A *predictor* is an object $[s; u]$ where $u \in \mathcal{F}$ and s is a positive u-sequence. s will be called the *path* of the predictor $[s; u]$, and u its *tree*. The predictor's *weight* is the length of its path, and its *height* is the length of its tree.

3·8 REMARK The weight of $[s; u]$ is not greater than its height.

3·9 DEFINITION We say that s is *tight* in u, or that u *tightly contains* s, if s is a u-sequence and $\max s = \ell h(u)$. In the contrary case we shall use the words *loose* and *loosely*. We may indeed define the *looseness of* u *over* s as $\ell h(u) - \max s$.

3·10 For each $u \in \mathcal{F}$ and each u-sequence s we shall define a finite sequence z_s^u of symbols. Our definition will proceed by a mode of induction that will also be used in proving our theorem, which we shall call *double induction*. To spell the method out in greater detail: we first consider the case $s = \circledcirc$. Then we suppose that $m \geqslant 1$ and that we have already treated all pairs u, s with s a u-sequence of length $< m$. On that supposition, we take an s of length m, and consider all $u \in \mathcal{F}$ for which s is a u-sequence, starting with those u for which $\ell h(u) = \max s$, and then progressively treating longer u; thus for given s we proceed by induction on the looseness of u over s. The following convention will be useful.

3·11 Definition We write s' for the sequence s with its last element removed—so that if s is of length 1, $s' = \circledcirc$—and we write u' for u with its last element removed.

We proceed to our definition of z_s^u by double induction, and first treat the case of $s = \circledcirc$.

3·12 Definition For $u \in \mathcal{F}$,

$$z_{\circledcirc}^u =_{\mathrm{df}} \big\langle [u_{\leqslant 1}], [u_{\leqslant 2}], \ldots, [u_{\leqslant \ell h(u)-1}], [u] \big\rangle.$$

3·13 Remark The length of $z_{\circledcirc}^u$ equals that of u.

3·14 Example

$$z_{\circledcirc}^{(0,0,2,1,0)} = \big\langle [(0)], [(0,0)], [(0,0,2)], [(0,0,2,1)], [(0,0,2,1,0)] \big\rangle$$

Now for $u \in \mathcal{F}$ and s a positive u-sequence we shall define z^u_s.

3·15 Definition

$$z^u_s =_{\mathrm{df}} \begin{cases} \langle [s;u] \rangle ^\frown z^u_{s'} & \text{if } \max s = \ell h(u); \\ z^{u'}_s {}^\frown \langle [s;u] \rangle ^\frown z^u_{s'} & \text{if } \max s < \ell h(u). \end{cases}$$

The first clause handles the case that u tightly contains s, and the second the cases when $\ell h(u)$ is strictly greater than $\max s$.

3·16 Remark Note that $[s;u]$ occurs only once in z^u_s; we shall refer to it as the *peak* of z^u_s. It is the only symbol in z^u_s with sum of weight and height equal to $\ell h(s) + \ell h(u)$.

We give several examples to illustrate that definition.

3·17 EXAMPLE If s is of length 1, then $z^u_s = \langle [s;u] \rangle^\frown z^u_\circledcirc$ if $\max s = \ell h(u)$ and $z^u_s = z^{u'}_s {}^\frown \langle [s;u] \rangle^\frown z^u_\circledcirc$ otherwise.

3·18 EXAMPLE If u is the sequence (0,0,2,1,0), then $z^u_{(5)}$ is

$$\langle [(5);(0,0,2,1,0)], [(0)], [(0,0)], [(0,0,2)], [(0,0,2,1)], [(0,0,2,1,0)] \rangle,$$

a sequence of six symbols, whereas $z^u_{(2)}$ is

$$\begin{aligned} \langle [(2);(0,0)], [(0)], [(0,0)], [(2);(0,0,2)], [(0)], [(0,0)], [(0,0,2)], \\ [(2);(0,0,2,1)], [(0)], [(0,0)], [(0,0,2)], [(0,0,2,1)], \\ [(2);(0,0,2,1,0)], [(0)], [(0,0)], [(0,0,2)], [(0,0,2,1)], [(0,0,2,1,0)] \rangle, \end{aligned}$$

which has eighteen, of which the heights, in order, are 2, 1, 2; 3, 1, 2, 3; 4, 1, 2, 3, 4; 5, 1, 2, 3, 4, 5.

$$
\begin{aligned}
z_{(1)}^{(0)} &= \Big\langle [(1);(0)],[(0)] \Big\rangle; \\
z_{(1)}^{(0,0)} &= \Big\langle [(1);(0)],[(0)],[(1);(0,0)],[(0)],[(0,0)] \Big\rangle; \\
z_{(1)}^{(0,0,2)} &= \Big\langle [(1);(0)],[(0)],[(1);(0,0)],[(0)],[(0,0)], \\
&\qquad\quad [(1);(0,0,2)],[(0)],[(0,0)],[(0,0,2)] \Big\rangle; \\
z_{(1)}^{(0,0,2,1)} &= \Big\langle [(1);(0)],[(0)],[(1);(0,0)],[(0)],[(0,0)], \\
&\qquad\quad [(1);(0,0,2)],[(0)],[(0,0)],[(0,0,2)], \\
&\qquad [(1);(0,0,2,1)],[(0)],[(0,0)],[(0,0,2)],[(0,0,2,1)] \Big\rangle; \\
z_{(1)}^{(0,0,2,1,0)} &= \Big\langle [(1);(0)],[(0)],[(1);(0,0)],[(0)],[(0,0)], \\
&\qquad\quad [(1);(0,0,2)],[(0)],[(0,0)],[(0,0,2)], \\
&\qquad [(1);(0,0,2,1)],[(0)],[(0,0)],[(0,0,2)],[(0,0,2,1)], \\
&\quad [(1);(0,0,2,1,0)],[(0)],[(0,0)],[(0,0,2)],[(0,0,2,1)],[(0,0,2,1,0)] \Big\rangle.
\end{aligned}
$$

$$
\begin{aligned}
z^{(0,0,2,1)}_{(1,4)} &= \big\langle [(1,4);(0,0,2,1)] \big\rangle ^\frown z^{(0,0,2,1)}_{(1)} \\
&= \big\langle [(1,4);(0,0,2,1)], \\
&\qquad\quad [(1);(0)], [(0)], [(1);(0,0)], [(0)], [(0,0)], \\
&\qquad\quad [(1);(0,0,2)], [(0)], [(0,0)], [(0,0,2)], \\
&\qquad\quad [(1);(0,0,2,1)], [(0)], [(0,0)], [(0,0,2)], [(0,0,2,1)] \big\rangle ;
\end{aligned}
$$

$$
\begin{aligned}
z^{(0,0,2,1,0)}_{(1,4)} &= z^{(0,0,2,1)}_{(1,4)} {}^\frown \big\langle [(1,4);(0,0,2,1,0)] \big\rangle {}^\frown z^{(0,0,2,1,0)}_{(1)} \\
&= \big\langle [(1,4);(0,0,2,1)], \\
&\qquad\quad [(1);(0)], [(0)], \\
&\qquad\quad [(1);(0,0)], [(0)], [(0,0)], \\
&\qquad\quad [(1);(0,0,2)], [(0)], [(0,0)], [(0,0,2)], \\
&\qquad\quad [(1);(0,0,2,1)], [(0)], [(0,0)], [(0,0,2)], [(0,0,2,1)], \\
&\qquad [(1,4);(0,0,2,1,0)], \\
&\qquad\quad [(1);(0)], [(0)], \\
&\qquad\quad [(1);(0,0)], [(0)], [(0,0)], \\
&\qquad\quad [(1);(0,0,2)], [(0)], [(0,0)], [(0,0,2)], \\
&\qquad [(1);(0,0,2,1)], [(0)], [(0,0)], [(0,0,2)], [(0,0,2,1)], \\
&\quad [(1);(0,0,2,1,0)], [(0)], [(0,0)], [(0,0,2)], [(0,0,2,1)], [(0,0,2,1,0)] \big\rangle.
\end{aligned}
$$

$$
\begin{aligned}
z_{(2,3)}^{(0,0,2)} &= \langle [(2,3);(0,0,2)] \rangle^\frown z_{(2)}^{(0,0,2)}; \\
z_{(2,3)}^{(0,0,2,1)} &= z_{(2,3)}^{(0,0,2)\frown} \langle [(2,3);(0,0,2,1)] \rangle^\frown z_{(2)}^{(0,0,2,1)}; \\
z_{(2,3)}^{(0,0,2,1,0)} &= z_{(2,3)}^{(0,0,2,1)\frown} \langle [(2,3);(0,0,2,1,0)] \rangle^\frown z_{(2)}^{(0,0,2,1,0)} \\
&= z_{(2,3)}^{(0,0,2)\frown} \langle [(2,3);(0,0,2,1)] \rangle^\frown z_{(2)}^{(0,0,2,1)\frown} \langle [(2,3);(0,0,2,1,0)] \rangle^\frown z_{(2)}^{(0,0,2,1,0)}
\end{aligned}
$$

which equals

$$
\begin{aligned}
\Big\langle & [(2,3);(0,0,2)], \\
& \qquad [(2);(0,0)],[(0)],[(0,0)], \\
& \qquad [(2);(0,0,2)],[(0)],[(0,0)],[(0,0,2)], \\
& [(2,3);(0,0,2,1)], \\
& \qquad [(2);(0,0)],[(0)],[(0,0)], \\
& \qquad [(2);(0,0,2)],[(0)],[(0,0)],[(0,0,2)], \\
& \qquad [(2);(0,0,2,1)],[(0)],[(0,0)],[(0,0,2)],[(0,0,2,1)], \\
& [(2,3);(0,0,2,1,0)], \\
& \qquad [(2);(0,0)],[(0)],[(0,0)], \\
& \qquad [(2);(0,0,2)],[(0)],[(0,0)],[(0,0,2)], \\
& \qquad [(2);(0,0,2,1)],[(0)],[(0,0)],[(0,0,2)],[(0,0,2,1)], \\
& \qquad [(2);(0,0,2,1,0)],[(0)],[(0,0)],[(0,0,2)],[(0,0,2,1)],[(0,0,2,1,0)]\Big\rangle.
\end{aligned}
$$

3·19 EXAMPLE Suppose that $3 + \max t = \ell h(v)$. Let $v_i = v_{\leqslant i+\max t}$, so that $v_0 = v_{\leqslant \max t}$ and $v_3 = v$.

Then z_t^v is

$$\langle [t; v_0] \rangle^\frown z_{t'}^{v_0} {}^\frown \langle [t; v_1] \rangle^\frown z_{t'}^{v_1} {}^\frown \langle [t; v_2] \rangle^\frown z_{t'}^{v_2} {}^\frown \langle [t; v] \rangle^\frown z_{t'}^{v},$$

which has precisely the four predictors shown of weight equal to the length of t; all other predictors in z_t^v will be of lesser weight.

Here is a first example of proof by double induction:

3·20 PROPOSITION *If s is not $\circledcirc$, then the first symbol of z_s^u is the predictor $[s; u_{\leqslant \max s}]$.*

Proof : If u tightly contains s, $z_s^u = \langle [s; u] \rangle ^\frown z_{s'}^u$ of which the first symbol is $[s; u]$, which equals $[s; u_{\leqslant \max s}]$. Otherwise $z_s^u = z_s^{u'} {}^\frown \langle [s; u] \rangle ^\frown z_{s'}^u$, of which the first symbol is that of $z_s^{u'}$, which, by the induction hypothesis, is the predictor $[s; u'_{\leqslant \max s}]$; but that in the context equals $[s; u_{\leqslant \max s}]$.

⊣ (3·20)

Notation for finite sequences

3·21 DEFINITION $t \preccurlyeq s \iff_{\mathrm{df}} t$ is an extension of s; $t \prec s \iff_{\mathrm{df}} t$ is an proper extension of s; $s \succcurlyeq t \iff_{\mathrm{df}} s$ is an initial segment of t; $s \succ t \iff_{\mathrm{df}} s$ is a proper initial segment of t.

3·22 REMARK Thus $s \succcurlyeq t \iff t \preccurlyeq s$, and so on. $\circledcirc$ has no proper initial segments, but is itself a proper initial segment of every finite sequence of positive length. Note that longer sequences are lower in this ordering.

3·23 DEFINITION We shall say that two finite sequences s and t *cohere* if either $s \succcurlyeq t$ or $t \succcurlyeq s$.

Properties of finite sequences

3·24 PROPOSITION *Let u and v be members of $\mathcal{F}$, and let t be both an u-sequence and a v-sequence.*

(i) $\ell h(u) = \ell h(z^u_{\circledcirc})$;

(ii) *for* $\ell \leqslant \ell h(v)$, $z^v_{\circledcirc} \restriction \ell = z^{v \restriction \ell}_{\circledcirc}$;

(iii) $v \prec u \Longrightarrow z^v_t \prec z^u_t$;

(iv) $z^v_t = z^u_t \Longrightarrow v = u$;

(v) $z^v_t \prec z^u_t \Longrightarrow v \prec u$.

Proof of 3·24 (iii): If $t = \circledcirc$, use (ii): otherwise use an earlier instance to note that $z_t^v \prec z_t^{v'} \preccurlyeq z_t^u$.
Proof of 3·24 (iv): Compare peaks.
Proof of 3·24 (v): The peak of z_t^v cannot be in z_t^u, for otherwise $u = v$; whence $z_t^u \succcurlyeq z_t^{v'}$, giving, inductively, $v' \preccurlyeq u$.

3·25 DEFINITION An *m-predictor* is a predictor of weight exactly m. An *m-stretch* is a finite sequence of symbols all of weight at most m.

3·26 LEMMA *Let* $u \in \mathcal{F}$, s *a* u*-sequence of weight* $> m$. *Let* $x \sqsubseteq z^u_s$ *be an* m*-stretch.*

(i) $x \sqsubseteq z^u_{s'}$;

(ii) *in fact* $x \sqsubseteq z^u_{s_{\leqslant m}}$.

Proof of 3·26 (i)*:* Its weight forbids the peak of z^u_s to lie in x.

Case 1: s is tight in u. Then $z^u_s = \langle [s;u] \rangle ^\frown z^u_{s'}$, whence $x \sqsubseteq z^u_{s'}$.

Case 2: otherwise. Then $z^u_s = z^{u'}_s {}^\frown \langle [s;u] \rangle ^\frown z^u_{s'}$, so either $x \sqsubseteq z^{u'}_s$ or $x \sqsubseteq z^u_{s'}$; if the second alternative is false, we may iterate the first, progressively shortening u till it does tightly contain s, and then apply Case 1. ⊣ (3·26·i)

Proof of 3·26 (ii)*:* By iterating Lemma 3·26 (i), progressively shortening s. ⊣ (3·26·ii)

Indeed we can sharpen that result:

3·27 PROPOSITION *Let x be an m-stretch with all symbols of height at most h. Suppose that $x \sqsubseteq z^u_s$. Then $x \sqsubseteq z^{u_{\leqslant h}}_{s_{\leqslant m}}$.*

Proof : For fixed x by double induction on s and u. If the peak of z^u_s occurs in x, then both the height and weight of x equal those of z^u_s, and then the proposition is trivially true. Otherwise $x \sqsubseteq z^{u'}_s$ or $x \sqsubseteq z^u_{s'}$; in the first case the height is less and in the second the weight. In either case we have a reduction to an earlier instance of the induction. ⊣ (3·27)

3·28 LEMMA *The recorders in z^u_s are those in $z^u_{\circledcirc}$: namely non-empty initial segments of u. Hence any two recorders in z^u_s cohere.*

Proof : By applying Proposition 3·27 to 0-stretches of length 1. ⊣ (3·28)

3·29 Lemma *If* $s \succcurlyeq t$ *and* t *is a* u*-sequence, then* z^u_s *is a final segment of* z^u_t*; if* $s \succ t$*, that final segment is immediately preceded by the predictor* $[s^+; u]$*, where* $s^+ = t_{\leqslant \ell h(s)+1}$.

Proof : Write $t_0 = t$, and progressively write $t_{k+1} = t'_k$ till we reach $t_n = s$. If $n = 0$ the Lemma is trivial; if $n > 0$, then we remark that for each k, $z^u_{t_k}$ ends in $z^u_{t_{k+1}}$ which is preceded by $[t_k; u]$; finally note that $t_{n-1} = t_{\leqslant \ell h(s)+1}$. ⊣ (3·29)

3·30 Lemma *if* $u \succcurlyeq v$ *and* s *is a* u*-sequence, then* $z^u_s \succcurlyeq z^v_s$*; if* $u \succ v$*, the term in* z^v_s *after that occurrence of* z^u_s *is* $[s; u^+]$*. where* $u^+ = v_{\leqslant \ell h(u)+1}$.

Proof : The first part is Proposition 3·24 (iii) rephrased; the second part holds if $v' = u$, and stays true for longer v by an easy induction, as then $u \succ v' \succ v$. ⊣ (3·30)

3·31 LEMMA *If* $[s; u]$ *occurs in* z_t^v *then* $s \succcurlyeq t$ *and* $u \succcurlyeq v$.

Proof : By a double induction on t and v. The lemma is true if $[s; u] = [t; v]$. Otherwise $[s; u]$ occurs in $z_{t'}^v$ or, provided t is loose in v, in $z_t^{v'}$; in either case we have a reduction to an earlier instance of the induction, to which we then link either the fact that $t' \succ t$ or that $v' \succ v$. ⊣ (3·31)

3·32 LEMMA *An occurrence of* $[s; u]$ *in* z_t^v *is followed by the whole of* $z_{s'}^u$.

Proof : By a similarly structured induction on t and v. ⊣ (3·32)

3·33 LEMMA *In any* z_s^u *the immediate successor of an* m*-predictor is a symbol of weight* $m - 1$.

Proof : Immediate from the definition if $m = 1$; by Proposition 3·20 otherwise. ⊣ (3·33)

3·34 LEMMA *If s is of length $m+1$, $\langle[s;u]\rangle^\frown x$ is a final segment of z_s^w and x is an m-stretch, then $u=w$ and $x=z_{s'}^u$.*

Proof : $[s;w]$ is the last symbol of weight $m+1$ in z_s^w. ⊣ (3·34)

3·35 PROPOSITION *If s is of length $m+1$, x is an m-stretch, and $y =_{\mathrm{df}} \langle[s;u]\rangle^\frown x^\frown\langle[s;v]\rangle \sqsubseteq z_r^w$, then $u=v'$ and $x=z_{s'}^u$.*

Proof by double induction: By Proposition 3·27, we can suppose $r=s$. If $v\neq w$, we have $z_s^w = z_s^{w'}{}^\frown\langle[s;w]\rangle^\frown z_{s'}^w$ and therefore $y\sqsubseteq z_s^{w'}$; thus we may reduce the length of w until $w=v$.

So our proposition is now reduced to the case that $y \sqsubseteq z_s^v$. We then have

$$\langle [s;u] \rangle ^\frown x ^\frown \langle [s;v] \rangle \sqsubseteq z_s^{v'} {}^\frown \langle [s;v] \rangle ^\frown z_{s'}^v;$$

since $[s;v]$ occurs in neither $z_s^{v'}$ nor in $z_{s'}^v$, we may be sure that the last symbol of y occurs as the peak of z_s^v; but then $\langle [s;u] \rangle ^\frown x$ forms a final segment of $z_s^{v'}$, so we may apply Lemma 3·34 to infer that $u = v'$ and $x = z_{s'}^u$. ⊣ (3·35)

3·36 Corollary *If* $y = \langle [s; u_1] \rangle^\frown x_1{}^\frown \langle [s; u_2] \rangle^\frown x_2{}^\frown \langle [s; u_3] \rangle \sqsubseteq z_r^w$, *where* s *is of length* $m + 1$ *and both* x_1 *and* x_2 *are* m*-stretches, then* $x_1 \succ x_2$*, and* $\ell h(u_2) = \ell h(u_1) + 1$*.*

Proof : In the circumstances, $x_1 = z_{s'}^{u_1}$, $x_2 = z_{s'}^{u_2}$, and $u_1 = (u_2)'$.

⊣ (3·36)

3·37 Lemma *If* s *is of length* $m+1$*,* x *is an* m*-stretch, and* $x^\frown \langle [s; v] \rangle \sqsubseteq z_t^w$*, then* x *is a final segment of* $z_s^{v'}$*.*

Proof : The hypotheses imply, by Proposition 3·27, that $x^\frown \langle [s; v] \rangle \sqsubseteq z_s^v$, in which the only occurrence of $[s; v]$ is the peak; but then x must be a final segment of the preceding sequence, which is $z_s^{v'}$. ⊣ (3·37)

3·38 LEMMA *If the recorder* $[e]$, *of height at least 2, occurs in* z_s^u, *its predecessor is* $[e_{\leqslant \ell h(e)-1}]$; *if of height 1, its predecessor, if any, will be a predictor of weight 1.*

3·39 PROPOSITION *If* $z_s^u(i)$ *and* $z_s^u(i{+}1)$ *are both recorders then* $\ell h(z_s^u(i+1)) = 1 + \ell h(z_s^u(i))$.

3·40 REMARK The unique longest m-stretch in z_s^u is at the end, namely $z^u_{s_{\leqslant m}}$: for if s is of weight m, z_s^u is itself an m-stretch; and if s is of greater weight, the m-stretches in z_s^u are those of $z^u_{s'}$ and, provided s is loose in u, of $z_s^{u'}$. By induction, the unique longest of those are $z^u_{s_{\leqslant m}}$ and $z^{u'}_{s_{\leqslant m}}$, of which two the first is in any case strictly longer. ⊣ (3·40)

3·41 PROPOSITION *Suppose that* $x =_{\mathrm{df}} \langle [s;u] \rangle^\frown z^u_{s'} \sqsubseteq z^w_r$ *but is not a final segment thereof. Then the first symbol after the segment* x *of* z^w_r *is of the form* $[t;v]$ *where* $v' = u$ *and* $t \preccurlyeq s$*, and if* $t \prec s$ *there will be a later occurrence in* z^w_r *of a symbol of weight that of* s*.*

3·42 REMARK $\langle [s;u] \rangle^\frown z^u_{s'}$ is a final segment of z^u_s, properly so if and only if s is loose in u.

Towards the proof of Proposition 3·41, we first prove a Lemma to cover the case $s = r$.

3·43 LEMMA $x =_{\mathrm{df}} \langle [s;u] \rangle^\frown z^u_{s'}$ *is a final segment of* z^w_s *if and only if* $u = w$*.*

Proof : One way is covered by Remark 3·42. For the other, since $z^w_s = z^{w'}_s {}^\frown \langle [s;w] \rangle^\frown z^w_{s'}$, the peak of z^w_s is its last symbol of weight $\ell h(s)$ and therefore if x is a final segment of z^w_s, the first symbol of x must be that peak, whence $z^u_{s'} = z^w_{s'}$, whence $u = w$. ⊣ (3·43)

Proof of Proposition 3·41*:* We consider s and u to be fixed and do a double induction on r and w.

As always, we have

$$z_r^w = z_r^{w'} {}^\frown \langle [r; w] \rangle {}^\frown z_{r'}^w$$

The hypotheses imply that $r \preccurlyeq s$ and, by Lemma 3·43, that $w \prec u$; hence the peak of z_r^w cannot lie in x, and therefore either $x \sqsubseteq z_{r'}^w$ or $x \sqsubseteq z_r^{w'}$.

If $x \sqsubseteq z_{r'}^w$, then x will not be a final segment of $z_{r'}^w$, and so the induction will apply.

If $x \sqsubseteq z_r^{w'}$, either $w' \prec u$, whence by Lemma 3·43 x is not final in $z_r^{w'}$, and the induction will again apply; or $w' = u$, x is final—again by Lemma 3·43—in $z_r^{w'}$ and the next symbol is $[r; w]$, which is of the desired form $[t; v]$ with $v' = u$ and $t \preccurlyeq s$.

The final clause follows from Lemma 3·33. ⊣ (3·41)

3·44 PROPOSITION *In any z_s^u, if the same symbol, of weight m, occurs twice, then between the two occurrences there must be an occurrence of a symbol of weight $m + 1$.*

Proof by double induction: The indicated symbol, that which repeats, cannot be the peak of z_s^u, which occurs only once there.

If s is tight in u, the two occurrences must both be in $z_{s'}^u$, and we have reduced to an earlier case.

Otherwise $z_s^u = z_s^{u'} \cap \langle [s; u] \rangle \cap z_{s'}^u$, and there are three possibilities: both occurrences are before the peak, when both lie in $z_s^{u'}$; both lie after, and therefore both lie in $z_{s'}^u$—both times we have a reduction to an earlier case—or one lies before the peak and the other after; but then the proposition is proved, for the peak is of weight greater than m, and, if of weight $> m + 1$, will by Lemma 3·33 immediately be followed by symbols of weights declining by 1 at each step, thus reaching a symbol of weight $m + 1$ before the second occurrence of the indicated symbol. ⊣ (3·44)

The remaining slides are taken from §4 of *Analytic sets under attack*.

Introducing infinite sequences

We have introduced two of our three kinds of symbol. For the third, the *markers*, we take infinitely many objects $[\mathsf{m}_0], [\mathsf{m}_1], \dots$ distinct from each other and from all recorders and predictors.

We define $\mathcal{Y}$ to be the space of all sequences of length ω of symbols. Here we return to normal set-theoretic convention by considering the domain of such sequences to be $\omega = \{0, 1, 2, \dots\}$.

On $\mathcal{Y}$ we may define the shift function, which we again denote by $\mathfrak{s}$: $\mathfrak{s}(\zeta)(n) = \zeta(n+1)$ for $n \geqslant 0$.

As in section 4 of *Delays* we write $\zeta \triangleright \xi$, read ζ *is near to* ξ, if $\zeta = \mathfrak{s}^n(\xi)$ for some $n \geqslant 0$.

4·0 Definition The *weight* of a point ζ of $\mathcal{Y}$ is the supremum of the weight of its predictors: thus either a natural number or ∞. The *height* of a point $\zeta \in \mathcal{Y}$ is the supremum of the height of its recorders and predictors: again either a natural number or ∞.

Introducing the real b

At last we are in a position to define our point b, which will lie in the space $\mathcal{Y}$.

4·1 Definition Enumerate all sequences z_s^u where $u \in \mathcal{F}$ and s is a u-sequence, in some recursive fashion as z_i $(i = 0, 1, \ldots)$.

Define

$$b \;=_{\mathrm{df}}\; z_0{}^\frown\langle[\mathsf{m}_0]\rangle^\frown z_1{}^\frown\langle[\mathsf{m}_1]\rangle^\frown \ldots$$

4·2 Theorem $\theta(b, \mathfrak{s}) = \omega_1$.

To classify the points of $\mathcal{Y}$ attacked by b, we shall use the infinite trees to which the members of $\mathcal{F}$ are codes of finite approximations.

Introducing infinite trees

4·3 DEFINITION $\mathcal{T} =_{\text{df}} \{\tau : \mathbb{N}^+ \longrightarrow \mathbb{N} \mid \text{ for all } n \geqslant 1,\ 0 \leqslant \tau(n) < n\}$

4·4 REMARK With the product topology of discrete finite spaces, $\mathcal{T}$ is a compact space.

4·5 REMARK If one regards $\mathcal{F}$ as a tree, $\mathcal{T}$ is the set of all infinite paths through it.

4·6 DEFINITION For $\tau \in \mathcal{T}$, a (*positive*) τ*-sequence* is a (non-empty) finite sequence of positive integers $p_1 < \cdots < p_k$ with $\tau(p_1) = 0$ and $\tau(p_{n+1}) = p_n$ for each $1 \leqslant n < k$. Thus $\circledcirc$ is a τ-sequence. A τ*-path* is an infinite sequence $\pi = (p_1, p_2, \ldots)$ with $\tau(p_1) = 0$ and $\tau(p_{n+1}) = p_n$ for each $n \geqslant 1$. For such π we write $\pi_{\leqslant k}$ for its initial segment $(p_1, p_2, \ldots, p_k)$, where $k \geqslant 1$.

We speak of τ as *well-founded* if there are no τ-paths: *ill-founded* if there are.

4·7 Remark We may regard each $\tau \in \mathcal{T}$ as coding a tree, of which the top point is 0 and $m <_\tau n$ if m is not 0 and for some $\ell > 0$, $\tau^\ell(m) = n$.

4·8 Remark Every countably infinite tree T of finite sequences under end-extension is coded by some $\tau \in \mathcal{T}$. To see that, partition ω into infinitely many infinite sets X_i. List the members of T as $v_0, v_1, v_2, \ldots$ We define a *first assignment* λ of natural numbers to members of T by induction on the length of each member as a finite sequence.

Assign 0 to the top point $\circledcirc$ of T. Once a natural number $\lambda(v_i)$ has been assigned to v_i, assign distinct members of $X_i \setminus \{m \mid m \leqslant \lambda(v_i)\}$ to the immediate extensions of v_i. Let $\mu : \mathrm{Im}\,(\lambda) \cong \omega$ be the order-preserving bijection of the set of all natural numbers used in the first assignment λ, so that $\mu \circ \lambda$ is a bijection between T and ω, which is the *final assignment*; let χ be its inverse.

Now set $\tau(n)$ to be the m such that $\chi(m) = \chi(n)'$. Then $\tau \in \mathcal{T}$, and $(\omega, <_\tau) \cong (T, \prec)$.

Properties of infinite sequences attacked by b

4·9 LEMMA *If the recorder* $[e]$, *of height at least 2, occurs in some* ζ *attacked by* b, *its predecessor in* ζ *is* $[e_{\leqslant \ell h(e)-1}]$; *if of height 1, its predecessor, if any, in* ζ *will be a predictor of weight 1.*

Proof : By Lemma 3·38. ⊣ (4·9)

4·10 PROPOSITION *If* $\zeta(i)$ *is a recorder then* $\zeta(i+1)$, *if a recorder, is of height one more than* $\zeta(i)$.

4·11 LEMMA *If* $b \curvearrowright_{\mathfrak{s}} \xi$, ξ *contains no markers: hence to each* ℓ *there are* u *and* s *with* $\xi \restriction \ell \sqsubseteq z^u_s$.

Proof : No marker occurs twice in b. ⊣ (4·11)

4·12 LEMMA *Any two recorders, d and e, in ξ cohere.*

Proof : Pick ℓ with both d and e occurring in $\xi \restriction \ell$, and let $\xi \restriction \ell \sqsubseteq z_t^v$. Then by Lemma 3·28 both d and e are initial segments of v. ⊣ (4·12)

4·13 LEMMA *If $b \curvearrowright \zeta$ and an m-predictor occurs in ζ, then m-predictors occur infinitely often in ζ.*

Proof : By Lemma 3·32 and Proposition 3·41.

4·14 PROPOSITION *If $b \curvearrowright_{\mathfrak{s}} \zeta$ then the height of ζ is ∞.*

Proof by cases, according to the weight of ζ: If ζ is of weight 0, then we use Proposition 4·10.

If on the other hand ζ is of positive finite weight, m, we consider the sequence of m-predictors in ζ. By Proposition 3·41, their height increases by one each time. Hence the $z_{\circledcirc}^u$'s that ζ contains are of unbounded length.

Finally, if ζ is of infinite weight, then by Remark 3·8 it must also be of infinite height. Hence it contains recorders of every height. ⊣ (4·14)

4·15 Thus if $b \curvearrowright_{\mathfrak{s}} \zeta$, ζ has recorders of unbounded height; they cohere to define a tree, which we shall call τ_ζ, in $\mathcal{T}$. This tree is uniquely determined by ζ; by the coherence property, Lemma 4·12, no $u \in \mathcal{F}$ other than the initial segments of τ_ζ may occur in ζ.

Points of finite weight attacked by b

We proceed to give an exact description of the points of finite weight attacked by b.

4·16 DEFINITION For $\tau \in \mathcal{T}$ and s a τ-sequence, set

$$\xi_s^\tau =_{\mathrm{df}} \bigcup_{k \geqslant \max s} z_s^{\tau \restriction k}$$

which will be a member of our symbol space $\mathcal{Y}$.

4·17 EXAMPLE $\xi_{\circledcirc}^\tau = \langle [\tau_{\leqslant 1}], [\tau_{\leqslant 2}], [\tau_{\leqslant 3}], \ldots, [\tau_{\leqslant k-1}], [\tau_{\leqslant k}], [\tau_{\leqslant k+1}], \ldots \rangle$, which has no predictors.

4·18 Example Suppose that s is a positive τ-sequence with $\max s = 5$. Then $\xi^\tau_s =$

$$\left\langle [s;\tau_{\leqslant 5}] \right\rangle ^\frown z_{s'}^{\tau_{\leqslant 5}} {}^\frown \left\langle [s;\tau_{\leqslant 6}] \right\rangle ^\frown z_{s'}^{\tau_{\leqslant 6}} {}^\frown \left\langle [s;\tau_{\leqslant 7}] \right\rangle ^\frown z_{s'}^{\tau_{\leqslant 7}} {}^\frown \ldots {}^\frown \left\langle [s;\tau_{\leqslant k}] \right\rangle ^\frown z_{s'}^{\tau_{\leqslant k}} {}^\frown \ldots$$

which has infinitely many predictors of weight 5 but none of weight 6 or more.

4·19 Remark The tree defined by ξ^τ_s equals τ.

Points of weight nought attacked by b

4·20 Proposition *For each* $\tau \in \mathcal{T}$, $b \curvearrowright_{\mathfrak{s}} \xi^{\tau}_{\circledcirc}$; *each* $\xi^{\tau}_{\circledcirc}$ *is of weight nought; no* γ *near* $\xi^{\tau}_{\circledcirc}$ *attacks itself.*

Proof : The first part holds since each $z^{u}_{\circledcirc}$ occurs infinitely often as a segment of b; the second is plain; and the third holds because no recorder occurs twice in any $\xi^{\tau}_{\circledcirc}$. ⊣ (4·20)

4·21 Proposition *If* $b \curvearrowright_{\mathfrak{s}} \zeta$ *and* ζ *is of weight 0, then* $\zeta \triangleright \xi^{\tau_\zeta}_{\circledcirc}$.

Proof : By Proposition 4·10, if $\zeta(0)$ is of height k then $\zeta = \mathfrak{s}^{k-1}(\xi^{\tau_\zeta}_{\circledcirc})$. ⊣ (4·21)

Points of positive finite weight attacked by b

4·22 PROPOSITION *For each positive τ-sequence s, $b \curvearrowright_{\mathfrak{s}} \xi^\tau_s$; each ξ^τ_s is of finite weight equal to $\ell h(s)$; and no γ near ξ^τ_s attacks itself.*

Proof of the last part: No predictor of weight $\ell h(s)$ occurs twice in ξ^τ_s.

⊣ (4·22)

4·23 Lemma *If the weight of* ζ*, attacked by* b*, is bounded, let* m *be the largest weight of a predictor occurring in* ζ*. Then:*

(i) ζ *has infinitely many predictors of weight* m;

(ii) *there is a unique sequence* s_ζ *of length* m *such that every predictor of weight* m *occurring in* ζ *is of the form* $[s_\zeta; v]$ *for some* $v \in \mathcal{F}$ *with* v *an initial segment of* τ_ζ *and* s_ζ *a* v*-sequence;*

(iii) *to each* ℓ *there are* $u \succ \tau_\zeta$ *and* $t \succcurlyeq s_\zeta$ *with* $\zeta \restriction \ell \sqsubseteq z^u_t$ *and the two stretches* $\zeta \restriction \ell$ *and* z^u_t *having the same height and weight.*

Proof : The first part is just Lemma 4·13. The second part is a consequence of the principle of coherence. The third follows from Proposition 3·27. ⊣ (4·23)

4·24 PROPOSITION *If* $b \curvearrowright_{\mathfrak{s}} \zeta$ *and* ζ *is of finite weight* $m > 0$, *then there is a unique* τ_ζ*-sequence* s_ζ, *of length* m, *such that* $\zeta \triangleright \xi^{\tau_\zeta}_{s_\zeta}$.

Proof : By comparing Lemma 4·23 with Example 4·18; in each case a segment $\langle [s;u] \rangle^\frown z^u_{s'}$ is promptly followed by a segment $\langle [s;v] \rangle^\frown z^v_{s'}$ where $v' = u$ and $s = s_\zeta$. The "missing" initial segment determines the shift required. $\dashv$ (4·24)

4·25 LEMMA *If* t *and* s *are* τ*-sequences with* $s = t'$, *then* $\xi^\tau_t \curvearrowright_{\mathfrak{s}} \xi^\tau_s$.

Proof : By examination of Example 4·18.

4·26 PROPOSITION *If* t *and* s *are* τ*-sequences with* $t \prec s$, *then* $\xi^\tau_t \curvearrowright_{\mathfrak{s}} \xi^\tau_s$.

Points of infinite weight attacked by b

Suppose that ζ, attacked by b, is of infinite weight. We know that if $[s;u]$ and $[t;v]$ occur in ζ then as they both lie in some z_r^w, u and v cohere, both being initial segments of w, and s and t cohere, both being initial segments of r. The union of the trees of the predictors in ζ will be the tree τ_ζ. The union of the paths of the predictors in ζ will be a τ_ζ-path that we shall call π_ζ. π_ζ is infinitely long because ζ is of infinite weight; hence τ_ζ is ill-founded.

Denote by s_k the τ_ζ-sequence $(\pi_\zeta)_{\leqslant k}$ and by γ_k the point $\xi^{\tau_\zeta}_{s_k}$ of $\mathcal{Y}$.

Plainly $\gamma_{k+1} \curvearrowright_{\mathfrak{s}} \gamma_k$ for each k. We wish to show that ζ lies at the end of the path $\gamma_0 \curvearrowleft_{\mathfrak{s}} \gamma_1 \curvearrowleft_{\mathfrak{s}} \ldots$ There are two things to be verified: that ζ is the limit of well-chosen finite shifts of the γ_k's, and that ζ attacks each γ_k.

First, given $\ell \in \omega$, let m be the greatest weight of any symbol occurring in the initial segment $\zeta \restriction \ell$, so that segment is an m-stretch. Then by Proposition 3·27 there are w and s such that $\zeta \restriction \ell \sqsubseteq z_s^w$ with w an initial segment of τ_ζ and s an initial segment of π_ζ, and therefore

$$\zeta \restriction \ell \sqsubseteq z^w_{s_{\leqslant m}} \sqsubseteq \gamma_m .$$

Thus $\zeta \restriction \ell$ will be an initial segment of an appropriate shift of γ_m.

Secondly, for given k and ℓ pick initial segments w and s of τ_ζ and π_ζ so that $\gamma_k \restriction \ell \sqsubseteq z_s^w$. Let $[r; u]$ be a predictor occurring as late in ζ as desired and of weight strictly exceeding the height of w. Then $u \prec w$ and $r \prec s$. Therefore

$$\gamma_k \restriction \ell \sqsubseteq z^w_s \sqsubseteq z^u_{r'} \sqsubseteq \zeta,$$

since $z^u_{r'}$ occurs as a segment of ζ immediately after the given occurrence of $[r; u]$.

Thus, remembering our remarks on slides 30 and 31 about points at the end of a path, we have proved:

4·30 Proposition *If* $b \curvearrowright_{\mathfrak{s}} \zeta$ *and* ζ *is of infinite weight, then there are unique* τ_ζ *and* π_ζ *defined by* ζ; τ_ζ *is ill-founded, and* ζ *lies at the end of the* τ_ζ*-path* π_ζ *and is therefore recurrent.*

So we have shown that all points of infinite weight attacked by b are recurrent. We now prove the converse.

4·31 PROPOSITION *If* $b \curvearrowright_{\mathfrak{s}} \rho \curvearrowright_{\mathfrak{s}} \rho$, *then* ρ *contains recorders and also contains predictors of every positive weight.*

Proof : ρ has no markers; hence for each ℓ there are u and s such that $\rho \restriction \ell \sqsubseteq z^u_s$. The immediate successor of a predictor of weight $m > 1$ will be a predictor of weight $m - 1$; the immediate successor of a predictor of weight 1 will be a recorder. Hence ρ must contain recorders.

Since ρ is recurrent, any symbol in it recurs infinitely often. We complete the proof by remarking, following Proposition 3·44, that between two occurrences of the same recorder, there must occur a predictor of weight one; and between two occurrences of the same predictor of weight m there must occur a predictor of weight $m + 1$. ⊣ (4·31)

Combining those two propositions yields this characterisation:

4·32 PROPOSITION *If* $b \curvearrowright_{\mathfrak{s}} \rho$, ρ *is recurrent if and only if it is of infinite weight.*

4·33 Example We illustrate the way in which recurrent points arise. Suppose that $\tau \in \mathcal{T}$ is ill-founded, and that π is an infinite τ-path. We choose strictly increasing integers n_k such that $\pi_{\leqslant k}$ is a $\tau_{\leqslant n_k - 1}$ path, so that $\pi(k) < n_k$ and $\pi_{\leqslant k}$ is not tight in $\tau_{\leqslant n_k}$.

The most "efficient" choice might be to set $n_k = \pi(k) + 1$, but other choices are of course possible.

Fix k, and suppose for the sake of example that n_{k+1} equals $n_k + 3$. Write s for $\pi_{\leqslant k}$, and t for $\pi_{\leqslant k+1}$, so that $t' = s$. Write u for $\tau_{\leqslant n_k}$, u^+ for $\tau_{\leqslant n_k + 1}$, u^{++} for $\tau_{\leqslant n_k + 2}$, and v for $\tau_{\leqslant n_{k+1}}$, so that $v' = u^{++}$.

Consider the following string of symbols:

$$\underbrace{\underbrace{\underbrace{\underbrace{z_s^{u'} {}^\frown \langle [s;u] \rangle {}^\frown z_{s'}^{u}}_{z_s^u} {}^\frown \langle [s;u^+] \rangle {}^\frown z_{s'}^{u^+}}_{z_s^{u^+}} {}^\frown \langle [s;u^{++}] \rangle {}^\frown z_{s'}^{u^{++}}}_{z_s^{u^{++}}} {}^\frown \langle [t;v] \rangle {}^\frown z_{t'}^{v}}_{\mathfrak{s}^\ell(z_t^v)}$$

To see that the entire string is expressible as a shift of z_t^v, note that $z_s^{u^{++}} = z_{t'}^{v'}$, which is an end-segment of $z_t^{v'}$, so we must choose ℓ to be 1 if t is tight in v', and to be $1 + \ell h(z_t^{v''})$ otherwise.

4·34 Remark For further variety in the construction of recurrent points, reflect that if s is loose in u and has successive extensions s^+, s^{++}, say, which are also u-sequences, the string $\langle[s;u]\rangle^\frown z^u_{s'}$ is an end-segment of z^u_s but also of $z^u_{s^+}$ and of $z^u_{s^{++}}$, and hence can be followed by $\langle[s;u^+]\rangle^\frown z^{u^+}_{s'}$, $\langle[s^+;u^+]\rangle^\frown z^{u^+}_{s}$ or $\langle[s^{++};u^+]\rangle^\frown z^{u^+}_{s^+}$, to yield, respectively, end-segments of $z^{u^+}_s$, $z^{u^+}_{s^+}$ or $z^{u^+}_{s^{++}}$.

Proof of the main result

4·35 Lemma *If* $b \curvearrowright_{\mathfrak{s}} \xi$, $\tau_\xi \neq \sigma \in \mathcal{T}$, *and* s *is a* σ*-sequence, then* $\xi \not\curvearrowright_{\mathfrak{s}} \xi^\sigma_s$.

Proof : Let e be an initial segment of σ that is not one of τ_ξ. Then $[e]$ occurs in ξ^σ_s but not in any $\xi^{\tau_\xi}_{\pi_\xi \restriction k}$, hence not in ξ. ⊣ (4·35)

4·36 Lemma *If* $b \curvearrowright_{\mathfrak{s}} \zeta \curvearrowright_{\mathfrak{s}} \gamma \triangleright \xi^\sigma_s$ *and* σ *is well-founded, then* ζ *is of finite weight.*

Proof : If ζ were of infinite weight, τ_ζ would be ill-founded. But $\tau_\zeta = \sigma$. ⊣ (4·36)

4·37 Proposition *Let* $\sigma \in \mathcal{T}$ *be well-founded. If* $b \curvearrowright_{\mathfrak{s}} \zeta \curvearrowright_{\mathfrak{s}} \gamma \triangleright \xi^\sigma_s$ *then there is a* $t \prec s$ *such that* ζ *is near* ξ^σ_t.

Proof : Take $t = s_\zeta$, as in Proposition 4·24. $t \preccurlyeq s$ since $\zeta \curvearrowright_{\mathfrak{s}} \gamma$; since $\gamma \not\curvearrowright_{\mathfrak{s}} \gamma$, $t \prec s$. ⊣ (4·37)

4·38 COROLLARY *For σ well-founded,* $\beta(\xi^\sigma_{\circledcirc}, b, \mathfrak{s}) = \varrho_\sigma(0)$.

Here β is as in Definition 1·1, and ϱ_σ is the rank function defined on the nodes of σ, as in section 1 of *Delays*. The number 0 is the top node in the tree relation $<_\sigma$ defined in Remark 4·7 above.

Proof : By lemmata 4·4 and 4·6 of *Delays*, taking T to be the tree coded by σ, x_T to be b and, for s a σ-sequence, x_s to be ξ^σ_s. Proposition 4·37 above shows that b plays the rôle required of x_T in lemma 4·6 of *Delays*. ⊣ (4·38)

Proof of Theorem 4·2: Let η be any countable ordinal and let $\sigma \in \mathcal{T}$ be well-founded with $\varrho_\sigma(0) = \eta$: such σ may be constructed following Remark 4·8 and *Delays*, proposition 4·1. Theorem 4·7 of *Delays* may now be applied, to show that $\theta(b, \mathfrak{s}) > \eta$. Since η was arbitrary, $\theta(b, \mathfrak{s}) \geqslant \omega_1$; by *Delays*, corollary 2·5, $\theta(b, \mathfrak{s}) \leqslant \omega_1$; thus $\theta(b, \mathfrak{s}) = \omega_1$. ⊣ (4·2)

4·39 Thus we arrive at the following attractive picture: the recurrent points attacked by b are all at the ends of paths through ill-founded trees, and they are all *maximal recurrent in* b in the sense of definition 3·21 of *Delays*; all other points attacked by b are near to some ξ_s^τ for uniquely determined τ and s; the points that escape are those near to ξ_s^τ with τ well-founded below s.

4·40 Remark The abode $A(b, \mathfrak{s})$ is a complete analytic set, since the assignment $\tau \mapsto \xi_{\circledcirc}^\tau$ is continuous, and τ is ill-founded if and only if $\xi_{\circledcirc}^\tau \in A(b, \mathfrak{s})$. Similarly $E(b, \mathfrak{s})$ is a complete co-analytic set.

4·41 Remark Our methods confirm a conjecture of Martin Goldstern: let

$$G =_{\mathrm{df}} \{\alpha \in \mathcal{N} \mid \overline{\overline{\omega_{\mathfrak{s}}(\alpha)}} \leqslant \aleph_0\}.$$

G is co-analytic since

$$\alpha \in G \Longleftrightarrow \forall \beta (\alpha \curvearrowright_{\mathfrak{s}} \beta \Longrightarrow \beta \text{ is hyperarithmetic in } \alpha).$$

We shall show that G is complete by exhibiting a continuous reduction of the collection of well-founded trees to it.

For $\tau \in \mathcal{T}$, define ξ^τ by modifying Definition 4·1: let $(w_i^\tau)_i$ list all z_s^u where $u \succ \tau$ and s is a u-sequence—plainly such a list may be found uniformly recursive in τ by deleting all z_s^u with $u \not\succ \tau$ from the recursive list $(z_i)_i$—and then set

$$\xi^\tau \;=_{\rm df}\; w_0^\tau {}^\frown \langle [\mathsf{m}_0] \rangle {}^\frown w_1^\tau {}^\frown \langle [\mathsf{m}_1] \rangle {}^\frown w_2^\tau {}^\frown \langle [\mathsf{m}_2] \rangle {}^\frown \cdots$$

If τ is well-founded, ξ^τ will be in G, since it attacks only points near to ξ_s^τ for some τ-sequence s.

If τ is ill-founded, then ξ^τ will attack some recurrent point at the end of a τ-path. The variety of construction of recurrent points indicated in Example 4·33 and Remark 4·34 may readily be exploited to prove that the set of recurrent points at the end of a given path is uncountable, and indeed contains a perfect set.

Thus if τ is ill-founded, ξ^τ will not be in G.

Since the association $\tau \mapsto \xi^\tau$ is continuous, indeed recursive, we have reduced a known complete co-analytic set to G, which must, therefore, itself be complete co-analytic.

Open problems

PROBLEM *Which ordinals are the scores under shift of recursive points of $\mathcal{N}$?*

Every recursive ordinal is possible; the first non-recursive ordinal is possible; ω_1 is possible; *are there any others ?* My candidate is ω_1^L.

A familiar concept in dynamics is that of a minimal recurrent point. Using AC and the lemma about points at the end of a path one can build maximal recurrent points:

PROBLEM *Are maximal recurrent points found in nature ?*

REMARK Matt Foreman and collaborators have papers applying descriptive set theory to problems in ergodic theory. For example, with Beleznay, he showed that the collection of distal flows is not Borel.

REMARK There are papers on uniformly recurrent sequences.

More recent work

[**2d**] A. R. D. Mathias, Choosing an attacker by a local derivation, *Acta Universitatis Carolinae - Math. et Phys.*, **45**(2004) 59–65.

That paper is the surviving fragment of attempts to prove that uncountable scores are impossible.

Problem (*James Cummings*) *In a system without points of uncountable score, is score a* Π^1_1 *norm ?*

Problem The paper

[**2e**] A. R. D. Mathias, A scenario for transferring high scores, *Acta Universitatis Carolinae - Math. et Phys.*, **45** (2004) 67–73.

shows that a dynamical system satisfying four hypotheses will contain a point of uncountable score: I know of systems satisfying three of those hypotheses but *are there any systems satisfying all four ?*

Yet more recent work

[**6a**] C. Delhommé. Transfer of scores to the shift's attacks of Cantor space.

[**6b**] C. Delhommé. Representation in the shift's attacks of Baire space. [*formerly* On embedding transitive relations in that of shift-attack.]

[**6c**] C. Delhommé. Completeness properties of the relation of attack.