Sweating the Small Stuff

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Outline

A Question of Hamkins and Löwe

This is joint work with Mohamaad Golshani

Question (Hamkins and Löwe)

Is it consistent that $V \equiv V[g]$ whenever g is generic for the Levy collapse, $\mathrm{Col}(\omega, \lambda)$*, of any cardinal λ.*

the rational for this question

- Say $\Diamond\phi$ (ϕ is possible) if $\exists\lambda \Vdash_{\mathrm{Col}(\omega,\lambda)} \phi$,
- and $\Box\phi$ (ϕ is necessary) if $\forall\lambda \Vdash_{\mathrm{Col}(\omega,\lambda)} \phi$.

This gives a modal logic semantics in which

- (the optimistic view) everything possible is true and
- (the realistic view) everything true is necessary.

A coarse equiconsistency

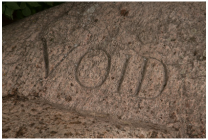
Definition

If U is a normal measure, then we say that U' is a U-normal measure if

- $x \in U' \iff U \in i^{U'}(x)$, and
- $x \in U \iff \{ W \mid \text{crit}(W) \in x \} \in U'$.

Theorem

(Assume that $V = K$) The following are equiconsistent:



1. *There is an inaccessible κ such that $V \equiv V^{\text{Col}(\omega,\lambda)}$ for any $\lambda < \kappa$.*
2. *There is κ with a normal measure U and κ^+ many U-normal measures.*

NOTE: I want to thank Woodin for pointing out that our proof from clause (2) to clause (1) is invalid.

1. The Easton product collapse needs to be interleaved with the Radin forcing. This can easily be fixed.
2. The forcing $\mathrm{Col}(\omega, \lambda)$ of $\lambda \in C$ will not absorb a Prikry sequence through λ. There for λ a limit member of C, there will be a Prikry sequence through λ, while if λ is a successor in C there will not be such a sequence.

Let U be the given ultrafilter and let

$$u = \langle\, u(0) = U, u(1), \ldots, u(\xi), \ldots \rangle_{\xi<\kappa^+}$$

be the $\lhd$-increasing enumeration of the U-normal measures.

We will begin with radin forcing, $\mathcal{R}(u)$, on the sequence u of ultrafilters.

the main step: Radin forcing

The result is a model $V[C, \mu]$ where

- C is a closed unbounded subset of κ.
- If $\lambda \in C \setminus \lim(C)$ then $u_\lambda = \lambda$.
- If $\lambda \in \lim(C)$ then μ_λ is a sequence $\mu_\lambda = \langle\, u(0) = U_\lambda, u(1), \dots, u(\xi), \dots \rangle_{\xi<\rho}$ of U_λ-normal measures of length $\rho < \lambda^+$.
- There is a condition $\langle\, (u, A) \,\rangle \in \mathcal{R}(u)$ such that $\langle\, (u, A) \,\rangle \parallel_{\mathcal{R}(u)} \sigma$ for every each sentence σ, and
- No cardinals are collapsed, and κ is still regular.

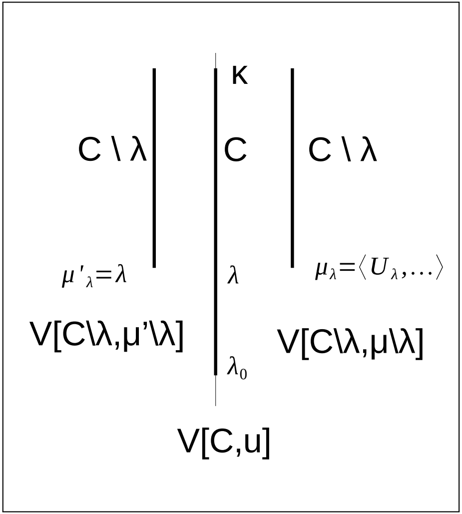
Claim

For any formula ϕ, possibly with parameters, there is $\lambda_0 \in C$ such that whenever $\lambda_0 \leq \lambda \leq \lambda'$ are in C then

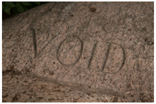
$$V[C]^{\mathrm{Col}(\omega,\lambda)} \equiv V[C]^{\mathrm{Col}(\omega,\lambda')}.$$

Proof of the claim

- If $\lambda \notin \lim(C)$ then $(C \setminus \lambda, \mu \setminus \lambda)$ is generic and is below $\langle (u, A) \rangle$.
- If $\lambda \in \lim(C)$ then $(C \setminus \lambda, \mu \setminus \lambda)$ is not generic,
- but $(C \setminus \lambda, \mu' \setminus \lambda)$ is generic and is below (u, A) —
- assuming $\lambda \in A$, which we can assume since every $u(\xi)$ is U-normal.



a collapse gives the Hamkins-Löwe Property



- Now forget about μ. If $\lambda \in C$ then
$$V[C] \equiv V[C \setminus \lambda].$$
- The final model is
$$V^* = V[C]^{\mathrm{COL}}$$
where COL is the Easton support product
$$\mathrm{Col}(\omega, \lambda_0) \times \prod_{\lambda \in C} \mathrm{Col}(\lambda^+, \min(C \setminus \lambda^+)),$$
using the homogeneity of the collapse forcing to see that it doesn't affect the equivalence.

This concludes the proof that the sequence μ can be used to

The other direction — A basic observation

Lemma (Essentially due to Philip Welch)

(Assuming weak covering) if the Hamkins-Löwe property holds then there is a closed unbounded class C such that

$$\forall\lambda \in C \quad K[C \setminus \lambda] \equiv K[C].$$

Proof of lemma

Proof sketch.

- If λ is a singular cardinal, then $\lambda^+ = (\lambda^+)^K$, a successor in K, so $V^{\mathrm{Col}(\omega,\lambda)} \models \omega_1$ is a successor in K,
- so all successor cardinals are successors in K.
- Set $C = \{\, \lambda \mid (\lambda^+)^K \text{ is a cardinal} \,\}$. Up (at least) to the first regular limit cardinal, C is club.
- Finally, $C \setminus \lambda$ is $\bar{C}$ as defined in $V[C]^{\mathrm{Col}(\omega,\lambda)}$ and K is absolute, so

$$K[C \setminus \lambda] \equiv K[C].$$

□

Getting indiscernibles

Corollary

Suppose $\phi(a)$ *is a formula, with parameters a and C. Then for any* $\eta \in C$ *there is* $\gamma < \eta$ *such that*

$\forall \lambda \in C \cap [\gamma, \eta)\ K[C \backslash \lambda] \models \phi(a)$ *or* $\forall \lambda \in C \cap [\gamma, \eta)\ K[C \backslash \lambda] \models \neg\phi(a)$

Proof sketch.

- Supposing this is false for ϕ, define two terms: set $a(\lambda)$ and $\eta(\lambda)$ to be the least pair giving a counterexample with $\eta > \lambda$.
- Then $\phi(a(\min(\bar{C})))$ is a sentence, so is either true in all $K[C \setminus \lambda]$ or false in all $K[C \setminus \lambda]$.
- But $a(\lambda) = a(\min(C))$ and $\eta(\lambda) = \eta(\min(C))$ for $\min(C) \leq \lambda < \eta(\min(C))$, contradicting the choice of $a(\lambda)$ and $\eta(\lambda)$.

Define the Measures on κ

Now we can define the measures on κ. The normal measures are straightforward:

$$(\forall\lambda \in \lim(C) \cup \{\kappa\})\ X \in U_\lambda \iff \sup(\lambda \cap C \setminus x) = \lambda$$

To define the measures $u(\xi)$ for $\xi > 0$, define

$$\begin{aligned} C_0 &= C \\ C_{\xi+1} &= \lim(C_\xi) \\ C_\lambda &= \Delta_{\xi'<\xi} C_\xi && \text{for } \xi \text{ a limit ordinal.} \end{aligned}$$

Then writing $\text{next}_\xi(\lambda) = min((C_\xi \setminus C_{\xi+1}) \setminus \lambda)$, set

$$x \in u(\xi) \iff \sup\{\lambda \in C \mid U_{\text{next}_\xi(\lambda)}\} \in x = \kappa.$$

(Note that while U_λ is never definable in $K[C \setminus \lambda]$, U_η is definable there for all $\eta \in C_1 \setminus \lambda + 1$.)

what strength is needed?

We've considered three assertions:

1. The Hamkins-Löwe property.
2. There is a closed unbounded class C such that
$$\forall\lambda \in C\ K[C \setminus \lambda] \equiv K[C].$$
3. For some normal measure U on the class Ω of ordinals there are κ^+ many U-normal measures.

- The proof in section 2 that Clause 1 implies Clause 2 is entirely in first class logic, as is the proof that a collapse forcing will give Clause 1 from Clause 2.
- The definition from Clause 2 of the ultrafilters on Ω is entirely first order — but the ultrafilters themselves are not. (And what is "Ω^+"?)

Sort of a mouse

- Define $J^{\mathcal{E}}_{\Omega_\xi} = \mathrm{ult}^{\Sigma_\omega}(K, u(\xi))$ for each $\xi < \Omega^+$.
- Truth for $J^{\mathcal{E}}_{\Omega_\xi}$ is definable in $K[C]$:

$$J^{\mathcal{E}}_{\alpha_\xi} \models \phi([f]_{u(\xi)}) \iff \sup\{\,\lambda \in C_\xi \setminus C_{\xi+1} \mid K \models \phi(U_{f(\mathrm{next}_\xi(\lambda))}\,\} = \Omega.$$

- Set $\Omega^* = \sup_{\xi<\Omega^+} \alpha_\xi$, where Ω^+ is the supremum of the well orderings definable in $K[C]$.
  Then $J^{\mathcal{E}}_{\Omega^*} = \mathrm{dir}\lim_{\xi<\Omega^+} J^{\mathcal{E}}$ is a model with Ω^+ many U-normal measures.
- Slightly more is needed for the Hamkins-L owe property: $\Sigma_{\omega+1} - \mathrm{cf}(K[C]) > \omega$ or, equivalently, Σ_1- $\mathrm{cf}(J^{\mathcal{E}}_{\Omega^*}) > \omega$.
- This also ensures that $J^{\mathcal{E}}_{\Omega^*}$ is well founded, which is conforting to a set theorist..

Conclusion

- This leaves only one remaining question: what about the Radin forcing making the Hamkins-Löwe property true?
- Obviously we can't do the full Radin forcing, but we can define a direct limit of the forcings $\mathcal{R}(u \restriction \xi)$ for $\xi < \Omega^+$. This will decide the truth of all first order formulas over the models $J^{\mathcal{E}}_{\Omega_\xi}$.
- Thus it decides all Σ_0 formulas over $J^{\mathcal{E}}_{\Omega^*}$, which is equivalent to all Σ_ω formulas over K.

Thank you for your attention

And thank you, Ronald, and best wishes for coming days.acchg