$\mathcal{L}$-forcing, I

plan: today: $\mathcal{L}$-version of namba forcing
($\mathcal{L}$-forcings without adding reals,) with cntbl. conditions

next time: $\mathcal{L}$-forcing with finite conditions,
use: increase $\underset{\sim}{\delta}^1_2$ ..

last time: fusion results.

$\mathcal{L}$-forcing due to jensen, i will present it with some modifications.

<u>namba forcing</u> adds a cont. system

$$(M_i, \pi_{ij} : i \le j \le \omega_1)$$

s.t. every M_i, $i<\omega_1$, is cntbl.,

$$M_{\omega_1} = H_{\omega_2} \qquad (\text{we assume } 2^{\aleph_1} = \aleph_2 \text{ in } V)$$

$$\operatorname{ran}(\pi_{i+1,\omega_1}) = \operatorname{Hull}^{H_{\omega_2}}(\operatorname{ran}(\pi_{i,\omega_1}) \cup \{\operatorname{crit}(\pi_{i\,i+1})\})$$

ran $\pi_{0\omega_1} \cap \omega_2$ is cofinal in ω_2.

namba forcing pres. stat. subsets of ω_1 and if CH holds, then it doesn't add any reals.

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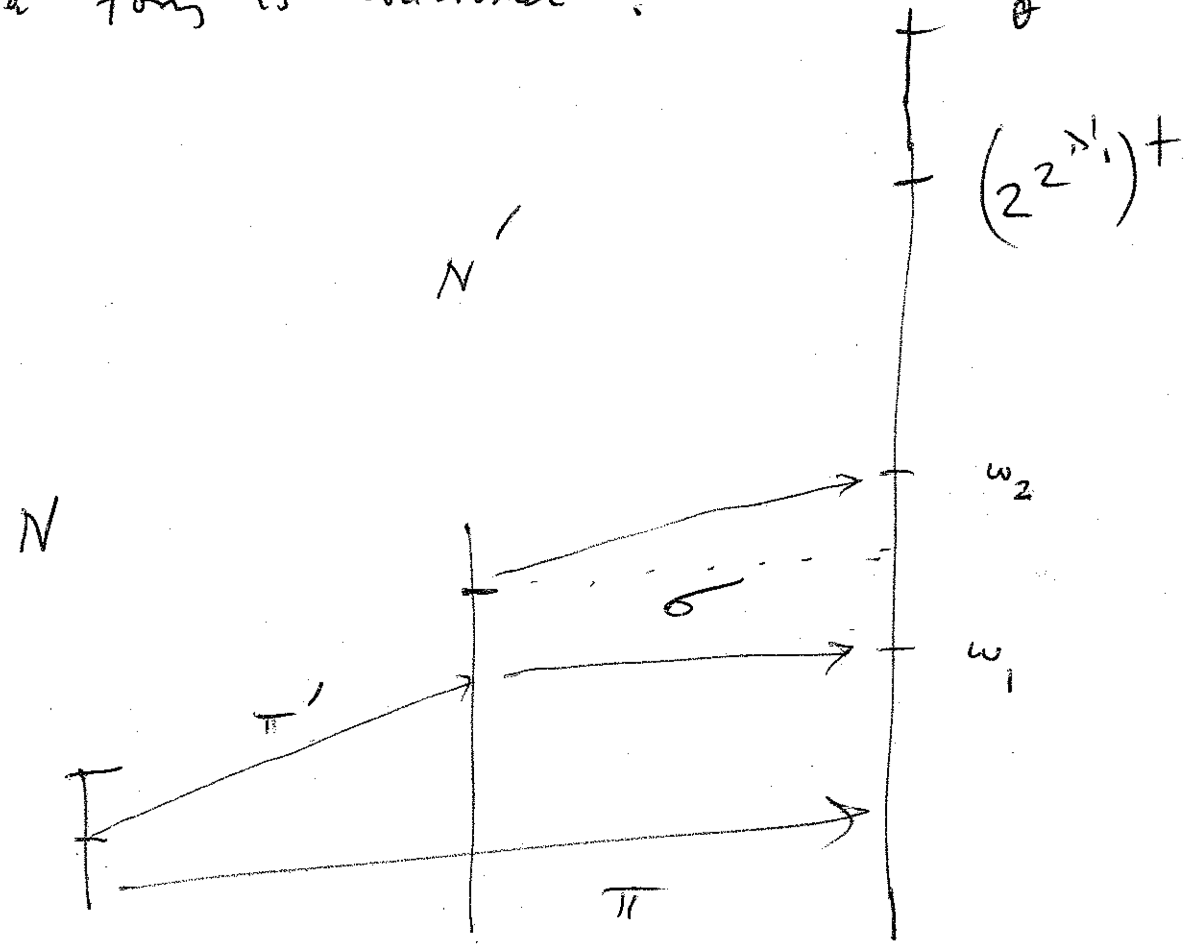
the $\mathcal{L}$-version of namba forcing directly produces such a system, as do all $\mathcal{L}$-forcings.

starting pt.: the result of forcing with Namba forcing is consistent:



let $N \stackrel{\pi}{\cong} \mathrm{Hull}^{H_\theta}(\emptyset) \prec H_\theta$, and

let $N' = \mathrm{ult}(N, \pi \upharpoonright (H_{\omega_2})^N)$.

consider pairs (a, f) s.t. f is a fct in N,
$a \in \pi(\mathrm{dom}(f))$,
$a \in H_{\omega_2}$, $f : \mathrm{dom}(f) \to N$. [$(H_{\omega_2})^N$]

$(a, f) \sim (b, g)$ iff

$(a, b) \in \pi(\{(u, v) : f(u) = g(v)\})$

for $[a, f]$.

$[a, f] \,\tilde{\in}\, [b, g]$ iff $(a, b) \in \pi(\{(u, v) : f(u) \in g(v)\})$.

elts. of $\mathrm{Ult}(N; \pi \restriction (H_{\omega_2})^N)$.

$\pi'(x) = [\emptyset, c_x]$.

may embed $\mathrm{Ult}(N; \pi \restriction (H_{\omega_2})^N)$ into H_θ

$$\sigma([a, f]) = \pi(f)(a).$$

let $\tau = \sup \pi'' \omega_2^N$.

<u>clm!</u> $\tau = \omega_2^{N'}$.

pf.: if $\xi < \tau$, then $\sigma(\xi) = \sigma([\xi, id])$

$= \pi(id)(\xi) = \xi$. so $\sigma \restriction \tau = id$.

so $\omega_2^{N'} \not< \tau$, i.e. $\tau \leq \omega_2^{N'}$.

let $\xi < \omega_2^{N'} = \pi'(\omega_2^N)$, say $\xi = [a, f]$.

so $[a, f] < [\emptyset, c_{\omega_2^N}]$, i.e.

$(a, 0) \in \pi(\{(u, v) : f(u) < \omega_2^N\})$

$a \in \pi(\{u : f(u) < \omega_2^N\})$, so

$f \in (H_{\omega_2})^N$ has values $< \omega_2^N$, $\mathrm{dom}(f) \in (H_{\omega_2})^N$, so f is bounded.

so $\sup(\operatorname{ran}(f)) = \eta < \omega_2^N$.

but then

$\xi = [a, f] < [\cancel{a}, c_\eta] = \pi'(\eta) < \tau$

i.e., $\omega_2^{N'} \leq \tau$. $\dashv$ (claim).

in N', let $X_0 = \operatorname{ran}(\pi') \cap (H_{\omega_2})^{N'}$

κ_i = the least (unc.) ordinal $\notin X_i$

$X_{i+1} = \operatorname{Hull}^{(H_{\omega_2})^{N'}}(X_i \cup \{\kappa_i\})$

$X_\lambda = \bigcup_{i<\lambda} X_i$.

gives $(M_i, \pi_{ij} : i \leq j \leq \omega_1)$ as above.

notice:

- if $S \in \mathcal{P}(\omega_1) \cap N'$ is stat. in N', then so in V.
- if CH holds, then $\mathbb{R} \subset N'$.

by absoluteness between

$$N'^{\,Col(\omega,\bar{\kappa})} \quad \text{and} \quad V$$

(where $\bar{\kappa} = \overline{\overline{(H_{2^{2^{\aleph_1}}})^{N'}}}^{N'}$), there is hence some

$\mathfrak{A} \in N'^{\,Col(\omega,\kappa)}$ s.t.

- $(H_{2^{2^{\aleph_1}}}, \dot{}\,)^{N'} \subset \mathfrak{A}$, $\mathfrak{A}$ is a trans. model of ZFC^-
- if $S \in \mathcal{P}(\omega_1) \cap N'$ is stat. in N', then so in $\mathfrak{A}$
- $\mathbb{R} \cap N' = \mathbb{R} \cap \mathfrak{A}$ (is ctbl)
- in $\mathfrak{A}$, there is $(M_i, \pi_{ij} : i \le j \le \omega_1)$ "Namba like".

by elementarity, in $V^{Col(\omega,\kappa)}$, where $\kappa = \overline{\overline{(H_{2^{2^{\aleph_1}}}, \dot{}\,)}}$, there is thus some $\mathfrak{A}$ s.t.

- $(H_{2^{2^{\aleph_1}}}, \dot{}\,) \subset \mathfrak{A}$, $\mathfrak{A}$ tr. model of ZFC^-
- if $S \in \mathcal{P}(\omega_1) \cap V$ is stat. in V, then so in $\mathfrak{A}$
- $\mathbb{R} \cap V = \mathbb{R} \cap \mathfrak{A}$ (is ctbl) • in $\mathfrak{A}$, have Namba seq.

forcing conditions :

$$\mathbb{P} \ni p = \left((\vec{M}^p_\xi, \vec{\pi}^p), (\bar{\tau}^p, \tau^p) \right)$$

$$\vec{M}^p_\xi = (M^p_i : i \le \alpha^p) \qquad \vec{\pi}^p = (\pi^p_{ij} : i \le j \le \alpha^p) \qquad \text{some } \alpha^p < \omega_1$$

s.t. p is certified by some σ as above in the obvious sense, and

$\exists x \in H_{2^{2^{\aleph_1}}}$ s.t. $\tau^p =$ the H_{ω_2}-type of x on $H_{(2^{2^{\aleph_1}})^+}$

$$= \{ \varphi(\vec{z}, v) : H_{(2^{2^{\aleph_1}})^+} \models \varphi(\vec{z}, x),\ \vec{z} \in H_{\omega_2} \}$$

$\bar{\tau}^p = {\pi^\sigma_{\alpha\omega_1}}^{-1} {}'' \tau^p$, and

$$\pi^\sigma_{\alpha\omega_1} : (M^p_\alpha ; \in, \bar{\tau}^p) \longrightarrow (H_{\omega_2} ; \in, \tau^p).$$

(i.e., $\pi^\sigma_{\alpha\omega_1}$ respects $\bar{\tau}^p, \tau^p$.)

here π^σ_{ij} are the embeddings for σ etc.

$q \leq p$ iff

$\vec{M}^p, \vec{\pi}^p = \vec{M}^q, \vec{\pi}^q \restriction \alpha^p$,

$\pi^q_{\alpha^p \alpha^q} : (M^p_{\alpha^p}; \in, \bar{\tau}^p) \to (M^q_{\alpha^q}; \in, \underbrace{(\pi^{\sigma}_{\alpha^q \omega_1})^{-1} {}'' \tau^p}$)

can be read off from $\bar{\tau}^q$ in a rec. way in the same way as τ^p can be read off from τ^q.

in fact: τ^p should be a section of τ^q, i.e., there is some n_0 s.t. $(k, \vec{x}) \in \tau^p$ iff $(n_0, (k, \vec{x})) \in \tau^q$.

we'll not state the more obvious facts about this forcing.

<u>Lemma</u> 1. Let p be a condition, $\pi^{\sigma}_{\alpha\omega_1}$ extends certified by σ.

~~uniquely to~~ to

$\tilde{\pi} : H \to H_{(2^{2^{\aleph_1}})^+}$ s.t. $x \in \operatorname{ran}(\tilde{\pi})$ if $\tau^p =$ the $H_{2^{2^{\aleph_1}}}$-type of x

$\Uparrow$
$\Downarrow$

s.t. $H = \operatorname{Hull}^H(M^p_\alpha \cup \{\bar{x}\})$, where $\bar{x} = \tilde{\pi}^{-1}(x)$

and $\bar{\tau}^p =$ the M^p_α-type of $\bar{x}$ in H.

$H, \bar{x}$ are independent from the choice of x.

pf.:

let $X = \mathrm{Hull}^{H_{2^{2^{\omega_1}}}}(\mathrm{ran}(\pi^{\sigma}_{\alpha\omega_1}) \cup \{x\})$.

let $z_0 \in X \cap H^{V}_{\omega_2}$, say

z_0 is the solution to

$$\exists z \quad H_{2^{2^{\omega_1}}} \models \varphi(z, \vec{y}, x) \quad .$$

$$\vec{y} \in \mathrm{ran}(\pi^{\sigma}_{\alpha\omega_1})$$

So $(\varphi, (z, \vec{y})) \in \tau^{p}$

as $\mathrm{ran}(\pi^{\sigma}_{\alpha\omega_1}) \prec (H_{\omega_2}; \in, \tau^{p})$

$z \in \mathrm{ran}(\pi^{\sigma}_{\alpha\omega_1})$.

so $X \cap H_{\omega_2} = \mathrm{ran}(\pi^{\sigma}_{\alpha\omega_1})$. so get $H, \bar{x}$.

we will also have $\bar{\tau}^{p}$ = the H^{p}_{α}-type of $\bar{x}$ over H, basically by elementarity.

for two different choices of x, the corr. H's are $\in$-isomorphic (sending $\bar{x}$ to $\bar{x}$), so equal.

⊣ (claim 1)

Lemma 2. If p, q are both certified by $\mathfrak{M}$, then $p \parallel q$.

pf.: let $\gamma \geq \alpha^p, \alpha^q$ s.t.

$$\pi^{\mathfrak{M}}_{\alpha^p \gamma} : (M^p_{\alpha^p}; \tilde\tau^p) \to (M^{\mathfrak{M}}_\gamma; \pi^{\mathfrak{M}\,-1}_{\gamma\,\gamma+1} {}'' \tau^p)$$

(where $\pi^{\mathfrak{M}\,-1}_{\gamma\,\gamma+1} {}'' \tau^p = \tilde\tau^p$)

$$\pi^{\mathfrak{M}}_{\alpha^q \gamma} : (M^q_{\alpha^q}; \tilde\tau^q) \to (M^{\mathfrak{M}}_\gamma; \pi^{\mathfrak{M}\,-1}_{\gamma\,\gamma+1} {}'' \tau^q).$$

(where $\pi^{\mathfrak{M}\,-1}_{\gamma\,\gamma+1} {}'' \tau^q = \tilde\tau^q$)

set $r = (M^{\mathfrak{M}}_i, \pi^{\mathfrak{M}}_{ij} : i \leq j \leq \gamma), (\tilde\tau^p \oplus \tilde\tau^q, \tau^p \oplus \tau^q)$.

$\mathfrak{M}$ certifies r. $\dashv$

note: every condition has size $2^{\aleph_1}$, so the forcing $\mathbb{P}$ has size $\leq 2^{2^{\aleph_1}}$.

therefore, $\mathbb{P} \in H_{(2^{2^{\aleph_1}})^+} \subset \mathfrak{M}$ for every certifying model $\mathfrak{M}$.

<u>lem</u> 3. $\mathbb{R} \cap V^{\mathbb{P}} = \mathbb{R} \cap V$.

pf.: let $p \Vdash \sigma \in {}^{\omega}\omega$.

let $\tau =$ the H_{ω_2}-type of x over $H_{(2^{\aleph_1})^+}$,

where $x = (\mathbb{P}, p, \Vdash, \sigma)$
rel. to ...

as in the pf. of lem 2, pick γ s.t. $\tilde{\tau}^p$

$$\pi^{\sigma}_{\alpha_p \gamma} : (M_{\alpha_p}; \tilde{\tau}^p) \to (M^{\sigma}_{\gamma}; \pi^{\sigma\,-1}_{\gamma\omega_1}{}''\tau^p)$$

$$\pi^{\sigma}_{\gamma\omega_1} : (M^{\sigma}_{\gamma}; \pi^{\sigma\,-1}_{\gamma\omega_1}{}''\tau) \to (H^V_{\omega_2}; \tau).$$

let $q = (\langle M^{\sigma}_i, \pi^{\sigma}_{ij} : i \le j \le \gamma \rangle, (\tilde{\tau}^p \oplus \pi^{\sigma\,-1}_{\gamma\omega_1}{}''\tau, \tau^p \oplus \tau))$

$q \in \mathbb{P}$, certified by σ.

let

$$\begin{array}{ccc} \pi^{\sigma}_{\gamma\omega_1} : M^q_{\gamma} & \longrightarrow & H_{\omega_2} \\ \cap & & \cap \\ \tilde{\pi} : H & \longrightarrow & H_{(2^{2^{\aleph_1}})^+} \\ \cap & & \\ V & & \end{array}$$

as in lem 1.

write $\bar{\mathbb{P}}, \bar{p}, \bar{\sigma} = \tilde{\pi}^{-1}(\mathbb{P}, p, \sigma)$.

let $g \in V$ be $\bar{\mathbb{P}}$-gen. $/H$, $\tilde{\pi}^{-1}(p) \in g$.

So $z = \bar{\sigma}^g$.

let

$$r = \big((\underbrace{M_i^g, \pi_{ij}^g}_{\text{given by } g} : i \le j \le \gamma), (\bar{\tau}^g, \tau^g) \big).$$

$r \in \mathbb{P}$, certified by $\mathcal{A}$. (!).

claim. $r \Vdash \sigma = \check{z}$.

Pf: deny. let $s \le r$, $s \Vdash \sigma(\check{n}) = \check{m} \ne z(n)$.

let $\mathcal{B}$ certify s. then

$$M_\gamma^r = M_\gamma^g = M_\gamma^{\mathcal{B}}$$

$$(M_\gamma^g ; \bar{\tau}^g) \longrightarrow (M^s_{\alpha^s} ; \pi^{\mathcal{B}}_{\alpha^s \omega_1}(\bar{\tau}^g)) \to (H_{\omega_2}; \tau^g)$$

$$\cap \qquad\qquad\qquad\qquad \cap$$

$$H \xrightarrow{\quad \tilde{\pi} \quad} H_{(2^{2^{\aleph_1}})^+}$$

same H as before by lemma 1. (?)

$\bar{\mathbb{P}}, \bar{p}, \bar{\sigma} \mapsto \mathbb{P}, p, \sigma$ by lemma 1.

let $t \in g$, $t \Vdash_{\bar{\mathbb{P}}} \bar{\sigma}(\check{n}) = (z(n))^\vee$.

$$\tilde{\pi}'(t) = \left((M_i^t, \pi_{ij}^t : i \le j \le \alpha^t), (\tau^t, \tilde{\pi}'(\tau^t)) \right)$$

is certified by $\mathcal{B}$,

as $\pi^{g}_{\alpha t_\gamma} = \pi^{\mathcal{B}}_{\alpha t_\gamma} : (M^{\mathcal{B}}_{\alpha t}; \bar{\tau}^t) \to (M^{\mathcal{B}}_\gamma; \tau^t) \xrightarrow{\tilde{\pi}'} (M_{\omega_2}; \tilde{\pi}'(\tau^t))$,

also $t \Vdash_{\bar{\mathbb{P}}} \bar{\sigma}(\check{n}) = (z(n))^\vee$,

$$\tilde{\pi}'(t) \Vdash_{\mathbb{P}} \sigma(\check{n}) = (z(n))^\vee .$$

but $\tilde{\pi}'(t)$, s both certified by $\mathcal{B}$,

so $\tilde{\pi}'(t) \parallel s$ by lemma 2.

but $s \Vdash_{\mathbb{P}} \sigma(\check{n}) \neq (z(n))^\vee$. ↯ ⊣

remarks: basically the same proof shows that $\mathbb{P}$ preserves club subsets of ω_1. (we don't need CH here.)

also, argument leads to the concept of

<u>subcompleteness</u>.