Ralf Schindler: Talks#5 on Logic Summer School of Fudan University, 2020

TODAY:

- Finish the last theorem of the last lecture: Force by a stationary set preserving forcing:

$$(M; \in, I) \xrightarrow[\text{of length } \omega_1]{\text{generic iteration}} (H^V_{\omega_2}; \in, \mathbf{NS}^V_{\omega_1}),$$

  where M is a generically iterable countable transitive structure.
- $\mathbb{P}_{\max}$ forcing and analysis of $L(\mathbb{R})^{\mathbb{P}_{\max}}$;
- $(*)$ and:

  Theorem 1 (Asperó-Schindler)**.** $\mathbf{MM}^{++} \implies (*)$.

We would like to force by a stationary set preserving forcing:

$$(M; \in, I) \xrightarrow[\text{of length } \omega_1]{\text{generic iteration}} (H^V_{\omega_2}; \in, \mathbf{NS}^V_{\omega_1}).$$

The idea is that we would like to force to create the countable model M such that it satisfies certain sentence.

Proof. (Sketch) Now we define the forcing $\mathbb{P}$. Recall that $\diamondsuit_{\omega_3}$ is the statement:

> There is a sequence $((Q_\alpha, A_\alpha) : \alpha < \omega_3)$ such that $(Q_\alpha : \alpha < \omega_3)$ is a tower of continuous transitive substructures of H_{ω_3} of size $\aleph_2$ with $\bigcup_\alpha Q_\alpha = H_{\omega_3}$; Moreover, for all $A \subset H_{\omega_3}$, $\{\alpha : (Q_\alpha, A_\alpha) \prec (H_{\omega_3,A})\}$ is stationary.

Since $\diamondsuit_{\omega_3}$ can be added by a forcing, we here assume, without loss of generality, that $V \models \diamondsuit_{\omega_3}$. Now we define $\mathbb{P}_\lambda$ for $\lambda \leq \omega_3$, and $\mathbb{P} = \mathbb{P}_{\omega_3}$, with $|\mathbb{P}| = \aleph_3$. Assume that $\mathbb{P}_\alpha$ are already defined for every $\alpha < \lambda$. $p \in \mathbb{P}_\lambda$ iff $p = (p_0, p_1)$ such that:

- p_0 is a finite set of formulae, starting to describe a transitive model of size $\aleph_1$ which can see the generic iteration $(M; \in, I) \to (H_{\omega_2}; \in, \mathbf{NS}_{\omega_1})$. p_0 remains the same for every pair in $\mathbb{P}_\lambda$.
- p_1 is a finite partial function from ω_1 to λ such that it maps some $\delta < \omega_1$ to some $\alpha < \lambda$. Moreover, there are finitely much information in p_1 about substructure $X_\delta \prec (Q_\alpha; A_\alpha)$.
- Moreover, every condition p needs to be certified such that there is a transitive model $\mathfrak{A} \in V^{Col(\omega,\theta)}$ of some large enough θ which can see:
  - a consistent complete theory containing p_0;
  - f.a. $\delta \in \text{dom}(f)$, there is $\alpha = f(\delta)$ and some countable $X_\delta \prec (Q_\alpha; A_\alpha)$ comparable with the information given by p_1 such that $X_\delta \cap \omega_1 = \delta$;
  - the sequence of g_δ for $\delta \in \text{dom}(f)$, where $g(\delta)$ is the collection of finite pieces of information $\mathfrak{A}$ gives restricted to X_δ, and $g_\delta \cap E \cap X_\delta \neq \emptyset$ f.a. $E \subset \mathbb{P}_\alpha$, E definable over X_δ, with E being dense in $\mathbb{P}_\alpha$.

We are here trying to convey the idea that, when we are trying to build the ω_1-sized model M, and we want it to satisfy a certain theory, we start to do it from $\emptyset$ of sentences, and trying to add each sentence or its negation to this set. Since the complete theory would be ω_1-large, we have to do it via forcing. The left element of the pair (p_0, p_1) provides some finite piece of information we need, and the right element, called the side condition, is hard to explain why it is used here. Readers can refer to the original paper [1] to see the details.

It is even non-trivial to see that the above defined forcing notion is not empty for each α. Consider $(\emptyset, \emptyset)$ as a condition of every $\mathbb{P}_\alpha$: As the definition, $(\emptyset, \emptyset)$ needs to be certified. We only need to show that the first condition is satisfied. In other words, we need some ω_1-sized, generic iterable model M which satisfies $\emptyset$ of sentences.This argument also implies that the forcing notion $\mathbb{P}$ actually adds the structure we want.

Now we see that this forcing notion preserves stationary sets.The easy part is: If G is V-generic for $\mathbb{P}_\lambda$, then G adds a complete consistent theory describing $M \to (H_{\omega_2}; G, \mathbf{NS}_{\omega_1})$. The hard part is: $\mathbb{P} = \mathbb{P}_{\omega_3}$ preserves stationary subset of ω_1. Details of such a construction may be found in [1].

Claim. $\mathbb{P} = \mathbb{P}_{\omega_3}$ preserves stationary subset of ω_1.

Proof. Let $S \subset \omega_1$ be stationary, and $p \Vdash_{\mathbb{P}}$ "$\dot{C}$ is a club". We shall use $\Diamond_{\omega_3}$ to guess some names in $\mathbb{P}$. In case that $\dot{C}$ might be complicated, we look at:

$$A = \{(p, \xi) \in \mathbb{P} \times \omega_1 : p \Vdash \check{\xi} \in \dot{C}\} \subset H_{\omega_3}.$$

Note that $\mathbb{P} \subset H_{\omega_3}$. Let $\alpha < \omega_3$ large enough such that $(Q_\alpha; A_\alpha) \prec (H_{\omega_3}; A)$. Our goal is to find $\delta \in S$, $q \leq p$, $q \Vdash \check{\delta} \in \dot{C}$. Pick $G \in V^{Col(\omega,\omega_2)}$ be V-generic for $\mathbb{P}_\alpha$. We may assume that A_α also codes $\mathbb{P}_\alpha = \mathbb{P} \cap \mathbb{Q}_\alpha$. Now since G adds the generic iterable structure M which iterates into $H^V_{\omega_2}$, we can pick some generic ultrapower of the structure $(H^V_{\omega_2}; \in, \mathbf{NS}^V_{\omega_1})$ such that S is in the filter, namely $\omega_1^V \in j(S)$.

$$\begin{array}{ccccc} & & V & \xrightarrow{\hat{j}} & (M^*; \dots) \\ & & \cup & & \cup \\ M & \longrightarrow & (H^V_{\omega_2}; \in, \mathbf{NS}^V_{\omega_1}) & \xrightarrow{j} & (\bar{M}^*; \dots) \end{array}$$

Now $\hat{j}(p) \in \hat{j}(\mathbb{P}_{\omega_3})$ in M^*. Let $\tilde{q}$ be the result of adding "$\omega_1^V \mapsto \hat{j}(\alpha)$" to the second component of $\hat{j}(p)$. We have that $\tilde{q} \in \hat{j}(\mathbb{P}_{\omega_3})$.

By pulling back argument via $\hat{j}$, $\exists \delta < \omega_1^V$ so that if q results from p by just adding "$\delta \mapsto \alpha$" to the second component, $q \in \mathbb{P}$, $\delta \in S$. Now the last step is to show $q \Vdash \check{\delta} \in \dot{C}$.

Otherwise, $\exists r \leq q \exists \eta < \delta [r \Vdash (\check{\eta}, \check{\delta}) \cap \dot{C} = \emptyset]$. r needs to be certified: There is a $X_\delta \prec (Q_\alpha; A_\alpha) \prec (H_{\omega_3}; A)$, where $X_\delta \cap \omega_1 = \delta$, and a g_δ such that $g_\delta \cap X_\delta \cap E \neq \emptyset$ f.a. X_δ-definable E. Let $D = \{s : \exists \eta' \geq \eta [s \Vdash \check{\eta}' \in \dot{C}]\}$. Then there exists $s \in D \cap X_\delta \cap g_\delta$ such that $s \parallel r$. This contradicts with $r \Vdash (\check{\eta}, \check{\delta}) \cap \dot{C} = \emptyset$. □

□

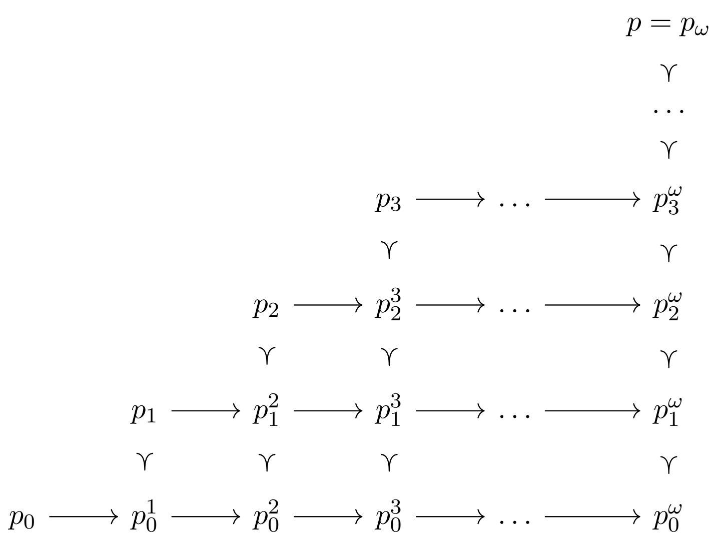
1 $\mathbb{P}_{\max}$ forcing

Definition (Hugh Woodin, [4])**.** $p \in \mathbb{P}_{\max}$ iff $p = (p; \in, I, a)$ where

- p is a countable transitive model of $\mathbf{ZFC}^-$ which is supposed to be generically iterable;
- $I \subset P(\omega_1^p) \cap p$ is a normal uniform ideal in p;
- $a \subset \omega_1^p$.

If $p, q \in \mathbb{P}_{\max}$, $p < q$ iff there is an ω_1^p-long generic iteration $q \to q^* = (q^*; \in, I^*, a^*)$ which can be seen by $p = (p; \in, I, a)$, and $I^* = I \cap q^*$, $a^* = a$.

Remark. $\mathbb{P}_{\max}$ is σ-complete, that is: for every ω-sequence $(q_i : i < \omega)$ such that $q_i > q_{i+1}$ for each $i < \omega$, then there exists $q \in \mathbb{P}_{\max}$ such that $p < p_i$ f.a. i. In particular, $\mathbb{P}_{\max}$ does not add any ω-sequence, and it does not add any reals. Moreover $\mathbb{P}_{\max}$ is also homogeneous, namely any sentences $\phi(\bar{x})$ holds in V with $\bar{x} \in V$ is forced by $1_{\mathbb{P}_{\max}}$.



We will force with $\mathbb{P}_{\max}$ over models of determinacy, like $L(\mathbb{R})$ with the presence of some large cardinals. We will need the following consequence of determinacy:

- f.a. $x \in \mathbb{R}$ and f.a. $A \subset \mathbb{R}$, there is a $\mathbb{P}_{\max}$ condition p such that $x \in p$, where p is A-iterable. i.e., $A \cap p \in p$ and $i(A \cap p) = A \cap p^*$ for all (countably long) generic iterations $i : p \to p^*$.

It is obvious that $L(\mathbb{R})^{\mathbb{P}_{\max}} \vDash \mathbf{ZF}$. Now we prove that the axiom of choice holds true in $L(\mathbb{R})^{\mathbb{P}_{\max}}$.

Lemma 2. *For every $X \subset \omega_1$ in the forcing extension, there is some generically iterable countable model M such that X is included in the ω_1-th iterate as being given by the generic. We shall use this key fact in later proof.*

Proof. Let G be $L(\mathbb{R})$-generic for $\mathbb{P}_{\max}$. Let $X = \tau^G$. Assume that

$$\tau = \{(p, \check{\xi}) : p \in \mathbb{P}_{\max} \wedge \xi < \omega_1 \wedge p \Vdash \check{\xi} \in \tau\}.$$

Basically τ is a set of reals, since each p can be coded by a real. From now on we identify τ as its corresponding set of reals. Let $q \in \mathbb{P}_{\max}$ and $r < q$ with r being τ-iterable. F.a. $\xi < \omega_1^r$, $D_\xi = \{t \in \mathbb{P}_{\max}^r : t \parallel \tau \cap r \ni \check{\xi}\}$ is dense in r. Work in r, we build a $\mathbb{P}_{\max}$-filter H which meets all these dense sets and hence q is in it. We may also assume $t > r$ for all $t \in H$.

By density, we have such a r together with such a $H \in r$ such that $r \in G$. Let $i : r \to r^*$ be the (unique) iteration of r as being given by G.

Claim. Let $A = \{\xi < \omega_1^r : \exists s \in H[s \Vdash \check{\xi} \in \tau \cap r]\}$. Then $\tau^G = i(A)$.

This immediately follows from everything in $i(H)$ is weaker than every iteration along $q \to q^*$. This proves in the $L(\mathbb{R})[G]$, if $X \subset \omega_1$, then there is a $p \in G$ together with its (unique) iteration $p \to p^*$ of length ω_1 as being given by G such that $X \in p^*$.

Recall $\psi_{\mathbf{AC}}$:

> If $S, T \subset \omega_1$ stationary and co-stationary, there is a $\alpha < \omega_2$, and a surjection $f : \omega_1 \to \alpha$, and a club C such that f.a. $\xi \in C$:
>
> $$f_\alpha(\xi) = \mathrm{otp}(f\text{"}\xi) \in T \iff \xi \in S.$$

Lemma 3 (Woodin). $\psi_{\mathbf{AC}} \implies$ *there is an injection* $P(\omega_1) \to \omega_2$

Proof. Let $\omega_1 = \bigsqcup_{i<\omega_1} S_i$, with each S_i stationary. Let $X \subset \omega_1$ which differs from $\emptyset$ and ω_1. Let $S^X = \bigsqcup_{i \in X} S_i$. Pick T stationary and co-stationary. Let α^X be the least α as in the statement of $\psi_{\mathbf{AC}}$ for (T, S^X). □

Theorem 4. $L(\mathbb{R})^{\mathbb{P}_{\max}} \models \psi_{\mathbf{AC}}$. *Thus* $L(\mathbb{R})^{\mathbb{P}_{\max}} \models \mathbf{AC}$.

The second statement holds since by the above lemma, $L(\mathbb{R})^{\mathbb{P}_{\max}}$ has a well-ordering of $P(\omega_1)$, so we can actually prove that $L(\mathbb{R})^{\mathbb{P}_{\max}}$ is $L(P(\omega_1))$, therefore it has a well-ordering of everything. Also since $\phi_{\mathbf{AC}}$ is proved by $L(\mathbb{R})^{\mathbb{P}_{\max}}$, it also proves $2^{\aleph_1} = \aleph_2$.

Proof. (Sketch) Here we use the key fact metioned above. Given S, T, we have a generic iteration $i : q \to q^*$ as being given by G, a $L(\mathbb{R})$-generic filter of $\mathbb{P}_{\max}$. Let $q_\alpha, \alpha < \omega_1$ denotes all the iterates. This gives that $i(S \cap \omega_1^q, T \cap \omega_1^q) = (S, T)$. By density argument, we will get a club C such that for every $\alpha \in C$, $\omega_1^{q_\alpha} \in S$ iff $\omega_1^{q_{\alpha+1}} \in T$. □

□

Comment. The same argument can be modified to show that $L(\mathbb{R})^{\mathbb{P}_{\max}} \models$ ACG. Thus, $L(\mathbb{R})^{\mathbb{P}_{\max}} \models \mathfrak{u}_2 = \omega_2$, which is exactly the original motivation of seeking compatibilities of $\mathfrak{u}_2 = \omega_2$ and large cardinals.

2 The axiom $(*)$

Definition. $(*)$ is the conjunction of the following statements:

- $L(\mathbb{R}) \vDash \mathbf{AD}$; and
- $P(\omega_1) \subset L(\mathbb{R})[G]$ for some $L(\mathbb{R})$-generic filter $G \subset \mathbb{P}_{\max}$.

Comment. Trivially, if we force with $\mathbb{P}_{\max}$ then $(*)$ holds in the universe. In the proof showed above, we add a function $P(\omega_1) \to \omega_2$ by forcing with $\mathbb{P}_{\max}$, and there are no more other structure added. In particular, $L(\mathbb{R})^{\mathbb{P}_{\max}}$ does not have any "big" large cardinals like measurable cardinals.

Theorem 5 (Asperó,Schindler). $\mathbf{MM}^{++} \implies (*)$.

Remark. Some words about the proof of this theorem:

- $L(\mathbb{R}) \vDash \mathbf{AD}$ does not requires the full power of $\mathbf{MM}^{++}$. In fact, Steel ([3]) has shown that $\mathbf{PFA} \implies L(\mathbb{R}) \vDash \mathbf{AD}$.
- The only candidate for G:

  Let $A \subset \omega_1$ s.t. $\omega_1^V = \omega_1^{L[A]}$. Then

  $$\begin{aligned} G = G_A &= \{p \in \mathbb{P}_{\max} : p > (H^V_{\omega_2}; \in, \mathbf{NS}^V_{\omega_1}, A)\}; \\ &= \{p \in \mathbb{P}_{\max} : \exists p \xrightarrow{\omega_1\text{-long}} p^* = (p^*; \in, I, a) \text{ s.t. } I = \mathbf{NS}^V_{\omega_1} \cap p^*, a = A\}. \end{aligned}$$

- After proving G is generic to $L(\mathbb{R})$, then $P(\omega_1) \subset L(\mathbb{R})[G]$. This statement follows from an argument like the proof of $\mathbf{MM}^{++} \implies \mathbf{NS}_{\omega_1}$ saturated.
- The key statement is to show that G intersects with all dense subsets in $\mathbb{P}_{\max}$ in $L(\mathbb{R})$. The forcing notion we introduced in the proof of Theorem 1 would be a good starting point. It actually adds an element of $G \cap D$.

Comment. Woodin has introduced an axiom called $(*)^+$, which says:

> There is a determinacy model(i.e. transitive model which contains all the reals and satisfies $\mathbf{AD}$) $W \supsetneq L(\mathbb{R})$ and a W-generic filter G of $\mathbb{P}_{\max}$ s.t. $P(\mathbb{R}) \subset W[G]$.

We can ask questions like "What is the relation between $(*)^+$ and axioms like $\mathbf{MM}^{++}$". There are some recent Woodin's preprints which show that $(*)^+$ is false in all known models of $\mathbf{MM}$. Readers can also look at [4] for more details.

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