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Varsovian models, II, cont'd

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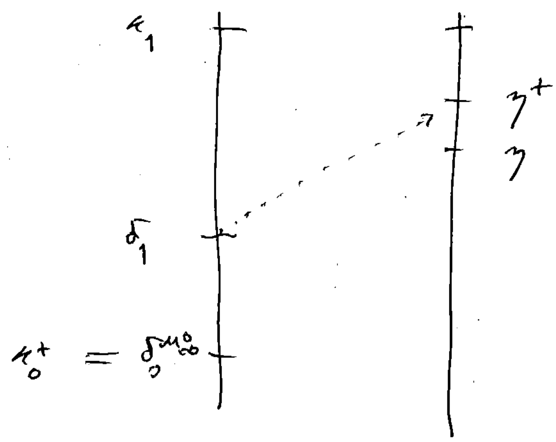
We are now going to define $\mathcal{M}_\infty^1$. Our starting point is $\mathcal{V}_0 = L[\mathcal{M}_\infty^0, p \mapsto p^*] = \mathrm{HOD}_{\mathcal{E}}^{M^{\mathrm{Col}(\omega, <\kappa_0)}} = \mathcal{P}$, where $\mathcal{P}$ is as on pp. 83–86. We will make use of Claim 24.

We will also make heavy use of another argument due to Farmer Schlutzenberg.



Let $\eta \in (\delta_1, \kappa_1)$ be a cutpoint of $\mathcal{V}_0$ in the sense that there is no $E_\iota^{\mathcal{V}_0} \neq \emptyset$ with $\mathrm{crit}(E_\iota^{\mathcal{V}_0}) < \eta$ and $\iota > \eta$.

Let us first consider a tree $\mathcal{T}$ on $\mathcal{V}_0$, $\mathcal{T} \in \mathcal{V}_0$, which lives on $(\delta_0^{\mathcal{M}_\infty^0}, \delta_1)$ and which starts making an initial segment of $\mathcal{V}_0|\eta^+$ generic over $\mathcal{M}(\mathcal{I})$. We aim to verify that if $\mathcal{I}$ is according to the strategy for $\mathcal{V}_0$ (as given by Claim 24), and if $\mathcal{I}$ has limit length, then $\mathcal{V}_0$ can compute the limit model $\mathcal{M}_b^{\mathcal{I}}$, when b is the correct branch thru b. (If $\mathcal{M}_b^{\mathcal{I}}$ comes with a Q-structure, b will be in $\mathcal{V}_0$; otherwise not.)

Let us also assume that $\mathcal{V}_0|\delta(\mathcal{I})$ is generic over $\mathcal{M}(\mathcal{I})$, and $\mathcal{I}$ (hence $\mathcal{M}(\mathcal{I})$) is definable over $\mathcal{V}_0|\delta(\mathcal{I})$ (otherwise we follow [SiLE]). We do a $\mathcal{P}$-construction over $\mathcal{M}(\mathcal{I})$ as follows:

- $\mathcal{P}|\delta(\mathcal{I}) = \mathcal{M}(\mathcal{I})$
- if $\mathcal{P}|\iota$ is constructed, and $E_\iota^{\mathcal{V}_0} \neq \emptyset$, $\mathrm{crit}(E_\iota^{\mathcal{V}_0}) > \delta(\mathcal{I})$, then we let $E_\iota^{\mathcal{V}_0} \cap \mathcal{P}|\iota$ be the top extender of $\mathcal{P}||\iota$
- if $\mathcal{P}|\iota$ is constructed, and $S_\iota^{\mathcal{V}_0} \neq \emptyset$ (cf. p. 86), then

$$\bar{S}_\iota = \{(\bar{\mathcal{I}}, \Sigma_{\mathcal{M}_\infty^{\mathcal{G}}|\delta_0^{\mathcal{M}_\infty}}(\bar{\mathcal{I}})) : \bar{\mathcal{I}} \in \mathcal{P}|\lambda^{\mathcal{P}|\iota}\}$$

is the top predicate of $\mathcal{P}||\iota$.

We stop the construction if $\delta(\mathcal{I})$ is not definably Woodin over $\mathcal{P}\|\nu$.

If $\delta(\mathcal{I})$ is overlapped in $\mathcal{L}_0$, then we do the construction in $ult(\mathcal{L}_0\|\alpha, F)$, where F is the least extender from the $\mathcal{L}_0$-sequence overlapping $\delta(\mathcal{I})$ and α is longest s.t. F measures $\mathcal{L}_0\|\alpha$.

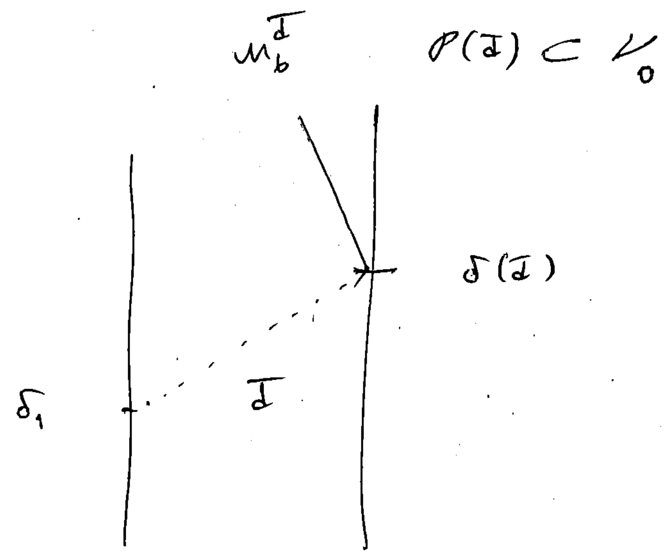
Let us write $\mathcal{P}(\mathcal{I})$ for the output of this construction. By arguments which are familiar by now, if $\mathcal{P}(\mathcal{I})$ is class sized, then $\mathcal{P}(\mathcal{I})$ is a ground for $\mathcal{L}_0$, in which case $\delta(\mathcal{I})$ can't be ~~Sm~~ between η and $\eta^+ = \eta^{+\mathcal{L}_0} = \eta^{+M}$. On the other hand, as we shall see, $\mathcal{P}(\mathcal{I})$ is an initial segment of the correct branch model for $\mathcal{I}$, so that if $\delta(\mathcal{I}) < \eta^+$, then $\mathcal{P}(\mathcal{I})$ gives the right Q-structure, and if $\delta(\mathcal{I}) = \eta^+$, then the process must stop in that $\mathcal{P}(\mathcal{I})$ is class sized (and in V, $\pi^{\mathcal{I}}_{ob}(\delta_1) = \delta(\mathcal{I})$, b the correct branch).

The following is the key claim.

Claim 25. Let $\Sigma_{\mathcal{V}_0}$ be the strategy for $\mathcal{V}_0$ given by Claim 24. Let $b = \Sigma_{\mathcal{V}_0}(\mathcal{I})$. Then

$$\mathcal{P}(\mathcal{I}) \trianglelefteq \mathcal{M}_b^{\mathcal{I}}.$$

Proof. We use an argument due to Farmer Schlutzenberg. By arguments which are familiar by now, it suffices to prove that $\mathcal{P}(\mathcal{I})$, $\mathcal{M}_b^{\mathcal{I}}$ may be successfully compared. By Claim 24, there is no issue about iterability; the worry is that we might get iterates of $\mathcal{P}(\mathcal{I})$, $\mathcal{M}_b^{\mathcal{I}}$ when the least disagreement is given by an S-predicate on one side.



$\mathcal{I}$ lifts to a tree, $\mathcal{I}'$, on $M = \mathcal{V}_0[M|\kappa_0^{+2}]$, and $(M, \mathcal{M}_b^{\mathcal{I}'}, \kappa_0^+)$ is iterable.

Let $\mathcal{U}_0$ on $(M, \mathcal{M}_b^{\mathcal{I}'}, \kappa_0^+)$ and $\mathcal{U}_1$ on M arise from a comparison where $E_\nu^{\mathcal{U}_0}, E_\nu^{\mathcal{U}_1}$ is always s.t. ν is least such that $\nu > \delta(\mathcal{I})$ and

- $E_\nu^{\mathcal{U}_0} \neq \emptyset = E_\nu^{\mathcal{U}_1}$ or
- $E_\nu^{\mathcal{U}_1} \neq \emptyset = E_\nu^{\mathcal{U}_0}$ or
- $\pi_{E_\nu^{\mathcal{U}_0}} \restriction \mathrm{crit}(E_\nu^{\mathcal{U}_0})^+ \neq \pi_{E_\nu^{\mathcal{U}_1}} \restriction \mathrm{crit}(E_\nu^{\mathcal{U}_0})^+$.

Here, $\pi_{E_\nu^{\mathcal{U}_h}}$ is the ultrapower map given by $E_\nu^{\mathcal{U}_h}$, $h = 0, 1$.

<u>Subclaim.</u> The final model $\mathcal{M}_\infty^{\mathcal{U}_0}$ of $\mathcal{U}_0$ is above $\mathcal{M}_b^{\mathcal{I}'}$ (not above M).

Assume the subclaim to be true. As κ_0 is the least strong cardinal of M (in particular, not a limit of cardinals strong up to it), the comparison then never uses an extender with critical point κ_0 on the $\mathcal{U}_0$-side, hence it also never uses an extender with critical point κ_0 on the $\mathcal{U}_1$-side.

In other words, the comparison is entirely above $\delta(I)$ on both sides (we may and shall assume for our purposes that $\delta(I)$ be not overlapped in $\mathcal{M}_b^{I\prime}$ by other extenders than ones with critical point $= \kappa_0$), and the comparison may be construed as a comparison of $\mathcal{M}_b^{I\prime}$ with M using the rules for "least disagreement" as above.

We have*) $M = L_0[M|\kappa_0^{+2}] = \mathcal{P}(I)[L_0|\delta(I)][M|\kappa_0^{+2}]$.

The forcing which adds $M|\kappa_0^{+2}$ over L_0 is also in $\mathcal{P}(I)$, so that this 2-step iteration is just a product, and we may also write

$$M = \mathcal{P}(I)[M|\kappa_0^{+2}][L_0|\delta(I)].$$

We have $\mathcal{M}_b^{I\prime} = \mathcal{M}_b^{I}[M|\kappa_0^{+2}]$. The tree $\mathcal{U}_0$ may then be construed as a tree, call it $\overline{\mathcal{U}_0}$, on $\mathcal{M}_b^{I}$, and the tree $\mathcal{U}_1$ may be

*) Let's assume $\mathcal{P}(I)$ is class sized. The argument in the other case is just a variant of what is to come.

construed as a tree, call it $\overline{\mathcal{U}}_1$, on $\mathcal{P}(\mathcal{I})$.

By the Consequences of the subclaim, $\overline{\mathcal{U}}_0$, $\overline{\mathcal{U}}_1$ are exactly the trees on $\mathcal{M}_b^{\mathcal{I}}$, $\mathcal{P}(\mathcal{I})$, resp., which arise from comparing these two models in the usual way (hitting the least extenders with disagreement); in particular, the fact that $\mathcal{U}_0$, $\mathcal{U}_1$ <u>never</u> use extenders with critical point κ_0 means that the least disagreement is never given by an S-predicate (cf. p. 86). Hence $\mathcal{M}_b^{\mathcal{I}}$, $\mathcal{P}(\mathcal{I})$ are in fact coiterable, as desired.

It thus remains to show the Subclaim on p. 99.

Deny. Let α be unique such that at stage α, $\mathcal{U}_0$ uses an extender F_0 with critical point κ_0. $\mathcal{U}_0 \restriction \alpha$ is then a tree on $\mathcal{M}_b^{\mathcal{I}'}$ which is above $\delta(\mathcal{I})$, and $\mathcal{U}_0 \restriction [\alpha+1, \mathrm{lh}(\mathcal{U}_0))$ is a tree on $\mathrm{ult}(M; F_0)$ which is above $\pi_{F_0}(\kappa_0)$.

By the definition of $\mathcal{U}_0$, $\mathcal{U}_1$, $\mathcal{U}_1$ must then also use an extender with critical point κ_0, and if β is unique s.t. $\mathcal{U}_1$ uses such an extender, F_1, at stage β, $\mathcal{U}_1 \restriction \beta$ is then a tree above $\delta(I)$, and $\mathcal{U}_1 \restriction [\beta+1, lh(\mathcal{U}_1))$ is a tree on $ult(M; F_1)$ which is above $\pi_{F_1}(\kappa_0)$.

$\mathcal{U}_0 \restriction \alpha$ may be construed as a tree on $\mathcal{M}_b^{I'}[\mathcal{L}_0|\delta(I)]$

$$= \mathcal{M}_b^{I}[M|\kappa_0^{+2}][\mathcal{L}_0|\delta(I)] = \mathcal{M}_b^{I}[\mathcal{L}_0|\delta(I)][M|\kappa_0^{+2}]$$

(cf. the displayed formula on p. 100). As $\mathcal{L}_0[M|\kappa_0^{+2}] = M$, this means that $\mathcal{M}_b^{I'}[\mathcal{L}_0|\delta(I)]$ has $M|\delta(I)$ as an initial segment.

$\mathcal{M}_\alpha^{\mathcal{U}_0}[\mathcal{L}_0|\delta(I)]$ then also has $M|\delta(I)$ as an initial segment, and by coherency of F_0 with $\mathcal{M}_\alpha^{\mathcal{U}_0}$, $\mathcal{M}_{\alpha+1}^{\mathcal{U}_0} = ult(M; F_0)[\mathcal{L}_0|\delta(I)]$ also has $M|\delta(I)$ as an initial segment.

More is true. $\mathcal{U}_0 \restriction \alpha$, $\mathcal{U}_1 \restriction \alpha$ may in fact be construed as the beginning of a comparison of $\mathcal{M}_b^{\mathcal{I}'}[\nu_0 | \delta(\mathcal{I})]$ with M in the "usual" sense.

If ν is the index of F_0, hence, this all means that $\mathcal{M}_\alpha^{\mathcal{U}_0}[\nu_0 | \delta(\mathcal{I})] \,|\, \nu = \mathcal{M}_\alpha^{\mathcal{U}_1} | \nu$, so that also

$$(\dagger) \qquad \mathrm{ult}(M; F_0)[\nu_0 | \delta(\mathcal{I})] \,\|\, \nu = \mathcal{M}_{\alpha+1}^{\mathcal{U}_1} \,\|\, \nu.$$

(We don't know if the map from M to $\mathrm{ult}(M; F_0)$ induces a map from M to $\mathrm{ult}(M; F_0)[\nu_0 | \delta(\mathcal{I})]$, but that doesn't matter.)

The rest of $\mathcal{U}_0, \mathcal{U}_1$, i.e. $\mathcal{U}_0 \upharpoonright [\alpha+1, \mathrm{lh}(\mathcal{U}_0))$, $\mathcal{U}_1 \upharpoonright [\alpha+1, \mathrm{lh}(\mathcal{U}_1))$ may then be constructed as a comparison in the usual sense, between $\mathrm{ult}(M; F_0)[\nu_0 | \delta(\mathcal{I})]$ and $\Phi(\mathcal{U}_1 \upharpoonright \alpha+1)$ (the latter being the phalanx produced by $\mathcal{U}_1 \upharpoonright \alpha+1$), or between the former and $\mathrm{ult}(M; F_1)$:

$$\begin{array}{ccc} \mathrm{ult}(M; F_0)[\nu_0 | \delta(\mathcal{I})] & \xrightarrow{i} & \\ & & Q \\ \mathrm{ult}(M; F_1) & \xrightarrow{j} & \end{array}$$

There must then be a common coiterate, Q (as both $ult(M; F_0)[\nu_0 | \delta(I)]$, $ult(M; F_1)$ think "I'm M_{swsw}.") Let i denote the map from $ult(M; F_0)[\nu_0 | \delta(I)]$ to Q, and let j denote the map from $ult(M; F_1)$ to Q.

Case 1. $\pi_{F_0}(\kappa_0) = \pi_{F_1}(\kappa_0)$.

In that case $ult(M; F_0)(\nu_0 | \delta(I))$ and $ult(M; F_1)$ are both their own hulls from $\pi_{F_0}(\kappa_0) = \pi_{F_1}(\kappa_0)$ together with indiscernibles for M, so that $i = j$ and $ult(M; F_0)[\nu_0 | \delta(I)] = ult(M; F_1)$.

$\pi_{F_0} \restriction \kappa_0^+$ is given by how $\mathcal{M}_\infty^0 | \delta_0^{\mathcal{M}_\infty^0}$ gets mapped into the direct lim of ~~$\mathcal{M}_\infty^0 | \delta_0^{\mathcal{M}_\infty^0}$~~ all iterates of $\mathcal{M}_\infty^0 | \delta_0^{\mathcal{M}_\infty^0}$ for trees in $\mathcal{M}_\alpha^{u_0} | \lambda^{F_0}$, which is the same as the direct lim of all iterates of $\mathcal{M}_\infty^0 | \delta_0^{\mathcal{M}_\infty^0}$ for trees in ~~$\mathcal{M}_\alpha^{u_0} | \lambda^{F_0}$~~

$\mathcal{M}_\alpha^{u_0} | \lambda^{F_0}[\nu_0 | \delta(I)] = \mathcal{M}_\alpha^{u_1} | \lambda^{F_1}$, so that F_0, F_1

have the same action on the ordinal. This is a contradiction.

<u>Case 2.</u> $\pi_{F_0}(\kappa_0) < \pi_{F_1}(\kappa_0)$.

Let us write I for the class of generating M-indiscernibles. We have

$$\mathrm{ult}(M;F_0)[\nu_0 \mid \delta(I)] \overset{i}{\cong} \mathrm{Hull}^{Q}(\pi_{F_0}(\kappa_0) \cup I),$$

and $$\mathrm{ult}(M;F_1) \underset{j}{\cong} \mathrm{Hull}^{Q}(\pi_{F_1}(\kappa_1) \cup I).$$

We thus get that

$$j^{-1} \circ i : \underbrace{\mathrm{ult}(M;F_0)[\nu_0 \mid \delta(I)]}_{\substack{\nabla \text{ by } (\dagger) \text{ on p. } 103 \\ \mathrm{ult}(M;F_1) \| \nu}} \longrightarrow \mathrm{ult}(M;F_1),$$

where ~~[illegible]~~ $j^{-1} \circ i(\pi_{F_0}(\kappa_0)) = \pi_{F_1}(\kappa_0)$.

This however means that $j^{-1} \circ i$ provides a counterexample to the initial segment condition of F_1.

Case 3. $\qquad \pi_{F_0}(\kappa_0) > \pi_{F_1}(\kappa_0)$.

This is symmetric to the previous case. We have that

$$i^{-1} \circ j : \text{ult}(M; F_1) \longrightarrow \text{ult}(M; F_0)[\kappa_0 | \delta(\mathcal{I})],$$

where $i_0^{-1} j(\pi_{F_1}(\kappa_0)) = \pi_{F_0}(\kappa_0)$.

We have $\beta < \alpha$, so $\mathcal{U}_0 \restriction \beta$, $\mathcal{U}_1 \restriction \beta$ are both just using extenders above $\delta(\mathcal{I})$, and these trees may be construed as the beginning of the comparison in the usual sense of M, $\mathcal{M}_b^{\mathcal{I}'}[\nu | \delta(\mathcal{I})]$. In particular, if $\bar{\nu}$ is the index of F_1, then

$$\begin{aligned} \text{ult}(M; F_1) \| \bar{\nu} = \mathcal{M}^{\mathcal{U}_1}_{\beta+1} \| \bar{\nu} &= \mathcal{M}^{\mathcal{U}_0}_{\beta+1}[\nu | \delta(\mathcal{I})] \| \bar{\nu}, \\ &= \mathcal{M}^{\mathcal{U}_0}_{\alpha+1}[\nu | \delta(\mathcal{I})] \| \bar{\nu} \\ &= \text{ult}(M; F_0)[\nu | \delta(\mathcal{I})] \| \bar{\nu}. \end{aligned}$$

Therefore, $\quad \text{ult}(M; F_0)[\nu | \delta(\mathcal{I})] \| \bar{\nu} \;=$

$H_{\bar{\nu}}^{ult(M;F_1)}$, so that writing $\mathcal{R} =$ $(i^{-1}\circ j)^{-1}$"$ult(M;F_0)$, $\mathcal{R}\|\bar{\nu} = ult(M;F_0)\|\bar{\nu}$, $\nu_0 | \delta(\mathcal{I})$ is generic over $\mathcal{R}$ for the extender algebra at $\delta(\mathcal{I})$, and

$$(i^{-1}\circ j)\restriction\mathcal{R} : \quad \mathcal{R} \longrightarrow ult(M;F_0) .$$

This yields a contradiction with the initial segment condition of F_0. $\dashv$ (Claim 25)

Let us now consider a more general tree $\mathcal{I}$ on ν_0. The elements of the system forming $\mathcal{M}^1_\infty$ will be $\mathcal{P}$-constructions over $\mathcal{M}(\mathcal{I})$ for such $\mathcal{I}$.

Let us fix a tree $\mathcal{I}$ on ν_0, $\mathcal{I} \in M$, which lives below δ_1. We suppose that there is some $\alpha < lh(\mathcal{I})$ s.t. $\mathcal{I}\restriction\alpha$ lives below $\delta_0^{\mathcal{M}^0_\infty} = \delta_0^{\nu_0}$ and there is some $F = E_\nu^M$ with $crit(F) = \kappa_0$ and $\nu < \kappa_1$, F total on M, such that $[0,\alpha)_{\mathcal{I}} \cap \mathcal{D}^{\mathcal{I}} = \emptyset$ and

$$\pi_{0\alpha}^{\mathcal{I}} = \pi_F^M \restriction \nu_0 .$$

(Any $\bar{\mathcal{T}} \in M|\delta_1$ on $\mathcal{L}_0$ living on $\mathcal{L}_0|\delta_0^{\mathcal{M}_\infty}$ can be absorbed by such a $\mathcal{T}\restriction\alpha+1$; this will be a crucial point.) Next, we assume that $\mathcal{T}\restriction[\alpha, lh(\mathcal{T}))$ is as $\mathcal{T}$ before, i.e., $\mathcal{T}$ lives on $(\delta_0^{\mathcal{M}_\alpha^{\mathcal{T}}}, \pi_{0\alpha}^{\mathcal{T}}(\delta_1))$ and $\mathcal{T}$ starts making an initial segment of $M|\eta^+$ generic over $\mathcal{M}(\mathcal{T})$, where $\eta \in (\pi_{0\alpha}^{\mathcal{T}}(\delta_1), \kappa_1)$ again is a cutpoint of $\mathcal{L}_0$.

Let us assume $\mathcal{T}$ is according to the strategy for $\mathcal{L}_0$ (as given by Claim 24).

Let us also assume that $lh(\mathcal{T})$ is a limit ordinal $\leq \eta^+ = \eta^{+M}$, $M|\delta(\mathcal{T})$ is generic over $\mathcal{M}(\mathcal{T})$, and $\mathcal{T}$ (and hence $\mathcal{M}(\mathcal{T})$) is definable over $M|\delta(\mathcal{T})$ (otherwise we follow [SiLE]).

We aim to verify that M can compute $\mathcal{M}_b^{\mathcal{T}}$, where b is the cofinal branch thru $\mathcal{T}$ acc. to the strategy for $\mathcal{L}_0$ (as given by Claim 24).

In order to compute $\mathcal{M}_b^{\mathcal{T}}$, we do a $\mathcal{P}$-construction over $\mathcal{M}(\mathcal{T})$ as follows.

- $\mathcal{P} | \delta(\mathcal{I}) = \mathcal{M}(\mathcal{I})$
- if $\mathcal{P}|\nu$ is constructed and $E_\nu^{\kappa_0} \neq \emptyset$ with $\mathrm{crit}(E_\nu^{\kappa_0}) > \delta(\mathcal{I})$ (equivalently, $E_\nu^{M} \neq \emptyset$, $\mathrm{crit}(E_\nu^{M}) > \delta(\mathcal{I})$), then we let $E_\nu^{\kappa_0} \cap \mathcal{P}|\nu = E_\nu^{M} \cap \mathcal{P}|\nu$ be the top extender of $\mathcal{P}||\nu$
- if $\mathcal{P}|\nu$ is constructed, and $S_\nu^{\kappa_0} \neq \emptyset$ (cf. p. 86) (equivalently, $E_\nu^{M} \neq \emptyset$ and $\mathrm{crit}(E_\nu^{M}) = \kappa_0$), then

$$\bar{S}_\nu = \{ (\bar{\mathcal{I}}, \Sigma_{\pi_{0\alpha}^{\mathcal{I}}(M_\infty^0 | \delta_0^{M_\infty^0})}(\bar{\mathcal{I}})) : \bar{\mathcal{I}} \in \mathcal{P} | \lambda^{\mathcal{P}|\nu} \}$$

is the top predicate of $\mathcal{P}||\nu$.

[$\bar{S}_\nu$ in this case is intertranslatable with the canonical map from $\pi_{0\alpha}^{\mathcal{I}}(M_\infty^0 | \delta_0^{M_\infty^0})$ into the direct limit of all non-dropping iterates of $\pi_{0\alpha}^{\mathcal{I}}(M_\infty^0 | \delta_0^{M_\infty^0})$ w.r.t. trees which are in $\mathcal{P}|\lambda^{\mathcal{P}|\nu}$.] ~~which is intertranslatable with~~ ~~E_ν^M~~

We let $\mathcal{P}(\mathcal{I})$ be defined much as on p. 97.

<u>Claim 26.</u> Let $\Sigma_{\mathcal{V}_0}$ be the strategy for $\mathcal{V}_0$ given by Claim 24. Let $b = \Sigma_{\mathcal{V}_0}(\mathcal{I})$. Then

$$\mathcal{P}(\mathcal{I}) \trianglelefteq \mathcal{M}_b^{\mathcal{I}} .$$

Proof: This is basically be the same argument as for Claim 25, but we present it slightly differently. In order to verify Claim 26, we need to see that $\mathcal{P}(\mathcal{I})$, $\mathcal{M}_b^{\mathcal{I}}$ may be successfully compared via trees which only use extenders with critical points above $\delta(\mathcal{I})$. Let us focus on the case that $\mathcal{P}(\mathcal{I})$ is a proper class.

By an argument as on pp. 28–32, $\mathcal{V}_0$, and hence M, is generic over $\mathcal{P}(\mathcal{I})$ for a forcing of size $\delta(\mathcal{I})$ which has the $\delta(\mathcal{I})$-c.c. Any tree $\mathcal{U}$ on $\mathcal{P}(\mathcal{I})$ which only uses extenders with critical points above $\delta(\mathcal{I})$ may therefore also be construed as a tree on $M = \mathcal{P}(\mathcal{I})[M|\delta(\mathcal{I})]$, and the indices $\nu > \delta(\mathcal{I})$ where $\mathcal{M}_\infty^{\mathcal{U}}$ (all construed as a tree on $\mathcal{P}(\mathcal{I})$ activates strategy information for

$\pi^{\mathcal{I}}_{0\alpha}(\mathcal{M}^0_\infty | \delta_0^{\mathcal{M}^0_\infty})$ are exactly the indices $\nu > \delta(\mathcal{I})$ where $\mathcal{M}^{\mathcal{U}}_\infty$ ($\mathcal{U}$ construed as a tree on M) has an extender with critical point κ_0.

Also, $\mathcal{I}$ may be construed as a tree on M (rather than $\mathcal{L}_0$) in the following way. Let $\mathcal{I}'$ be the tree on M which first uses F (see p. 107) and then lets $\mathcal{I} \restriction [\alpha, lh(\mathcal{I}))$ act on $ult(M; F) = \mathcal{M}^{\mathcal{I}'}_1$. I.e., $\mathcal{I}'$ is just $\mathcal{I}$ with $\mathcal{I} \restriction \alpha+1$ amalgamated to the application of one (long) extender $\pi^{\mathcal{I}}_{0\alpha} = \pi^M_F \restriction \mathcal{M}^0_\infty | \delta_0^{\mathcal{M}^0_\infty}$.

We have that $M = \mathcal{L}_0[g]$ via Bukowský, so that

$$\mathcal{M}^{\mathcal{I}'}_1 = ult(M; F) = \mathcal{L}_0^{ult(M;F)}[\pi^M_F(g)] .$$

But $\mathcal{L}_0^{ult(M;F)} = \mathcal{M}^{\mathcal{I}}_\alpha$, and hence

$$\mathcal{M}^{\mathcal{I}'}_1 = \mathcal{M}^{\mathcal{I}}_\alpha [\pi^M_F(g)] .$$

This then gives that

$$\mathcal{M}^{\mathcal{I}'}_\infty = \mathcal{M}^{\mathcal{I}}_\infty [\pi^M_F(g)] .$$

Any tree $\mathcal{U}$ on $\mathcal{M}_\infty^{\mathcal{I}} = \mathcal{M}_b^{\mathcal{I}}$ which only uses extenders with critical points above $\delta(\mathcal{I})$ may therefore also be construed as a tree on $\mathcal{M}_\infty^{\mathcal{I}'}$, which is an iterate of M, and the indices $\nu > \delta(\mathcal{I})$ where $\mathcal{M}_\infty^{\mathcal{U}}$ ($\mathcal{U}$ construed as a tree on $\mathcal{M}_b^{\mathcal{I}}$) activates strategy information for $\pi_{0\alpha}^{\mathcal{I}}(\mathcal{M}_\infty^0 | \delta_0^{\mathcal{M}_\infty^0})$ are exactly the indices $\nu > \delta(\mathcal{I})$ when $\mathcal{M}_\infty^{\mathcal{U}}$ (construed as a tree on $\mathcal{M}_\infty^{\mathcal{I}'}$) has an extender with critical point $\pi_F^M(\kappa_0)$.

In order to show that $\mathcal{P}(\mathcal{I})$, $\mathcal{M}_b^{\mathcal{I}}$ may be successfully compared via trees which only use extenders with critical points above $\delta(\mathcal{I})$, it thus suffices to prove the following:

(†) Let us compare M, $(\mathcal{M}_1^{\mathcal{I}'}, \mathcal{M}_\infty^{\mathcal{I}'}, \delta(\mathcal{I}))$ in the following way, producing trees $\mathcal{U}^1$, $\mathcal{U}^2$ on M, $(\mathcal{M}_1^{\mathcal{I}'}, \mathcal{M}_\infty^{\mathcal{I}'}, \delta(\mathcal{I}))$, respectively. Suppose $\mathcal{M}_\beta^{\mathcal{U}^1}$, $\mathcal{M}_\beta^{\mathcal{U}^2}$ are given. Let ν be least s.t. one of $E_\nu^{\mathcal{M}_\beta^{\mathcal{U}^1}}$, $E_\nu^{\mathcal{M}_\beta^{\mathcal{U}^2}}$ is nonempty and has critical point $\notin \{\kappa_0, \pi_F^M(\kappa_0)\}$ and $E_\nu^{\mathcal{M}_\beta^{\mathcal{U}^1}}$ and $E_\nu^{\mathcal{M}_\beta^{\mathcal{U}^2}}$

act differently on the local cardinal successors of their critical points, or one of $E_\nu^{m_\beta^{u^1}}$, $E_\nu^{m_\beta^{u^2}}$ is non-empty and the other not, or both $E_\nu^{m_\beta^{u^1}}$, $E_\nu^{m_\beta^{u^2}}$ are nonempty, $\mathrm{crit}(E_\nu^{m_\beta^{u^1}}) = \kappa_0$, $\mathrm{crit}(E_\nu^{m_\beta^{u^2}}) = \pi_F^M(\kappa_0)$, and

$$E_\nu^{m_\beta^{u^1}} \quad \text{and} \quad E_\nu^{m_\beta^{u^2}} \circ F$$

act differently on the cardinal successor of κ_0.

Then $E_\beta^{\mathcal{U}^1} = E_\nu^{m_\beta^{\mathcal{U}'}}$ and $E_\beta^{\mathcal{U}^2} = E_\nu^{m_\beta^{\mathcal{U}^2}}$.

Both $\mathcal{U}'$, $\mathcal{U}^2$, thus defined, only use extenders with critical points $> \delta(\mathcal{T})$.

The comparison $\mathcal{U}^1, \mathcal{U}^2$ terminates, and if the conclusion of (†) is wrong, then there is some ε s.t. $\mathcal{U}^2 \upharpoonright \varepsilon$ is an iteration of $m_\infty^{\mathcal{T}'}$, $\mathrm{crit}(E_\varepsilon^{\mathcal{U}^2}) = \pi_F^M(\kappa_0)$, and $\mathcal{U}^2 \upharpoonright [\varepsilon+1, \mathrm{lh}(\mathcal{U}^2))$ is an iteration of $\mathrm{ult}(m_1^{\mathcal{T}'}; E_\varepsilon^{\mathcal{U}^2})$ entirely above ε, and also there is some γ s.t. $\mathrm{crit}\left(E_\gamma^{\mathcal{U}^1}\right) = \kappa_0$ and $\mathcal{U}^1 \upharpoonright [\gamma+1, \mathrm{lh}(\mathcal{U}^1))$ is an iteration of $\mathrm{ult}(~~m_1~~\, M; E_\gamma^{\mathcal{U}^1})$ entirely above γ. Moreover, $M|\delta(I)$ is generic over $\mathcal{M}_\infty^{I'}|\delta(I) \lhd \mathrm{ult}(\mathcal{M}_1^{I'}; E_\varepsilon^{\mathcal{U}^2})$, so that the tail end $\mathcal{U}^1\restriction[\varepsilon+1,\infty)$, $\mathcal{U}^2\restriction[\varepsilon+1,\infty)$ of the comparison may be construed as a regular comparison between M and $\mathrm{ult}(M_1^{I'}; E_\varepsilon^{\mathcal{U}^2})[M|\delta(I)]$, ending in the same model, Q:

$$\begin{tikzcd}
M \arrow[r] & \mathrm{ult}(M; E_\gamma^{\mathcal{U}^1}) \arrow[rr] & & Q \\
M \arrow[r] & \mathrm{ult}(M;F) = \mathcal{M}_1^{I'} \arrow[r] & \mathrm{ult}(\mathcal{M}_1^{I'}; E_\varepsilon^{\mathcal{U}^2}) \arrow[d, phantom, "\cap"] & \\
 & & \mathrm{ult}(\mathcal{M}_1^{I'}; E_\varepsilon^{\mathcal{U}^2})[M|\delta(I)] \arrow[uur] &
\end{tikzcd}$$

If $\mathrm{lh}(E_\gamma^{\mathcal{U}^1}) > \mathrm{lh}(E_\varepsilon^{\mathcal{U}^2})$, we get a contradiction with the initial segment condition of $E_\gamma^{\mathcal{U}^1}$.

If $\mathrm{lh}(E_\gamma^{\mathcal{U}^1}) = \mathrm{lh}(E_\varepsilon^{\mathcal{U}^2})$, then we get a contradiction with the choice of $E_\gamma^{\mathcal{U}^1}$, $E_\varepsilon^{\mathcal{U}^2}$ (when $\gamma = \varepsilon$) in the comparison process $\mathcal{U}^1, \mathcal{U}^2$.

If $lh(E_\gamma^{\mathcal{U}^1}) < lh(E_\varepsilon^{\mathcal{U}^2})$, then we may derive a canonical map k from $ult(M; E_\gamma^{\mathcal{U}^1})$ to $ult(m_1^{\mathcal{I}'}; E_\varepsilon^{\mathcal{U}^2})[M|\delta(\mathcal{I})]$ from the above diagram, which, when restricted to k^{-1} " $ult(m_1^{\mathcal{I}'}; E_\varepsilon^{\mathcal{U}^2})$ gives a contradiction with the initial segment condition of $E_\varepsilon^{\mathcal{U}^2}$. $\dashv$ (Claim 26)

Let us now define the system which will produce m_∞^1. The elements of this system, $\mathcal{I}$, are of the form $\mathcal{P}(\mathcal{I})$ (cf. p. 97 and 109), where $\mathcal{I}$ is a tree on ν_0 s.t.

- there is some α such that $[0,\alpha]_{\mathcal{I}} \cap \mathcal{D}^{\mathcal{I}} = \emptyset$ and there is some $F = E_\nu^M$ with $crit(F) = \kappa_0$ and $\nu < \delta_1$ s.t. $\pi_{0\alpha}^{\mathcal{I}} = \pi_F^M \restriction \nu_0$, and
- $\mathcal{I} \restriction [\alpha, lh(\mathcal{I})]$ eventually makes $M|\eta^+$ generic over $m(\mathcal{I})$, where $\eta \in (\delta_1, \kappa_1)$ is a cutpoint of ν_0, $\eta^+ = \eta^{+\nu_0} = \eta^{+M}$, and $lh(\mathcal{I}) = \eta^+$.

By Claim 26, in V, every such $\mathcal{P}(\mathcal{I})$ is a class sized iterate of $\mathcal{L}_0$ by the right iteration strategy. If $\mathcal{P}, \mathcal{Q} \in \mathcal{F}$, then we write $\pi_{\mathcal{P},\mathcal{Q}}$ for the canonical map from $\mathcal{P}$ to $\mathcal{Q}$ (if it exists), and we let

$$\mathcal{M}^1_\infty = \text{dir lim}\left(\mathcal{P}, \pi_{\mathcal{P}\mathcal{Q}} : \mathcal{P}, \mathcal{Q} \in \mathcal{F}\right).$$

We write $\rho^{**} = \min\{\pi_{\mathcal{P},\infty}(\rho) : \mathcal{P} \in \mathcal{F}\}$, where $\pi_{\mathcal{P},\infty}$ is the canonical map from $\mathcal{P}$ into $\mathcal{M}^1_\infty$.

By arguments as before, $\mathcal{M}^1_\infty$ and $\rho \mapsto \rho^{**}$ are both definable in $\mathcal{L}_0$ (hence in M), and $\mathcal{L}_0$ (and hence M) is a generic extension of

$$\mathcal{L}_1 = L[\mathcal{M}^1_\infty, \rho \mapsto \rho^{**}]$$

for some forcing which has the (κ_1^+)-c.c. in $\mathcal{L}_1$.

We now have to analyze $\mathcal{L}_1$.

to be cont'd