Ralf Schindler March 2018

Groszek and Slaman on Prikry's problem

<u>Theorem</u> (Groszek – Slaman) Let $W \subset V$ be an inner model, $W \models CH$. Let P be a perfect set. There is then a perfect set $\bar{P} \subset P$ with $\bar{P} \subset V \setminus W$, provided that $(\mathbb{R} \cap V) \setminus W \neq \emptyset$.

We try to present the beautiful proof by Groszek–Slaman, see "A basis theorem for perfect sets," Bull. Symb. Logic 4 (2), 1998, pp. 204 – 209.

Proof: We may assume that $\omega_1^W = \omega_1^V$, as otherwise W has only countably many reals. This buys us that if $A \subset \mathbb{R} \cap W$ is countable in V, $A \in V$, then there is some $B \supset A$, $B \in W$, $W \models$ "B is countable."

Let us fix a perfect set $P \subset {}^{\omega}2$ in V. Say $P = [T]$, where T is on 2. We call $x \in [T]$ <u>eventually trivial</u> iff there is some $s \subsetneq x$ such that above s, x is the leftmost or the rightmost branch thru T.

<u>Case 1.</u> There is some $s \in T$ s.t. if $x \in [T_s]$ is not eventually trivial, then $x \in V \setminus W$.

(Here, $[T_s] = \{x : s \subset x\}$.)

$[T_s]$ has only countably many eventually trivial elements, so that $[T_s]$ has a perfect subset, say $\bar{P}$, consisting only of elements of $V \setminus W$. But then $\bar{P} \subset [T_s] \subset P$.

The following is then our main case.

<u>Case 2.</u> For all $s \in T$ there is some $x \in [T_s] \cap W$ which is not eventually trivial.

For each $s \in T$, pick $x_s \in [T_s] \cap W$ not eventually trivial. Let $\vec{g} = (g_n : n < \omega) \in W$ be a sequence of elements of ${}^{\omega}2 \cap W$ s.t. $\forall s \in T \, \exists n < \omega \; x_s = g_n$. $\vec{g}$ exists by the covering property from p. 1.

We shall also assume that $g_0 = x_{s_0}$, some $s_0 \in T$.

We shall now fix $\underline{S} \in ({}^{\omega}2 \cap V) \setminus W$ and construct $x, y \in {}^{\omega}2$ and subsequences $\vec{g}^x, \vec{g}^y$ of $\vec{g}$ s.t. $x, y \in [T]$ and

$(1)^*$ $\underline{S} \leq_T x, \vec{g}^x$, and

$(2)^*$ $\vec{g}^x, \vec{g}^y \leq_T x, y, \vec{g}$,

so that $\underline{S} \leq_T x, y, \vec{g}$. But then $x \in V \setminus W$ or $y \in V \setminus W$, and hence P will have a member in $V \setminus W$.

In a second round we will actually produce a perfect $\bar{P} \subset P$, $\bar{P} \subset V \setminus W$.

We call $s \in T$ a <u>splitting node</u> iff $s^\frown 0, s^\frown 1 \in T$.

We recursively produce strict initial segments of $x, \vec{g}^x, y, \vec{g}^y$, as follows.

$$\vec{g}^x = (\vec{g}^x_n : n < \omega) \qquad \vec{g}^y = (\vec{g}^y_n : n < \omega)$$

We start with $\vec{g}^x_0 = \vec{g}_0 = \vec{g}^y_0$.

Suppose now we are given $x \restriction m$, $\vec{g}_n^x$, $y \restriction k$, $\vec{g}_n^y$ such that

(a) $x \restriction m \subsetneq \vec{g}_n^x$,

(b) $\vec{g}_n^x = x_s$, some $s \in T$,

(c) $y \restriction k \subsetneq \vec{g}_n^y$, and

(d) $\vec{g}_n^y = x_{s'}$, some $s' \in T$.

(For $n=0$, we may just let $m = 0 = k$, and then (a) thru (d) will be satisfied.)

Now say $\vec{g}_n^y = \vec{g}_j$. Pick*) $m' > m, k$ such that $\vec{g}_\ell \restriction m' \neq \vec{g}_j \restriction m'$, all $\ell < j$. By (b), we may also assume that $\vec{g}_n^x \restriction m'$ is a splitting node in T and $\vec{g}_n^x(m') \neq \underline{s}(n)$.

Then set

$$x \restriction m'+1 = \vec{g}_n^x \restriction m' \frown \underline{s}(n),$$

and pick $\vec{g}_{n+1}^x$ s.t.

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*) we'll inductively have $m \leq k$.

there is some $s'' \in T$ with $\vec{g}^x_{n+1} = x_{s''}$

and $x \restriction m'+1 \subsetneq x_{s''}$.

Say $\vec{g}^x_{n+1} = \vec{g}_i$. Pick $k' > m'+1$

such that $\vec{g}_\ell \restriction k' \neq \vec{g}_i \restriction k'$, all $k < i$.

By (d), we may also assume that $\vec{g}^y_n \restriction k'$

is a splitting node.

Then set

$$y \restriction k'+1 = \vec{g}^y_n \restriction k' {}^\frown (1 - \vec{g}^y_n(k'))$$

and pick $\vec{g}^y_{n+1}$ s.t.

there is some $s''' \in T$ with $\vec{g}^y_{n+1} = x_{s'''}$

and $y \restriction k'+1 \subsetneq x_{s'''}$.

We're back to (a) thru (d) with $x \restriction m'+1$, $\vec{g}^x_{n+1}$, s'', $y \restriction k'+1$, $\vec{g}^y_{n+1}$, s''' replacing $x \restriction m$, $\vec{g}^x_n$, s, $y \restriction k$, $\vec{g}^y_n$, s', resp.

This finishes to construction of x, $\vec{g}^x$, y, $\vec{g}^y$.

For every $n<\omega$, $\underline{s}(n) = 1 - \vec{g}^x_n(m')$, where m' is maximal such that $x \restriction m' = \vec{g}^x_n \restriction m'$.

This shows $(1)^*$ on p. 3.

To show $(2)^*$ on p. 3, notice that $\vec{g}^y_n = \vec{g}_j$ for the least j s.t. $y \restriction m' = \vec{g}_j \restriction m'$, where m' is maximal with $x \restriction m' = \vec{g}^x_n \restriction m'$; also, $\vec{g}^x_{n+1} = \vec{g}_i$ for the least i s.t. $x \restriction k' = \vec{g}_i \restriction k'$, where k' is maximal with $y \restriction k' = \vec{g}^y_n \restriction k'$.

We have shown that $P \cap (V \setminus W) \neq \emptyset$.

Let us now prove the full theorem, varying the above argument.

By recursion on the length of $s \in {}^{<\omega}2$ we construct $x^s, y^s \in T$ and subsequences $\vec{g}^{x^s}$, $\vec{g}^{y^s}$ of $\vec{g}$ s.t.

(1) $x^{s^\frown 0}$, $x^{s^\frown 1}$ and $y^{s^\frown 0}$, $y^{s^\frown 1}$ are incompatible,

(2) $x^s \subsetneq x^{s'}$, $y^s \subsetneq y^{s'}$ for $s \subsetneq s'$,

(3) $\vec{g}^{x^s}$, $\vec{g}^{y^s}$ are sequences of elements from $\vec{g}$, in fact from $\{x_s : s \in T\}$, of length $\mathrm{lh}(s)+1$,

(4) $\vec{g}^{x^s} \subsetneq \vec{g}^{x^{s'}}$, $\vec{g}^{y^s} \subsetneq \vec{g}^{y^{s'}}$ for $s \subsetneq s'$,

and writing $x^z = \bigcup\{x^s : s \subset z\}$, $\vec{g}^{x^z} = \bigcup\{\vec{g}^{x^s} : s \subset z\}$, $y^z = \bigcup\{y^s : s \subset z\}$, $\vec{g}^{y^z} = \bigcup\{\vec{g}^{y^s} : s \subset z\}$ for $z \in {}^\omega 2$,

(5) $\underline{S} \leq_T x^z, \vec{g}^{x^z}$, and

(6) $\vec{g}^{x^z}, \vec{g}^{y^{z'}} \leq_T x^z, y^{z'}, \vec{g}$

for all $z, z' \in {}^\omega 2$.

In particular, $\underline{S} \leq_T x^z, y^{z'}, \vec{g}$ for all $z, z' \in {}^\omega 2$.

But then $\{x^z : z \in {}^\omega 2\} \subset V \setminus W$ or $\{y^z : z \in {}^\omega 2\} \subset V \setminus W$, because if $x^z, y^{z'} \in W$, we'd have $S \in W$. By (1), both $\{x^z : z \in {}^\omega 2\}$ and $\{y^z : z \in {}^\omega 2\}$ are perfect, so one of them is a perfect $\bar{P} \subset P$ consisting only of reals $\in V \setminus W$, as desired.

The construction of x^s, $\vec{g}^{x^s}$, y^s, $\vec{g}^{y^s}$ is basically as above, just building in (1).

Suppose we already defined x^s, $\vec{g}^{x^s}$, y^s, $\vec{g}^{y^s}$ for all $s \in {}^{<\omega}2$ of length $\leq n$. (Again, we start out with $x^\emptyset = \emptyset = y^\emptyset$, $\vec{g}^{x^\emptyset} = \langle \vec{g}_0 \rangle = \vec{g}^{y^\emptyset}$.)

Fix s of length n, and let's define $x^{s^\frown 0}$, $\vec{g}^{x^{s^\frown 0}}_{n+1}$, $x^{s^\frown 1}$, $\vec{g}^{x^{s^\frown 1}}_{n+1}$.

Let $j = \max \{ \bar{j} : \vec{g}^{y^t}_n = \vec{g}_{\bar{j}},\ \mathrm{lh}(t) = n \}$, and pick $m' > \max \{ \mathrm{lh}(x^t), \mathrm{lh}(y^t) : \mathrm{lh}(t) = n \}$ s.t. $\vec{g}_\ell \restriction m' \neq \vec{g}_{\ell'} \restriction m'$ for all $\ell, \ell' \leq j$, $\ell \neq \ell'$ and $m_1 > m_0 \geq m'$ are both s.t. $\vec{g}^{x^s}_n \restriction m_0$, $\vec{g}^{x^s}_n \restriction m_1$ are splitting nodes in T and $\vec{g}^{x^s}_n(m_0) \neq s(n) \neq \vec{g}^s_n(m_1)$.

Then set

$$x^{s^\frown 0} = \vec{g}^{\,s}_n \restriction m_0 {}^\frown \underline{s}(n),$$

$$x^{s^\frown 1} = \vec{g}^{\,s}_n \restriction m_1 {}^\frown \underline{s}(n),$$

and pick $\vec{g}^{\,x^{s^\frown 0}}_{n+1}$, $\vec{g}^{\,x^{s^\frown 1}}_{n+1}$ s.t. there are s'', $\bar{s}''$ $\in T$ with $x^{s^\frown 0} \subsetneq x_{s''} = \vec{g}^{\,x^{s^\frown 0}}_{n+1}$, $x^{s^\frown 1} \subsetneq x_{\bar{s}''} = \vec{g}^{\,x^{s^\frown 1}}_{n+1}$.

This defines all x^t, $\vec{g}^{\,x^t}_{n+1}$, $lh(t) = n+1$.

Again, fix s of length n, and let's define $y^{s^\frown 0}$, $\vec{g}^{\,y^{s^\frown 0}}_{n+1}$, $y^{s^\frown 1}$, $\vec{g}^{\,y^{s^\frown 1}}_{n+1}$.

Let $i = \max\{\bar{\imath} : \vec{g}^{\,x^t}_{n+1} = \vec{g}_{\bar{\imath}}, \ lh(t) = n+1\}$,

and pick $k' > \max\{lh(y^{\bar{t}}), lh(x^t) : lh(\bar{t}) = n \wedge lh(t) = n+1\}$,

s.t. $\vec{g}_l \restriction k' \neq \vec{g}_{l'} \restriction k'$ for all $l, l' \leq i$, $l \neq l'$,

and $k_1 > k_0 \geq k'$ are both s.t. $\vec{g}^{\,y^s}_n \restriction m_0$,

$\vec{g}_n^{\,y^s} \restriction m$, are splitting nodes in T.

Then set

$$y^{s^\frown 0} = \vec{g}^{\,s}_n \restriction k_0 {}^\frown (1 - \vec{g}^{\,s}_n(k_0)),$$

$$y^{s^\frown 1} = \vec{g}^{\,s}_n \restriction k_1 {}^\frown (1 - \vec{g}^{\,s}_n(k_1)),$$

and pick $\vec{g}_{n+1}^{\,y^{s^\frown 0}}$, $\vec{g}_{n+1}^{\,y^{s^\frown 1}}$ s.t. there are s''', $\bar{s}'''$ $\in T$ with $y^{s^\frown 0} \subsetneq x_{s'''} = \vec{g}_{n+1}^{\,y^{s^\frown 0}}$, $y^{s^\frown 1} \subsetneq$ $x_{\bar{s}'''} = \vec{g}_{n+1}^{\,y^{s^\frown 1}}$.

This defines all y^t, $\vec{g}_{n+1}^{\,y^t}$, $lh(t) = n+1$.

This finishes the construction.

The proofs of (5) and (6) on p. 7 are like the proofs of $(1)^*$ and $(2)^*$ on p. 3.

For each n, $\underline{S}(n) = x^z(m)$, where m is largest s.t. $z^z \restriction m = \vec{g}_n^{\,x^z} \restriction m$ (here, $z \in \omega_2$). This shows (5).

Moreover, $\vec{g}_n^{\,y^z} = \vec{g}_j$ for the least j s.t. $y \restriction m' = \vec{g}_j \restriction m'$, where m' is maximal with $x^{z'} \restriction m' = \vec{g}_n^{\,x^{z'}} \restriction m'$; also, $\vec{g}_{n+1}^{\,x^z} = \vec{g}_i$ for the least i s.t. $x^{z'} \restriction k' = \vec{g}_i \restriction k'$, where k' is maximal with $y^{z'} \restriction k' = \vec{g}_n^{\,y^{z'}} \restriction k'$. This shows (6).