Hamel bases and the principle of dependent choice

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Abstract

We construct, relative to the consistency of ZF, a model of ZF plus DC in which there is a Hamel basis but no well–ordering of $\mathbb{R}$.

1 Introduction

By a "Hamel basis" we mean a basis of $\mathbb{R}$, construed as a vector space over $\mathbb{Q}$. Answering a question from [4], we showed in [1] that the Cohen–Halpern–Lévy model has a Hamel basis. However, DC, the principle of dependent choice, fails badly in the Cohen–Halpern–Lévy model, so that [1] left open the question as to whether there is a model of ZF plus DC in which there is a Hamel basis but no well–ordering of $\mathbb{R}$. The current paper answers this in the affirmative, by further developing methods from [2] and [3].

Let us informally refer to a model M as a "Solovay model" iff M is obtained via a symmetric collapse over a model in which what is to become ω_1^M is either inaccessible or a limit of large cardinals (e.g., Woodin cardinals). The paper [2] shows that if U is a selective ultrafilter on ω which was added by forcing over a Solovay model M, then $M[U]$ satisfies the Open Coloring Axiom (see [2, p. 247]), hence $M[U]$ inherits from M the property that every uncountable set of reals that a perfect subset and in particular $M[U]$ does not contain a well–ordering of the reals, see [2, Theorem 5.1].

The paper [3] further explores this topic and studies which consequences of having a well–ordering of $\mathbb{R}$ remain false when adding certain ultrafilters on ω over a Solovay model or when adding a Vitali set. Also, [3] produces a model of ZF plus DC plus "there is a Hamel basis" plus "there is no well–ordering of the reals." The verification in [3] that the extension of the Solovay model via forcing with

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countable linearly independent sets of reals doesn't have a well–ordering of its reals uses Woodin's stationary tower forcing.

The current paper proposes a new version of a forcing to add a Hamel basis, cf. Definition 2.1. If ω_1^V is inaccessible to the reals, the forcing of Definition 2.1 is dense in the forcing to add a Hamel basis via forcing with countable linearly independent sets of reals, but in general these two forcings are different from each other. To prove the following theorem, it will be important that we use the forcing of Definition 2.1.

Theorem 1.1 *There is a symmetric extension of L in which the following hold true.*

(1) ZF *plus* DC.

(2) There is a Hamel basis.

(3) There is no well–ordering of the reals.

2 A Hamel basis

This section is devoted to a

Proof of Theorem 1.1.

Definition 2.1 *Let R be the set of reals of a given class–sized transitive model of set theory. We denote by $\mathbb{H}(R)$ the following forcing. $p \in \mathbb{H}(R)$ iff there is some $x \in R$ such that*

(a) $p \in L[x]$, and

(b) p is a Hamel basis for $\mathbb{R} \cap L[x]$.

We set $p \leq_{\mathbb{H}(R)} q$ iff $p \supseteq q$.

Of course, for any given R, $\mathbb{H}(R) \in W$ for all inner models W with $R \in W$, in particular if $R = \mathbb{R} \cap W$.

The following is easy to see. As $g \notin L(\mathbb{R}^0)$, the statement of Claim 2.2 only makes sense inside $L[g]$, though.

Claim 2.2 *Inside $L[g]$, the set of all $q \in \mathbb{H}(\mathbb{R}^0)$ such that there are $x \in \mathbb{R}^0$ and $\alpha < \omega_1$ such that*

(a) $q \in L[x]$,

(b) q is a Hamel basis for $\mathbb{R} \cap L[x]$, and

(c) $L[x] = L[g \restriction \alpha]$

is dense in $\mathbb{H}(\mathbb{R}^0)$.

Let g be $\mathbb{C}(\omega_1)$–generic over L. Here, $\mathbb{C}(\omega_1)$ is the forcing for adding ω_1 Cohen reals, cf. [5, Definition 6.27]. Let us write $\mathbb{R}^0 = \mathbb{R} \cap L[g]$.

Claim 2.3 *(a) In $L[g]$, $L(\mathbb{R}^0)$ is closed under ω–sequences.*
(b) For all $x \in \mathbb{R}^0$ and all $f \in {}^\omega L[x]$ there is some $y \in \mathbb{R}^0$ with $f \in L[y]$.

Proof. (a) Inside $L(\mathbb{R}^0)$, every set is ordinal definable from a real. As $\mathbb{R}^0 = \mathbb{R} \cap L[g]$, it suffices to show that ${}^\omega\mathrm{OR} \cap L[g] \subseteq L(\mathbb{R}^0)$. But if $\tau^g \in {}^\omega\mathrm{OR}$, where $\tau \in L^{\mathbb{C}(\omega_1)}$, say $p \Vdash_L^{\mathbb{C}(\omega_1)} \tau \colon \check{\omega} \to \mathrm{OR}$, where $p \in g$, then we may assume that

$$\tau = \{(q, (n,\xi)^\vee) \colon n < \omega,\, \xi \in \mathrm{OR},\, q \in A_n\},$$

where each A_n is an antichain in $\mathbb{C}(\omega_1)$. Because $\mathbb{C}(\omega_1)$ satisfies the c.c.c., this buys us that $\tau^g \in L[g \restriction \eta] \subset L(\mathbb{R}^0)$ for some $\eta < \omega_1$.
(b) This is established by the proof of (a). $\square$ (Claim 2.3)

By Claim 2.3 (a), $L(\mathbb{R}^0)$ is also the Chang model of $L[g]$ and $L(\mathbb{R}^0)$ is a model of ZF plus DC.

Claim 2.4 *(a) Two conditions p, $q \in \mathbb{H}(\mathbb{R}^0)$ are compatible iff $p \cup q$ is linearly independent.*
(b) $\mathbb{H}(\mathbb{R}^0)$ is ω–closed in $L[g]$ as well as in $L(\mathbb{R}^0)$.

Proof. (a) Let $p \in L[x]$, $q \in L[y]$, x, $y \in \mathbb{R}^0$. Then $\{p, q\} \subset L[x \oplus y]$ and if $r \in L[x \oplus y]$ is a Hamel basis for the reals of $L[x \oplus y]$, then $r \leq_{\mathbb{H}(\mathbb{R}^0)} p$ and $r \leq_{\mathbb{H}(\mathbb{R}^0)} q$.
(b) This is by a slight extension of the argument for (a). By Claim 2.3, it suffices to show that this is true in $L[g]$. Let $(p_n \colon n < \omega)$ be such that $p_{n+1} \leq_{\mathbb{H}(\mathbb{R}^0)} p_n$ for all $n < \omega$. If $p_n \in L[x_n]$, $x_n \in \mathbb{R}^0$, then $(p_n \colon n < \omega) \subset L[x]$ for $x = \oplus_{n<\omega} x_n$. But then by Claim 2.3 (b) there is some $y \in \mathbb{R}^0$ with $(p_n \colon n < \omega) \in L[y]$. Working in $L[y]$, let p be a Hamel basis for the reals of $L[y]$ which extends $\bigcup_{n<\omega} p_n$. Then $p \leq_{\mathbb{H}(\mathbb{R}^0)} p_n$ for all $n < \omega$. $\square$ (Claim 2.4)

Let h be $\mathbb{H}(\mathbb{R}^0)$–generic over $L[g]$ and hence also over $L(\mathbb{R}^0)$. By Claim 2.4 (b), $\mathbb{R} \cap L(\mathbb{R}^0)[h] = \mathbb{R} \cap L(\mathbb{R}^0) = \mathbb{R}^0$, and it is easy to see that $\bigcup h$ is a Hamel basis for $\mathbb{R}^0$. We are left with having to verify the following.

Claim 2.5 *In $L(\mathbb{R}^0)[h]$, there is no well–ordering of the reals.*

Proof. Assume that $L(\mathbb{R}^0)[h]$ has a well–ordering of $\mathbb{R}^0$. There is then some bijection $f \colon \omega_1 \to \mathbb{R}^0$ inside $L(\mathbb{R}^0)[h]$. Let $f = \tau^h$, where $\tau \in L(\mathbb{R}^0)^{\mathbb{H}(\mathbb{R}^0)}$. As $\tau \in L(\mathbb{R}^0)$, we may pick a formula φ, $z \in \mathbb{R}^0$, and an ordinal γ such that $\tau = \{u \colon L(\mathbb{R}^0) \models \varphi(u, z, \gamma)\}$.
Let $p_0 \in h$ be such that

$$(1) \qquad p_0 \Vdash_{L(\mathbb{R}^0)}^{\mathbb{H}(\mathbb{R}^0)} \{u \colon L(\mathbb{R}^0) \models \varphi(u, z, \gamma)\} \colon \omega_1 \to \mathbb{R}^0 \text{ is bijective.}$$

By Claim 2.2 we may and shall assume that there is some $\alpha < \omega_1$ such that $p_0 \in L[g \restriction \alpha]$ is a Hamel basis for $\mathbb{R} \cap L[g \restriction \alpha]$. In addition, we may certainly assume that $z \in L[g \restriction \alpha]$. We then have by (1), making use of the notation of [5, Definition 6.27],

$$(2) \qquad \Vdash_{L[g\restriction\alpha]}^{\mathbb{C}(\omega_1 \setminus \alpha)} \text{“}\check{p}_0 \Vdash_{L(\mathbb{R})}^{\mathbb{H}(\mathbb{R})} \{u \colon L(\mathbb{R}) \models \varphi(u, \check{z}, \check{\gamma})\} \colon \omega_1 \to \mathbb{R} \text{ is bijective,”}$$

where $\mathbb{R}$ refers to the set of reals of the forcing extension.

Let $\bar{g}$ be $\mathbb{C}(\omega_1 \setminus \alpha)$–generic over $L[g]$, and let $g' = g \restriction \alpha \frown \bar{g}$. Let $\mathbb{R}^1 = \mathbb{R} \cap L[g']$. By (2),

$$p_0 \Vdash^{\mathbb{H}(\mathbb{R}^1)}_{L(\mathbb{R}^1)} \{u \colon L(\mathbb{R}^1) \models \varphi(u, z, \gamma)\} \colon \omega_1 \to \mathbb{R}^1 \text{ is bijective.} \tag{3}$$

Let us also write $\mathbb{R}^2 = \mathbb{R} \cap L[g, \bar{g}]$. As $\mathbb{C}(\omega_1 \setminus \alpha) \cong \mathbb{C}(\omega_1 \setminus \alpha) \times \mathbb{C}(\omega_1 \setminus \alpha)$, (2) also yields

$$p_0 \Vdash^{\mathbb{H}(\mathbb{R}^2)}_{L(\mathbb{R}^2)} \{u \colon L(\mathbb{R}^2) \models \varphi(u, z, \gamma)\} \colon \omega_1 \to \mathbb{R}^2 \text{ is bijective.} \tag{4}$$

Let h' be $\mathbb{H}(\mathbb{R}^1)$–generic over $L(\mathbb{R}^1)$. As $\mathbb{R}^0 \neq \mathbb{R}^1$, by (3) and (4) we may certainly pick $\xi < \omega_1$, $n < \omega$, m, $m' < \omega$ with $m \neq m'$, $p \in h$, and $p' \in h'$ such that

$$p \Vdash^{\mathbb{H}(\mathbb{R}^0)}_{L(\mathbb{R}^0)} \{u \colon L(\mathbb{R}^0) \models \varphi(u, z, \gamma)\}(\xi)(n) = m, \tag{5}$$

and

$$p' \Vdash^{\mathbb{H}(\mathbb{R}^1)}_{L(\mathbb{R}^1)} \{u \colon L(\mathbb{R}^1) \models \varphi(u, z, \gamma)\}(\xi)(n) = m'. \tag{6}$$

By Claim 2.2 we may and shall assume that there are β, $\beta' < \omega_1$ such that $p \in L[g \restriction \beta]$ is a Hamel basis for $\mathbb{R} \cap L[g \restriction \beta]$ and $p' \in L[g' \restriction \beta']$ is a Hamel basis for $\mathbb{R} \cap L[g' \restriction \beta']$. (5) and (5) then yield

$$\Vdash^{\mathbb{C}(\omega_1 \setminus \beta)}_{L[g \restriction \beta]} \text{“}\check{p} \Vdash^{\mathbb{H}(\mathbb{R})}_{L(\mathbb{R})} \{u \colon L(\mathbb{R}) \models \varphi(u, \check{z}, \check{\gamma})\}(\check{\xi})(\check{n}) = \check{m}, \tag{7}$$

and

$$\Vdash^{\mathbb{C}(\omega_1 \setminus \beta')}_{L[g \restriction \beta']} \text{“}\check{p}' \Vdash^{\mathbb{H}(\mathbb{R})}_{L(\mathbb{R})} \{u \colon L(\mathbb{R}) \models \varphi(u, \check{z}, \check{\gamma})\}(\check{\xi})(\check{n}) = \check{m}', \tag{8}$$

where $\mathbb{R}$ refers to the set of reals of the respective forcing extension.

The following is the key fact.

Claim 2.6 *$p_0 \cup p \cup p'$ is linearily independent.*

Proof. Let

$$\sum_{n=1}^{k} r_n \cdot x_n + \sum_{n=1}^{l} s_n \cdot y_n + \sum_{n=1}^{m} t_n \cdot z_n = 0,$$

where $\{r_n \colon 1 \leq n \leq k\} \cup \{s_n \colon 1 \leq n \leq l\} \cup \{t_n \colon 1 \leq n \leq m\} \subset \mathbb{Q}$, $\{x_n \colon 1 \leq n \leq k\} \subset p_0 \subset L[g \restriction \alpha]$, $\{y_n \colon 1 \leq n \leq l\} \subseteq p \setminus p_0 \subset L[g]$, and $\{z_n \colon 1 \leq n \leq m\} \subset p' \setminus p_0 \subset L[g']$. Then

$$\sum_{n=1}^{m} t_n \cdot z_n \in L[g] \cap L[g'] = L[g \restriction \alpha],$$

so that we may write

$$\sum_{n=1}^{m} t_n \cdot z_n = \sum_{n=1}^{i} u_n \cdot v_n,$$

where $\{u_n \colon 1 \leq n \leq i\} \subset \mathbb{Q}$ and $\{v_n \colon 1 \leq n \leq i\} \subset p_0$. But then $p_0 \cup p'$ is not linearily independent, whereas $p_0 \cup p' \subseteq \bigcup h'$. Contradiction! $\square$ (Claim 2.6)

By (the proof of) Claim 2.4 (a) (with $\mathbb{R}^2$ replacing $\mathbb{R}^0$), Claim 2.6 ensures that we may now pick $q \in \mathbb{H}(\mathbb{R}^2)$ such that $q \leq_{\mathbb{H}(\mathbb{R}^2)} p_0$, p, p'. We may and shall assume without loss of generalization that $\alpha \leq \beta = \beta'$. We have that

$$\mathbb{C}(\omega_1 \setminus \beta) \cong \mathbb{C}(\omega_1 \setminus \beta) \times \mathbb{C}(\omega_1 \setminus \beta') \cong \mathbb{C}(\omega_1 \setminus \beta') \ ,$$

so that by (7) and (8),

$$p \Vdash^{\mathbb{H}(\mathbb{R}^2)}_{L(\mathbb{R}^2)} \{u \colon L(\mathbb{R}^2) \models \varphi(u, z, \gamma)\}(\xi)(n) = m, \tag{9}$$

and

$$p' \Vdash^{\mathbb{H}(\mathbb{R}^2)}_{L(\mathbb{R}^2)} \{u \colon L(\mathbb{R}^2) \models \varphi(u, z, \gamma)\}(\xi)(n) = m', \tag{10}$$

a contradiction with (4) and the existence of $q \leq_{\mathbb{H}(\mathbb{R}^2)} p_0$, p, p'. $\square$ (Theorem 1.1)

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