Ideal Extenders

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Abstract

We introduce the concept of an "ideal extender" as a system of precipitous ideals in much the same way as classical extenders correspond to systems of measures. We investigate the consistency strength of having one or more cardinals which carry ideal extenders.

This paper is devoted to the analysis of various generic embeddings. In the first section, we recapitulate the analysis of the relationship between precipitous ideals and $< \kappa$-complete ultrafilters. In the second section we define the concept of an "ideal extender" which we think is a natural counterpart to precipitous ideals in the strong context. That is, ideal extenders are to strong cardinals what precipitous ideals are to measurable cardinals, and the standard forcing technique to produce such ideal extenders is the same as in the case of precipitous ideals. It will be shown that the existence of these ideal extenders correspond consistency–wise to strong cardinals. In the third section we will discuss how to produce finitely many of such ideal extenders simultaneously. In the forth and last section we use the techniques developed in the previous sections to show that given ω supercompact cardinals, we can construct a model in which every $\aleph_n$, $0 < n < \omega$, is generically strong.

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1 In the case of a measurable

Let κ be a cardinal. The levy collapse of κ to ω_1, $\mathrm{col}(\omega, <\kappa)$, is the set of all finite function p such that $\mathrm{dom}(p) \subseteq \kappa \times \omega$ and for all $\langle \alpha, n \rangle \in \mathrm{dom}(p)$ $p(\langle \alpha, n \rangle) < \alpha$. We say that p is stronger than q, $p \leqslant_{\mathrm{col}(\omega,<\kappa)} q$, if $q \subseteq p$.

Fact 1.1. *If G is $\mathrm{col}(\omega, <\kappa)$-generic over V, $V[G] \vDash \kappa = \omega_1$.*

Let $\mathbb{P} = \mathrm{col}(\omega, <\kappa)$ and for $\nu < \kappa$, we set

$$\mathbb{P}^\nu = \{ p \in \mathbb{P}; \forall \langle \alpha, n \rangle \in \mathrm{dom}(p)\ \alpha \geqslant \nu \},$$

similarly

$$\mathbb{P}_\nu = \mathrm{col}(\omega, <\nu) = \{ p \in \mathbb{P}; \forall \langle \alpha, n \rangle \in \mathrm{dom}(p)\ \alpha < \nu \}.$$

It is easy to see that $\mathbb{P}$ is isomorphic to the product $\mathbb{P}_\nu \times \mathbb{P}^\nu$, for all ν.

Let κ be a measurable cardinal and U a normal $<\kappa$-complete ultrafilter on κ. Let

$$\pi : V \to M = \mathrm{Ult}(V, U)$$

be the ultrapower generated by U. We can split $\pi(\mathbb{P})$ in $\mathbb{P}$ and $\pi(\mathbb{P})^\kappa$. Let G be $\mathbb{P}$-generic over V and H be $\pi(\mathbb{P})^\kappa$-generic over $V[G]$. Every condition in $q \in \pi(\mathbb{P})^\kappa$ can be represented by a family $\langle q_\alpha; \alpha < \kappa \rangle$, that is $[\langle q_\alpha; \alpha < \kappa \rangle]_U = q$, where all the q_α are in $\mathbb{P}$. Moreover for U-almost all α $q_\alpha \in \mathbb{P}^\alpha$, since $[q] \in \pi(\mathbb{P})^\kappa \iff \{ \alpha; q_\alpha \in \mathbb{P}^\alpha \} \in U$.

In $V[G \times H]$ we can define a new $V[G]$-ultrafilter W by:

$$\tau^G \in W \iff \kappa \in (\pi(\tau))^{G \times H}.$$

Let $\dot{W}$ be the canonical name for W. For every $p \in \mathbb{P}$ and $q \in \pi(\mathbb{P})^\kappa$,

$$\langle p, q \rangle \Vdash \dot{X} \in \dot{W} \iff \text{ for } U\text{-measure one many } \alpha, p \cup q_\alpha \Vdash \check{\alpha} \in \dot{X}.$$

We will use this last remark to show that W is generic over $V[G]$ for the following forcing: $\mathbb{Q} = \{ X \in V[G]; \forall Y \in U\ Y \cap X \neq \varnothing \}$, where $X \leqslant_{\mathbb{Q}} Y$ if and only if $X \subseteq Y$. We already know that W is a $V[G]$ ultrafilter, so we only have to prove that it is generic. Suppose $X = \{ X_i; i < \theta \}$ is a maximal antichain in $V[G]$ and for all $i < \theta$, $X_i \notin W$. Let $\dot{X}, \dot{X}_i$ be names for X and X_i. Let $p \in G$ and $q \in H$ be such that:

$$\langle p, q \rangle \Vdash \forall i < \theta\ \dot{X}_i \notin \dot{W}$$

By the last remark $q = [\langle q_\alpha; \alpha < \kappa\rangle]_U$ and for each i there is a set $A_i \in U$ such that for all $\alpha \in A_i$ $p \cup q_\alpha \Vdash \alpha \notin \dot{X}_i$. Now let $T = \{\alpha, q_\alpha \in G\}$. We first prove that $T \cap X_i \notin \mathbb{Q}$. For each $i < \theta$, if $\alpha \in T \cap A_i$ we have that $q_\alpha \in G$ and $\alpha \notin X_i$. Therefore $T \cap X_i \cap A_i = \varnothing$ but $A_i \in U$ hence $T \cap X_i \notin \mathbb{Q}$. Thus T is incompatible with all X_i. If we can prove that $T \in \mathbb{Q}$ we would have that $\{X_i; i < \theta\}$ wasn't a maximal antichain, a contradiction. Let $Z \in U$, we have to prove that $T \cap Z \neq \varnothing$. We want to show that $q_\alpha \in G$ for some $\alpha \in Z$. Let

$$E = \{r \in \mathbb{P}; r \leqslant q_\alpha \text{ for some } \alpha \in Z\}$$

Let us show that E is dense. Take some $p \in \mathbb{P}$ and let β be the minimal such that $p \in \mathbb{P}_\beta$. Now since Z is unbounded in κ there is a $\alpha \in Z \setminus \beta$. But then, p and q_α have disjoint domains in a way that $p \cup q_\alpha \in \mathbb{P}$ and thus $r = p \cup q_\alpha$ is the strengthening of p that we were looking for. Thus $E \cap G \neq \varnothing$ and $T \cap Z \neq \varnothing$.

What we basically did is, starting with some embedding:

$$\pi : V \to M = \mathrm{Ult}(V, G)$$

to lift up π to some

$$\tilde{\pi} : V[G] \to M[G, H],$$

moreover if W is the ultrafilter derived from $\tilde{\pi}$, W is generic over $V[G]$ for the forcing $\mathbb{Q}$. This case was easy, because the forcing adding G was basically below κ, the critical point of π. But there are ways to lift up embeddings even when forcing above of a large cardinal. Let us first show a way to deal with it in the case of a measurable.

Lemma 1.2. *Assume* GCH. *Let κ be measurable, X_κ the set of all cardinals less or equal to κ and $\mathbb{P}$ the easton support iteration of* $\mathrm{col}(\xi, \xi)$*, the forcing adding a cohen subset of ξ, for all $\xi \in X_\kappa$. Let G be $\mathbb{P}$-generic over V, then in $V[G]$, κ is still measurable.*

Proof. Let U be an normal ultrafilter witnessing the measurability of κ. Let

$$j : V \to M = \mathrm{Ult}(V, U)$$

be the associated ultrapower map. $j(\mathbb{P})$ is the easton support iteration of $\mathrm{col}(\xi, \xi)$ for all $\xi \in j(X_\kappa) = X_{j(\kappa)}$, let $j(\mathbb{P})^\kappa$ for the part of the forcing starting after κ, that is $j(\mathbb{P}) = \mathbb{P} * \mathbb{P}^\kappa$. Let G be $\mathbb{P}$-generic over V, since $(H_\kappa)^V = (H_\kappa)^M$ and $\mathcal{P}(\kappa) \cap V = \mathcal{P}(\kappa) \cap M$ we have that G is $\mathbb{P}$-generic over M. If we can show that there is an $\tilde{G} \in V[G]$ such that $G * \tilde{G}$ is $j(\mathbb{P})$-generic over M and $j"G = \tilde{G} \cap \mathrm{ran}(j)$, we will be able to lift the embedding j to an embedding

$$\tilde{j} : V[G] \to M[G \times \tilde{G}]$$

By the Factor Lemma [Jec03, Lemma 21.8 pp. 396] it suffices to define $\tilde{G}$ such that it is $j(\mathbb{P})^\kappa$-generic over $M[G]$.

By [Kan03, Proposition 5.7 (b)], $2^\kappa \leq (2^\kappa)^M < j(\kappa)$. Notice that for the same reasons, we also have $j(\kappa^+) < (2^\kappa)^+ = \kappa^{++}$. Hence

$$\mathrm{card}^V(\{D \in M, D \text{ is dense in } j(\mathbb{P})^\kappa\}) = \kappa^+ = 2^\kappa$$

Every dense set of $j(\mathbb{P})^\kappa$ in $M[G]$ is of the form $j(f)(\kappa)^G$, where f is a function from κ to $V^{\mathbb{P}}$. Let $\langle f_i; i < \kappa^+ \rangle$ be an enumeration in V of functions representing all open dense sets of $j(\mathbb{P})^\kappa$ in $M[G]$. There is an enumeration with size κ^+ since in $V[G]$ there are at most κ^+ many such dense sets and $\mathbb{P}$ has the κ^+-c.c. Since M is κ-closed each initial segment $\langle j(f_i)(\kappa)^G; i < \alpha \rangle$ is in M, for $\alpha < \kappa^+$. But $j(\mathbb{P})^\kappa$ is also κ-closed, hence $\bigcap_{i<\alpha} j(f_i)(\kappa)^G$ is a dense set of $j(\mathbb{P})^\kappa$ in M. Now one can construct in $V[G]$ a sequence $\langle p_\alpha; \alpha < \kappa^+ \rangle$ with the following properties:

i. $p_0 = \bigcup j"(G \cap \mathrm{col}(\kappa, \kappa))$

ii. $p_\alpha < p_\beta$ for $\alpha < \beta$

iii. $p_\alpha \in \bigcap_{i<\alpha} j(f_i)(\kappa)^G$

Each element of the sequence is in M, and the sequence itself is in $V[G]$. Moreover $j"G \subseteq G * \tilde{G}$. Now we can lift j by using the classical definition:

$$j(\tau^G) = j(\tau)^{G * \tilde{G}} \qquad \dashv$$

for τ a $\mathbb{P}$-name in V.

2 One ideally strong cardinal

2.1 The definition of ideal extenders

Definition 2.1. Let κ be a cardinal, $\lambda > \kappa$ an ordinal and let X be a set. For every finite subset a of X let us fix one bijection between a and its cardinality. We identify finite sets of ordinals with their increasing enumeration, finite subsets of X with their previously fixed bijection.

i. A $\langle\kappa, X\rangle$-*system of filters* is a set

$$F \subseteq \left\{ \langle a, x\rangle \in [X]^{<\omega} \times \mathcal{P}([\kappa]^{<\omega}); x \subseteq [\kappa]^{\overline{\overline{a}}} \right\},$$

such that for all $a \in [X]^{<\omega}$, $F_a = \{x; \langle a, x\rangle \in F\}$ is a non trivial filter that is $F_a \neq \mathcal{P}([\kappa]^{\overline{\overline{a}}})$. We set $\mathrm{supp}(F) = \{a \in [X]^{<\omega}; F_a \neq \{X\}\}$.

ii. Let F be a $\langle\kappa, X\rangle$-system of filters. Let $a, b \in \mathrm{supp}(F)$, such that $a \subseteq b$. Let $s : \overline{\overline{a}} \to \overline{\overline{b}}$ be such that $a(n) = b(s(n))$. For a set $x \in \mathcal{P}([\kappa]^{\overline{\overline{a}}})$, we define

$$x_{a,b} = \left\{ \langle u_i; i < \overline{\overline{b}}\rangle \in [\kappa]^{\overline{\overline{b}}}; \langle u_{s(j)}; j < \overline{\overline{a}}\rangle \in x \right\}.$$

For a function $f : [\kappa]^{\overline{\overline{a}}} \to V$, we define $f_{a,b} : [\kappa]^{\overline{\overline{b}}} \to V$ by

$$f_{a,b}(\langle u_i; i < \overline{\overline{b}}\rangle) = f(\langle u_{s(j)}; j < \overline{\overline{a}}\rangle).$$

iii. A $\langle\kappa, X\rangle$-system of filters F is called *compatible* if for all $a \subseteq b \in \mathrm{supp}(F)$

$$x \in F_a \iff x_{a,b} \in F_b.$$

iv. Let $a \in [X]^{<\omega}$ and $x \in [\kappa]^{\overline{\overline{a}}}$, we say that $F' = \mathrm{span}\{F, \langle a, X\rangle\}$ is the *span of* F *and* $\langle a, x\rangle$ if it is the smallest compatible system of filters such that $F \subseteq F'$ and $\langle a, x\rangle \in F'$.

v. Let F be a $\langle\kappa, \lambda\rangle$-system of filters. The *associated forcing* $\mathbb{P}_F$ consists of all conditions $p = F^p$, where F^p is a compatible $\langle\kappa, \lambda\rangle$-system of filters, $\mathrm{supp}(p) = \mathrm{supp}(F^p) \subseteq \mathrm{supp}(F)$ is finite and F^p is generated by one point $x \in (F_a)^+$ for some $a \in \mathrm{supp}(p)$, i.e. F^p is the span of F and $\langle a, x\rangle$. $p \leqslant_{\mathbb{P}} q$ if and only if $\mathrm{supp}(q) \subseteq \mathrm{supp}(p)$ and for all $a \in \mathrm{supp}(q)$, $F^q_a \subseteq F^p_a$, that is if $F^q \subseteq F^p$.

Let F be a compatible (κ, λ)-systems of filters and G be $\mathbb{P}_F$-generic over V. Set $\dot{E}_F = \bigcup \dot{G}$, where $\dot{G}$ is the canonical name for the generic filter. Clearly $\dot{E}^G_F$ is a system of filters again. For any $a \in [\lambda]^{<\omega}$ and $X \in F^+_a$ we have that

$$A = \left\{ \mathrm{span}\{F, \langle a, X\rangle\}, \mathrm{span}\left\{F, \langle a, [\kappa]^{\overline{\overline{a}}} \setminus X\rangle\right\}\right\}$$

is an antichain in $\mathbb{P}_F$. This shows that each $\dot{E}^G_{F,a} = (\bigcup \dot{G})_a$ is an ultrafilter. Moreover $\dot{E}_F$ has the compatibility property. Let us now look how we can translate the normal and ω-closed concept to this situation.

Definition 2.2. Let κ, λ be as in the previous definition.

i. We call a $\langle\kappa,\lambda\rangle$-system of filters *potentially normal* if for every $p \in \mathbb{P}_F$, for every $a \in \operatorname{supp}(p)$ and for every $f : [\kappa]^{\overline{\overline{a}}} \to V$ if there is a $j < \overline{\overline{a}}$ such that
$$\{u, f(u) \in u_j\} \in F_a^p,$$
it follows that there is a dense set D below p such that for every $p' \in D$ there is a $\xi \in \operatorname{supp}(p')$ with
$$\left\{v, f_{a,a\cup\{\xi\}}(v) = v_i\right\} \in F_{a\cup\{\xi\}}^{p'},$$
where i is such that $s(i) = \xi$, s being the enumeration of $a \cup \{\xi\}$.

ii. We call a $\langle\kappa,\lambda\rangle$-system of filter *precipitous* if for all $p \in \mathbb{P}_F$ and for all systems $\langle\langle p_s, X_s, a_s\rangle; s \in {}^{<\omega}\theta\rangle$ such that:

  (a) $p_\varnothing = p$,
  (b) $a_s \subseteq a_{s^\frown i}$ for all $i < \theta$,
  (c) $p_{s^\frown i}$ contains the span of p_s and $\langle a_{s^\frown i}, X_{s^\frown i}\rangle$ for all i,
  (d) $\{p_{s^\frown i}; i < \theta\}$ is a maximal antichain below p_s,

there is an $x \in {}^\omega\theta$ and a $\tau : \bigcup_{s\subseteq x} a_s \to \kappa$ such that $\tau" a_s \in X_s$ for all $s \subseteq x$.

Definition 2.3. Let $\kappa < \lambda$ be ordinals. F is a $\langle\kappa,\lambda\rangle$-*ideal extender* if it is a compatible and potentially normal $\langle\kappa,\lambda\rangle$-system of filters such that for each $a \in \operatorname{supp}(F)$, F_a is $<\kappa$-closed.

Let F be a compatible $\langle\kappa,\lambda\rangle$-systems of filters and G be $\mathbb{P}_F$-generic over V. By compatibility and potential normality, we can see that $\dot{E}_F^G$ is a $\langle\kappa,\lambda\rangle$-extender over V. Hence we can construct the formal ultrapower, regardless of it being well-founded or not.

Lemma 2.4. *Let F be a $\langle\kappa,\lambda\rangle$-ideal extender and G be $\mathbb{P}_F$-generic over V. Let $\varphi(u)$ be a formula in the language of set theory in one free variable u. Łoś's theorem holds for generic ultrapowers, that is* $\operatorname{Ult}(V, \dot{E}_F^G) \models \varphi([a, f])$ *if and only if*
$$\left\{\vec{\alpha} \in [\kappa]^{\overline{\overline{a}}}; V \models \varphi(f(\vec{\alpha}))\right\} \in \dot{E}_{F,a}^G.$$

PROOF. We proceed by induction on the rank of the formula. For atomic formulae this holds by definition. We only prove the lemma for the negation and the existencial quantifier as the other cases are easy.

Let $\varphi \equiv \neg\psi$. It follows from the fact that each $\dot{E}_F^G$ is a system of ultrafilters, that is if $\langle a, x\rangle \notin \dot{E}_F^G$ then $\langle a, [\kappa]^{\overline{\overline{a}}} \setminus x\rangle \in \dot{E}_F^G$.

Let $\varphi([c,g]) \equiv \exists v\psi(v,[c,g])$. We first show that:

$$\mathrm{Ult}(V,\dot{E}_F^G) \vDash \varphi([c,g]) \implies \left\{\vec{\alpha} \in [\kappa]^{\overline{\overline{c}}}; V \vDash \varphi(g(\vec{\alpha}))\right\} \in \dot{E}_{F,c}^G$$

Let $[b,f]$ be such that $\psi([b,f],[c,g])$ holds. By induction hypothesis, there is a $\langle b\cup c, x\rangle \in G$ witnessing that $\psi([b,f],[c,g])$ is true, that is

$$x = \left\{\vec{\alpha} \in [\kappa]^{\overline{\overline{b\cup c}}}; V \vDash \psi(f_{b,b\cup c}(\vec{\alpha}), g_{c,b\cup c}(\vec{\alpha}))\right\} \in \dot{E}_{F,b\cup c}^G.$$

Since $\dot{E}_{F,b\cup c}^G$ is a filter, by a compatibility argument we can show that:

$$x_c^{b\cup c} \subseteq \left\{\vec{\beta} \in [\kappa]^{\overline{\overline{c}}}; V \vDash \exists x\psi(x, g(\vec{\beta}))\right\} \in \dot{E}_{F,c}^G.$$

Let us now prove the other direction, that is:

$$\left\{\vec{\alpha} \in [\kappa]^{\overline{\overline{c}}}; V \vDash \varphi(g(\vec{\alpha}))\right\} \in \dot{E}_{F,c}^G \implies \mathrm{Ult}(V,\dot{E}_F^G) \vDash \varphi([c,g]).$$

Let

$$y = \left\{\vec{\beta} \in [\kappa]^{\overline{\overline{c}}}; V \vDash \exists x\psi(x, g(\vec{\beta}))\right\} \in \dot{E}_{F,c}^G,$$

and f the function that assigns to some $\vec{\beta}$ some set x such that

$$V \vDash \psi(x, g(\vec{\beta})),$$

if one exists and the empty set else. f gives a witness for the fact that $\exists v\psi(v, g(\vec{\beta}))$ on a $\dot{E}_{F,c}^G$ measure one set. Thus by induction hypothesis

$$\mathrm{Ult}(V,\dot{E}_F^G) \vDash \psi([c,f],[c,g]). \qquad \dashv$$

Lemma 2.5. *A $\langle\kappa,\lambda\rangle$-ideal extender is precipitous if and only if the generic ultrapower given by any generic over the associated forcing is well-founded.*

PROOF. Suppose first that F is precipitous and that there is a condition $p \in \mathbb{P}_F$ such that $p \Vdash$" the ultrapower by $\dot{E}_F$ is ill-founded". That is there is a system $\langle [\dot{a}^n, \dot{f}^n], n < \omega\rangle$ such that

$$p \Vdash [\dot{a}^n, \dot{f}^n] > [\dot{a}^{n+1}, \dot{f}^{n+1}].$$

Without loss of generality we can fix a system $\langle\langle p_s, X_s, a_s\rangle, s \in {}^{<\omega}\theta\rangle$ with $p_\varnothing = p$ such that $\{p_{s^\frown i}; i < \theta\}$ is a maximal antichain below p_s and

$$p_s \Vdash \mathrm{dom}(\dot{f}_n) = \check{X}_s \in \dot{E}_{\check{a}_s} \wedge \dot{f}_n = \check{f}_s \wedge \check{a}_s = \dot{a}_n.$$

By precipitousness we then have a $x \in {}^\omega\theta$ and a $\tau : \bigcup_{s\subseteq x} a_s \to \kappa$ such that $\tau\text{"}a_s \in X_s$ for all $s \subseteq x$. Since all conditions are below p,

$$p_{x\restriction n+1} \Vdash \text{“}[\dot{a}^n, \dot{f}^n] > [\dot{a}^{n+1}, \dot{f}^{n+1}]\text{”}.$$

Moreover

$$p_{x\restriction n+1} \Vdash \text{“}\mathrm{dom}(\dot{f}_n) = \check{X}_{x\restriction n} \in \dot{E}_{\check{a}_{x\restriction n}} \wedge \check{a}_{x\restriction n} = \dot{a}_n\text{”}$$

and

$$p_{x\restriction n+1} \Vdash \text{“}\mathrm{dom}(\dot{f}_{n+1}) = \check{X}_{x\restriction n+1} \in \dot{E}_{\check{a}_{x\restriction n+1}} \wedge \check{a}_{x\restriction n+1} = \dot{a}_{n+1}\text{”}.$$

Thus $\tau\text{"}a_{x\restriction n} \in \mathrm{dom}(\check{f}_{x\restriction n})$ and $f_{x\restriction n}(\tau\text{"}a_{x\restriction n}) > f_{x\restriction n+1}(\tau\text{"}a_{x\restriction n+1})$, but this is a descending sequence of ordinals in V, contradiction!

Suppose now that for every generic, the ultrapower is well-founded. Consider the system $\mathcal{T} = \langle\langle p_s, X_s, a_s\rangle; s \in {}^{<\omega}\theta\rangle$ such that:

i. $p_\varnothing = p$,

ii. $a_s \subseteq a_{s^\frown i}$ for all $i < \theta$,

iii. $p_{s^\frown i}$ contains the span of p_s and $\langle a_{s^\frown i}, X_{s^\frown i}\rangle$ for all i,

iv. $\{p_{s^\frown i}; i < \theta\}$ is a maximal antichain below p_s.

Let us show that x, τ exists such that $\tau'' a_s \in X_s$ for $s \subseteq x$. Let G be a generic filter such that $p_\varnothing \in G$. Since for all $n < \omega$ the set $\{p_s, \mathrm{lh}(s) = n\}$ is a maximal antichain below $p_\varnothing$, there is one s such that $p_s \in G$, let x be the union of all such s, notice that $x \in {}^\omega\theta$ is well defined. Let $\pi : V \to \mathrm{Ult}(V, G)$ be the ultrapower map. We write $a_s^{\pi(\mathcal{T})}$ for the second components of the condition at the s-node of $\pi(\mathcal{T})$, similarly for $X_s^{\pi(\mathcal{T})}$ and $p_s^{\pi(\mathcal{T})}$. Let

$$\tau : \bigcup_{s\subseteq x} \pi(a_s) \to \pi(\kappa)$$

be defined as follows:

> if $\xi \in \bigcup_{s\subseteq x} \pi(a_s)$ then there is a s such that $\xi \in \pi(a_s)$, since a_s is finite there is a $\overline{\xi} \in a_s$ such that $\xi = \pi(\overline{\xi})$, let $\tau(\xi) = \overline{\xi}$.

Hence we have that:

$$\mathrm{Ult}(V,G) \models \text{“}\tau : \bigcup_{s\subseteq x} \pi(a_s) \to \pi(\kappa)\text{”}$$

and

$$\tau\text{''}\pi(a_s) \in \pi(X_s) \text{ for all } s \subseteq x.$$

By elementarity $\pi(a_s) = a^{\pi(\mathcal{T})}_{\pi(s)}$ and $\pi(X_s) = X^{\pi(\mathcal{T})}_{\pi(s)}$.

Let us argue why x and τ exists in $\mathrm{Ult}(V,G)$: let T be the tree of height ω, with finite conditions searching[1] for a x' and a τ' such that

$$\tau' : \textstyle\bigcup_{s\subseteq x'} a^{\pi(\mathcal{T})}_s \to \pi(\kappa) \text{ and } \tau'\text{''} a^{\pi(\mathcal{T})}_s \in X^{\pi(\mathcal{T})}_s \text{ for all } s \subseteq x'.$$

This tree is in $V[G]$ as well as in $\mathrm{Ult}(V,G)$, setting $x' = \pi'' x$ we can see that it is ill-founded in $V[G]$, hence it is ill-founded in $\mathrm{Ult}(V,G)$. A branch through the tree gives some x and τ with the above properties, hence

$$\mathrm{Ult}(V,G) \models \text{“}\exists x \exists \tau \text{ such that } \tau : \textstyle\bigcup_{s\subseteq x} a^{\pi(\mathcal{T})}_s \to \pi(\kappa) \text{ and } \tau\text{''} a^{\pi(\mathcal{T})}_s \in X^{\pi(\mathcal{T})}_s \text{ for all } s \subseteq x\text{”}.$$

By elementarity

$$V \models \text{“}\exists x \exists \tau \text{ such that } \tau : \textstyle\bigcup_{s\subseteq x} a_s \to \kappa \text{ and } \tau\text{''} a_s \in X_s \text{ for all } s \subseteq x\text{”}. \qquad \dashv$$

2.2 Forcing ideal extenders and ideally strong cardinals

Lemma 2.6. *Let κ be α-strong in V, $\mu < \kappa$ some cardinal and let E be the $\langle\kappa,\lambda\rangle$-extender derived by the ultrapower map witnessing the α-strongness. Let $W = V[G]$ where G is $\mathrm{col}(\mu, <\kappa)$-generic over V. Set*

$$F = \left\{ \langle a, x\rangle ; x \subseteq [\kappa]^{\overline{\overline{a}}} \textit{ and } \exists y \textit{ such that } \langle a, y\rangle \in E \ y \subseteq x \right\}$$

then F is precipitous.

Proof. Let us first start with a simple general consideration that is useful in many cases when considering ultrapowers and the Levy collapse:

Claim 1. *Suppose $V \models$ “E is a (κ,λ)-extender”. Let $\pi : V \to M = \mathrm{Ult}(V,E)$ be the associated ultrapower map. Then for each G $\mathrm{col}(\mu,<\kappa)$-generic over V and each condition $q \in \mathrm{col}(\mu, <\pi(\kappa))^M$ such that $q \restriction \mu \times \kappa \in G$, there is a M-generic G^* such that $\{q\} \cup G \subseteq G^*$, moreover there is a canonical map $\tilde{\pi} : V[G] \to M[G^*]$ such that $\pi \subseteq \tilde{\pi}$.*

[1] we already constructed such a type of tree in **??**

Proof. Since $(H_{\kappa^+})^V = (H_{\kappa^+})^M$, G is also generic over M. In $M[G]$ we can look for a $\mathrm{col}(\mu,]\kappa, \pi(\kappa)[)$-generic filter $\tilde{G}$ such that $q \restriction \mu \times]\kappa, \pi(\kappa)[\in \tilde{G}$. Let G^* be the filter generated by $G \cup \tilde{G}$, now we can define an embedding $\tilde{\pi} : V[G] \to M[G^*]$ as follow: for every name $\tau \in V^{\mathrm{col}(\mu,<\kappa)}$, let $\tilde{\pi}(\tau^G) = (\pi(\tau))^{G^*}$. It is easy to check that $\tilde{\pi}$ is well defined and an embedding. $\dashv$

Let us now turn to F, we first want to prove that for each $\mathrm{col}(\mu, <\pi(\kappa))$-generic over M filter G^*, we can construct an extender E^{G^*} that extends F such that the following diagram commutes:

$$\begin{tikzcd} V[G] \arrow[rr, "\tilde{\pi}"] \arrow[dr, "j"'] & & M[G^*] \\ & \mathrm{Ult}(V[G], E^{G^*}) \arrow[ur, "k"'] & \end{tikzcd}$$

where j is the associated ultrapower map and k still needs to be defined and $G = G^* \cap \mathrm{col}(\mu, <\kappa)$. We define E^{G^*} by:

$$\langle a, x\rangle \in E^{G^*} \iff a \in \tilde{\pi}(x),$$

for $a \in [\lambda]^{<\omega}$ and $x \subseteq \mathcal{P}([\kappa]^{\overline{\overline{a}}})$ and k by:

$$k([f, a]) = \tilde{\pi}(f)(a),$$

where a is as before and $f : \kappa^{\overline{\overline{a}}} \to V[G]$. It is easy to check that k is well defined. Hence $\mathrm{Ult}(V, E^{G^*})$ is transitive.

Let us do a few remark similar to the case of a measurable before turning to the genericity of E^{G^*}. Each condition in $\mathrm{col}(\mu, <\pi(\kappa))$ can be split in $p \in \mathrm{col}(\mu, <\kappa)$ and a $q \in \mathrm{col}(\mu, [\kappa, \pi(\kappa)[)$, moreover q can be represented in the ultrapower by $a_q \in [\lambda]^{<\omega}$ and a function $f^q : \kappa^{\overline{\overline{a}}_q} \to \mathrm{col}(\mu, <\kappa)$. Let s be an enumeration of $a_q \cup \{\kappa\}$ and i such that $s(i) = \kappa$, we have:

$$\left\{\vec{\xi}; f^q_{a_q, a_q \cup \{\kappa\}}(\vec{\xi}) \in \mathrm{col}(\mu, [\xi_i, \kappa[)\right\} \in E_{a_q \cup \{\kappa\}}.$$

Let $\dot{E}$ be the canonical name for E^{G^*}, $a \in [\kappa]^{<\lambda}$ and $\dot{X} \in V^{\mathrm{col}(\mu,<\kappa)}$ some set such that there are $\langle p, q\rangle \in \mathbb{P} * j(\mathbb{P})^\kappa$ with

$$p \cup q \Vdash^{j(\mathbb{P})} \langle \check{a}, \dot{X}\rangle \in \dot{E}.$$

By definition of π we then have:

$$p \cup q \Vdash^{j(\mathbb{P})} \check{a} \in \pi(\dot{X})^2.$$

[2]notice that $\pi(\dot{X})$ is a $\mathrm{col}(\mu, \pi(\kappa))$ name

Setting $\mathrm{id}^a : [\kappa]^{\overline{\overline{a}}} \to [\kappa]^{\overline{\overline{a}}}$, this leeds to:

$$\left\{\vec{\xi}; p \cup f^q_{a_q, a\cup a_q}(\vec{\xi}) \Vdash^{\mathbb{P}} \mathrm{id}^a_{a, a\cup a_q}(\vec{\xi}) \in \dot{X}\right\} \in E_{a\cup a_q}.$$

Let $G_F = \left\{p \in \mathbb{P}_F; F^p \subseteq E^{G^*}\right\}$. We want to prove that G_F is $\mathbb{P}_F$-generic over $V[G]$. Let $p \in G$ and $q \in G^* \upharpoonright \mathrm{col}(\mu[\kappa, \pi(\kappa)[)$ such that

$$p \Vdash \text{“}\dot{A} = \left\{\dot{F}^i; i < \theta\right\} \subseteq \mathbb{P}_F \text{ is an antichain”}$$

moreover for each $i < \theta$, $p \cup q \Vdash$ “$F^i \nsubseteq E^{G^*}$”. Let each $\dot{F}^i$ be generated by $\langle \check{a}_i, \dot{X}_i\rangle$, we have

$$p \cup q \Vdash \text{“}\dot{X}_i \notin \dot{E}_{\check{a}_i}\text{”}.$$

By the previous observation, we have sets $A_i \in E_{a_i \cup a_q}$ such that for all $\vec{\xi} \in A_i$:

$$p \cup f^q_{a_q, a_i\cup a_q}(\vec{\xi}) \Vdash \mathrm{id}^{a_i}_{a_i, a_i\cup a_q}(\vec{\xi}) \notin \dot{X}_i.$$

Let $T = \left\{\vec{\xi};\ f^q(\vec{\xi}) \in G\right\}$ and let F' be the span of F and $\langle a_q, T\rangle$. We first show that F' is a condition: for a $Z \in E_{a_q}$, we have to show that $Z \cap T \neq \varnothing$. Let

$$D = \left\{r; r \leqslant_{\mathrm{col}(\mu, <\kappa)} q_{\vec{\xi}} \text{ for some } \vec{\xi} \in Z\right\}.$$

D is dense, since each condition has size less then μ, Z is unbounded and μ is regular, therefore we can choose some $q_{\vec{\xi}}$ such that

$$\sup(\mathrm{dom}(r)) < \min(\mathrm{dom}(q_{\vec{\xi}})).$$

Let $r \in D \cap G$, there is a $\vec{\xi} \in Z$ such that $r \leqslant_{\mathrm{col}(\mu,<\kappa)} q_{\vec{\xi}}$, thus $\vec{\xi} \in T$, and we have $T \cap Z \neq \varnothing$. Let us now show that $T \cap X_i \notin F^+$, it suffices to prove that there is a set $X \in E_{a_i \cup a_q}$ such that

$$T_{a_q, a_i\cup a_q} \cap X_{i a_i, a_i \cup a_q} \cap X = \varnothing.$$

Let $\vec{\xi} \in A_i$. If $\xi \in T_{a_q, a_i\cup a_q}$, $q_\xi \in G$. Since

$$p \cup f^q_{a_q, a_i\cup a_q}(\vec{\xi}) \Vdash \mathrm{id}^{a_i}_{a_i, a_i\cup a_q}(\vec{\xi}) \notin \dot{X}_i.$$

We have that $\xi \notin X_{i a_i, a_i\cup a_q}$, hence the A_i where the set we sought, and $\langle X_i; i < \theta\rangle$ isn't a maximal antichain, a contradiction!

Claim 2. *Let G be $\mathrm{col}(\mu, <\kappa)$-generic over V. For each condition $p \in \mathbb{P}_F$, there is a G^* $\mathrm{col}(\mu, <\pi(\kappa))$-generic that extends G, such that $F^p \subseteq E^{G^*}$.*

PROOF. Let $\dot{p} \in V$ be a name for a condition in $\mathbb{P}_F$. Fix $\tau \in V$ and $q \in \mathrm{col}(\mu, < \kappa)$ such that $q \Vdash$ "$F^{\dot{p}}$ is the span of $\check{F}$ and $\langle \check{a}, \tau \rangle$", for some finite set of ordinals $a \in [\lambda]^{<\omega}$. Without loss of generality we can assume that $\tau = \left\{ \langle p, \check{\vec{\alpha}} \rangle ; p \Vdash \check{\vec{\alpha}} \in \tau \right\}$. We want to show that we can find a $q' < q \in \mathrm{col}(\mu, < \pi(\kappa))$ such that $a \in \tilde{\pi}(\tau)$, for every $\mathrm{col}(\mu, < \pi(\kappa))$-generic G^* with $q' \in G^*$. Let

$$y = \{ \vec{\alpha} ; \exists r < q \langle r, \vec{\alpha} \rangle \in \tau \} .$$

Clearly, $\langle a, y \rangle$ has to be in E, else τ would be a null set in $V[G]$. Hence

$$a \in \pi(y) = \pi(\{ \vec{\alpha} ; \exists r < q \langle r, \vec{\alpha} \rangle \in \tau \}) = \{ \vec{\alpha} ; \exists r \langle r, \vec{\alpha} \rangle \in \pi(\tau) \} .$$

This shows that there is a $q' \in \mathrm{col}(\mu, < \pi(\kappa))$, $q' < q$ such that $\langle q', a \rangle \in \pi(\tau)$. Let G^* be $\mathrm{col}(\mu, < \pi(\kappa))$-generic with $q' \in G^*$, G^* has the desired properties. $\dashv$

Let us prove now that F is potentially normal and precipitous. Suppose first that F is not precipitous, then there is a generic over $\mathbb{P}_F$ such that the associated ultrapower is ill-founded. This is then forced by a condition $p \in \mathbb{P}_F$. By the previous result we can find G^* $\mathrm{col}(\mu, < \pi(\kappa))$-generic such that $F^p \subseteq E^{G^*}$. Thus $\mathrm{Ult}(V[G], E^{G^*})$ should be ill-founded, a contradiction since you can embed it in $M[G^*]$. Similarly suppose that F is not potentially normal. Let $p \in \mathbb{P}_F$ such that there is $f : [\kappa]^{\overline{\overline{a}}} \to V$ with

$$\{ u, f(u) \in u_j \} \in F^p_a$$

for some $a \in \mathrm{supp}(p)$, such that for no $q \leqslant_{\mathbb{P}_F} p$ there is a ξ with $a \cup \{\xi\} \subseteq \mathrm{supp}(q)$ and

$$\left\{ v, f_{a, a \cup \{\xi\}}(v) = v_i \right\} \in F^q_{a \cup \{\xi\}}.$$

Let G^* be $\mathrm{col}(\mu, < \pi(\kappa))$-generic such that $F^p \subseteq E^{G^*}$. E^{G^*} is a normal extender, since it is an extender derived from an embedding. Hence there is a ξ such that

$$V[G^*] \vDash A = \left\{ v, f_{a, a \cup \{\xi\}}(v) = v_i \right\} \in E^{G^*}_{a \cup \{\xi\}}.$$

Let F^p be generated by $\langle b, x \rangle$, and define

$$y = x_{b, b \cup a \cup \{\xi\}} \cap A_{a \cup \{\xi\}, b \cup a \cup \{\xi\}}.$$

Let $q \in \mathbb{P}_F$ be such that F^q is the filter generated by $\langle b \cup a \cup \{\xi\}, y \rangle$. Then $q \leqslant_{\mathbb{P}_F} p$ and

$$\left\{ v, f_{a, a \cup \{\xi\}}(v) = v_i \right\} \in F^q_{a \cup \{\xi\}},$$

a contradiction! $\dashv$

Definition 2.7. Let κ be a regular cardinal. We call a regular cardinal κ *ideally strong* if and only if for all $A \subseteq OR$, $A \in V$, there is some $\langle \kappa, \nu \rangle$-ideal extender E such that, whenever G is E-generic over V, $A \in \mathrm{Ult}(V, G)$

Theorem 2.8. *Let κ be a strong cardinal in V and λ be a cardinal. Let G be* $\mathrm{col}(\lambda, < \kappa)$*-generic over V. In $V[G]$, κ is ideally strong.*

Proof. Let $A \subseteq V[G]$. There is a name $\tau \in V$ for A. Let $\tilde{E}$ be the extender witnessing the strongness of κ with respect to τ. That is $\tilde{\jmath} : V \to \mathrm{Ult}(V, \tilde{E})$ is such that $\tau \in \mathrm{Ult}(V, \tilde{E})$.

Now let E be the ideal extender derived by $\tilde{E}$ in $V[G]$, as we have seen previously if H is E-generic over V and $j : V[G] \to \mathrm{Ult}(V[G], H)$ is the associated ultrapower, then $j \restriction V = \tilde{\jmath}$. Moreover $G \in \mathrm{Ult}(V[G], H)$ thus we have that $A = \tau^G \in \mathrm{Ult}(V[G], H)$, which finishes the proof.

2.3 Iteration of ideal extenders

Let us now discuss the iteration of generic ultrapower by ideal extender.

Definition 2.9. A sequence:

$$\langle \langle M_i, E_i, \pi_{i,j}; i \leqslant j \leqslant \theta \rangle, \langle G_i; i < \theta \rangle \rangle$$

is a putative generic iteration of M (of length θ+1) if and only if the following holds:

i. $M_0 = M$,

ii. for all $i < \theta$ $M_i \vDash$ "E_i is an ideal extender",

iii. for all $i < \theta$ G_i is E_i-generic over M_i,

iv. for all $i + 1 \leqslant \theta$ $M_{i+1} = \mathrm{Ult}(M_i, G_i)$ and $\pi_{i,i+q}$ is the associated generic ultrapower,

v. for all $i \leqslant j \leqslant k \leqslant \theta$ $\pi_{j,k} \circ \pi_{i,j} = \pi_{i,k}$,

vi. if $\lambda < \theta$ is a limit ordinal, then $\langle M_\lambda, \pi_{i,\lambda}; i < \lambda \rangle$ is the direct limit of the system $\langle M_i, \pi_{i,j}; i \leqslant j < \lambda \rangle$.

We call

$$\langle \langle M_i, E_i, \pi_{i,j}; i \leqslant j \leqslant \theta \rangle, \langle G_i; i < \theta \rangle \rangle$$

a generic iteration of M (of length $\theta + 1$) if M_θ is well-founded. We call

$$\langle\langle M_i, E_i, \pi_{i,j}; i \leqslant j \leqslant \theta\rangle, \langle G_i; i < \theta\rangle\rangle$$

a putative generic iteration of $\langle M, E\rangle$ if the following additional clause holds true:

vii. for all $i + 1 < \theta$ $E_{i+1} = \pi_{i,i+1}(E_i)$.

Let E be an ideal extender. We say that G is E-generic if G is a $\mathbb{P}_E$-generic filter.

Lemma 2.10. *Let M be a countable transitive* ZFC *model and F be a precipitous $\langle\kappa, \lambda\rangle$-ideal extender over M. Let $\theta < \sup\left\{M \cap \mathrm{OR}, \omega_1^V\right\}$. Then M is $< \theta$-iterable by F. That is every putative iteration of $\langle M, F\rangle$ of length less or equal to θ is an iteration.*

Proof. This proof is an adaptation of Woodin's proof to the current context. By absoluteness if $\langle M, E\rangle$ is not generically $\theta + 1$ iterable, it is not generically $\theta + 1$ iterable in $M^{\mathrm{col}(\omega,<\delta)}$ for some δ. Let $\langle\kappa_0, \eta_0, \gamma_0\rangle$ be the least tripe in the lexicographical order such that:

i. $\kappa < \omega_1^M$ is regular in M,

ii. $\eta_0 < \kappa_0$

iii. there is a δ and a putative iteration

$$\langle\langle M_i, E_i, \pi_{i,j}; i \leqslant j \leqslant \gamma_0\rangle, \langle G_i; i < \gamma_0\rangle\rangle$$

of $\langle H^M_{\kappa_0}; \in, E\rangle$ inside $M^{\mathrm{col}(\omega,<\delta)}$ such that $\pi_{0,\gamma_0}(\eta_0)$ is ill-founded.

Since I is precipitous, γ_0 has to be a limit ordinal, η_0 has to be a limit ordinal in any case. Let $i^* < \gamma_0$ and $\eta^* < \pi_{i^*,\gamma_0}(\eta_0)$ be such that $\pi_{i^*,\gamma_0}(\eta^*)$ is ill-founded. Since κ_0 is regular we can consider

$$\langle\langle M_i, E_i, \pi_{i,j}; i^* \leqslant i \leqslant j \leqslant \gamma_0\rangle, \langle G_i; i^* \leqslant i < \gamma_0\rangle\rangle$$

as a putative iteration of $H^{M_{i^*}}_{\pi_{0,i^*}(\kappa_0)}$

By elementarity, $\langle\pi_{0,i^*}(\kappa_0), \pi_{0,i^*}(\eta_0), \pi_{0,i^*}(\gamma_0)\rangle$ is the least triple $\langle\kappa, \eta, \gamma\rangle$ such that condition i. to iii. holds with respect to M_{i^*}.

However as showed before the triple $\langle\pi_{0,i^*}(\kappa_0), \eta^*, \gamma_0 - i^*\rangle$ also fullfils i. to iii. and is lexicographically smaller than $\langle\pi_{0,i^*}(\kappa_0), \pi_{0,i^*}(\eta_0), \pi_{0,i^*}(\gamma_0)\rangle$, a contradiction! ⊣

2.4 The consistency strength of one ideally strong cardinal

Lemma 2.11. *Suppose* $\neg(0^{¶})$*. Let* κ *be ideally strong in* V*, then* κ *is strong in the core model.*

Proof. Let $K = K^V$ be the core model below $(0^{¶})$ as in [Jen]. Let $\lambda \in \mathrm{OR}$. We have to show that there is an embedding $j : K \to M$ such that $K|\lambda \in M$.

Let $\lambda \in \mathrm{OR}$, by the ideal strongness of κ, there is an ideal extender E such that if G is E-generic over V:

$$K|\lambda \in \mathrm{Ult}(V, G) = M.$$

Claim 1. *In* $V[G]$*,* K *iterates to* $K^{\mathrm{Ult}(V,G)} = K^M = K^*$*.*

j exists in $V[G]$ and $K = K^{V[G]}$, hence by [Jen, §5.3 Lemma 5 p. 7] K^* is an iterate of K and $j \restriction K$ is the iteration map.

Claim 2. $K|\lambda = K^*|\lambda$.

By the previous claim, we already know that $K|\nu = K^*|\nu$, where ν is the length of the first extender, F, of the iteration j. Since F was used in the iteration, $F \notin K^*$. Suppose $K|\lambda \neq K^*|\lambda$, then $\mathrm{lh}(F) < \lambda$. Since $\mathrm{lh}(F) < \lambda$, $F \in K|\lambda \subseteq M$. By [Jen, §5.2 Lemma 2 p. 3] we have that $\langle K|\mathrm{lh}(F), F\rangle$ is a generalized beaver for K^* and hence $\mathrm{Ult}(K^*, F)$ is well-founded. Let us coiterate K^* and $\mathrm{Ult}(K^*, F)$:

$$\begin{tikzcd}
K^* \arrow[r, "i_F"'] & \mathrm{Ult}(K^*, F) \arrow[dr, "k"] & \\
& & W \\
& K^* \arrow[ur, "i"'] &
\end{tikzcd}$$

Since $F \in M$, we can apply [Jen, §5.3 Lemma 5 p. 7] to $k \circ i_F$ in M. We get that $k \circ i_F = i$ and thus $k = \mathrm{id}$, $i = i_F$. This shows that F is on the K^*-sequence, a contradiction!

Thus we can assume that $\mathrm{lh}(F) \geqslant \lambda$ and so we have:

$$K|\lambda \lhd K^*.$$

Hence $j \restriction K : K \to K^*$ and $K|\lambda \in K^*$, which finishes the proof. ⊣

Corollary 2.12. *The existence of an ideally strong cardinal is equiconsistent to the existence of a strong cardinal.*

3 More ideally strong cardinals

As we have seen in the last section, lifting existing embeddings after forcing has been a very fruitful method to construct ideally strong cardinals. In this section, the lifting of various embeddings will be our main concern, especially when forcing "above" a large cardinal. In the last part we prove that such generic embeddings implies the existence of strong cardinals in the core model, giving a lower bound to our construction. Let us first put some light on the problems that arise, when constructing more than one ideally strong cardinal. The key problem is that, while forcing with so called "small forcings" preserves large cardinal properties, forcing above a strong cardinal κ will, in general, destroy its strongness, even if we don't add a new subset of κ.

Remark 3.1. Let κ be a strong cardinals and $\beta > 2^{(2^\kappa)^+}$, then in $V^{\mathrm{col}(\beta,\beta^+)}$ κ is not necessarily β^{++V}-strong anymore.

Proof. Let $K = V$ be the minimal core model for one strong cardinal. Let κ be strong in V and β as in the remark. Let E be an extender witnessing the β^{++V}-strongness of κ, and G a $\mathrm{col}(\beta,\beta^+)$-generic filter. Since G does not add any ω-sequence, E is still an ω-closed extender in the forcing extension. Let M be the ultrapower of $V[G]$ by E and j the ultrapower map. Suppose E is witnessing β^{++V}-strongness in $V[G]$. Then G would be in $H^M_{\beta^{++V}}$ and thus M believes that there is a $\mathrm{col}(\gamma,\gamma^{+K})$-generic filter over K for some cardinal $\gamma \leq j(\kappa)$, hence K believes that there is a $\mathrm{col}(\gamma,\gamma^{+K})$-generic filter over K for some cardinal $\gamma \leq \kappa$, a contradiction! $\dashv$

3.1 Lifting of generic embeddings

With some more detailed analysis of the ultrapower by an extender we may lift the original issue.

Lemma 3.2. *Suppose GCH. Let κ be a strong cardinal and $\lambda > \kappa$ a regular cardinal such that $2^{<\lambda} = \lambda$. Let $j : V \to M$ be an ultrapower by a $\langle\kappa, V_\lambda\rangle$-extender witnessing the λ-strongness of κ. Then for every M-sequence of ordinals $\lambda < \mu_i < \nu_i < \mu_{i+1} < j(\kappa)$ for $i < j(\kappa)$ such that $M \models$ "μ_i, ν_i are regular cardinals", there is a $G \in V$ that is $\mathbb{P}$-generic over M, where $\mathbb{P}$ is the easton iteration of all $\mathrm{col}(\mu_i, < \nu_i)^M$.*

Proof. Remark that since each $\mathrm{col}(\mu_i, < \nu_i)^M$ is λ-closed in M, so is $\mathbb{P}$. Since j is an ultrapower by a $\langle\kappa, V_\lambda\rangle$-extender[3], we have that:

i. M is closed under sequence of length κ: ${}^\kappa M \cap V \subseteq M$,

ii. $H_\lambda \subseteq M$,

iii. $\lambda < j(\kappa) < \lambda^{+V}$.

Hence every dense set of $\mathbb{P}$ in M is of the form $j(f)(a)$, for an $f : [V_\kappa]^{\overline{\overline{a}}} \to V_\kappa$ and some $a \in [V_\lambda^V]^{<\omega} \subseteq M$. By GCH we can count in V all such f in a sequence of order type κ^+. Let

$$\langle f_\xi; \xi < \kappa^+\rangle$$

be such a sequence. Moreover V_λ^V has cardinality λ in M as well as in V. Using the fact that for any given ξ, $j(f_\xi) \in M$, in M we can look at the set

$$X_\xi = \left\{j(f_\xi)(a);\ a \in V_\lambda \wedge j(f_\xi)(a) \text{ is a dense set in } \mathbb{P}\right\}.$$

Since M believes that the forcing iteration is an iteration of levy collapses of strong cardinals above λ, $\mathbb{P}$ is λ-closed in M. Now define the sequence p_ξ for $\xi < \kappa^+$ as follows

i. p_0 be the empty condition,

ii. $p_{\xi+1}$ is a condition below p_ξ and below each element of X_ξ,

iii. if $\nu < \kappa^+$ is a limit ordinal, let p_ν be some condition below each p_ξ for $\xi < \nu$.

The successor steps works in M because X_ξ and p_ξ are both in M and $\mathbb{P}$ is λ-closed. For the limit steps: we can define the sequence $\langle p_\xi;\ \xi < \nu\rangle$ in V. Since $\nu < \kappa^+$ and M is κ-closed, the sequence is in M as well. Hence by the λ-closedness of $\mathbb{P}$ in M, there is a p_ν less than all the p_ξ in M.

Now the sequence $\langle p_\xi; \xi < \kappa^+\rangle \subseteq M$ is definable in V, let $G \in V$ be the filter generated by all this points. G is $\mathbb{P}$-generic over M. $\dashv$

Lemma 3.3. *Let E be a $\langle\kappa, V_\lambda\rangle$-extender, where λ is such that ${}^\kappa V_\lambda \subseteq V_\lambda$ and $\mathbb{P}$ a κ-distributive forcing. Let*

$$j : V \to M = \mathrm{Ult}(V, E)$$

[3] that is $E \subseteq \left\{\langle a, x\rangle; a \in [V_\lambda]^{<\omega} \text{ and } x \subseteq \mathcal{P}([\kappa]^{\overline{\overline{a}}})\right\}$ for more on this type of extender see [MS89, p. 83 ff.]

be the ultrapower map. Let G be $\mathbb{P}$-generic over V, then j can be lifted to an embedding

$$\tilde{j} : V[G] \to M[G'],$$

where G' is the completion of $j''G$ in $j(\mathbb{P})$.

Proof. Let E be as in the theorem and $j : V \to M = \mathrm{Ult}(V, E)$. We have that M is closed under κ-sequences, that is ${}^{\kappa}M \cap V \subseteq M$. Let $\mathbb{P}$ be a κ-distributive forcing and G be $\mathbb{P}$-generic over V. Let

$$G' = \{q \in j(\mathbb{P});\ \exists p \in G, j(p) \leq q\}.$$

We claim that G' is already generic over M! Let $D = j(f)(a)$ be some dense open set in M. This implies

$$\left\{u \in [V_\kappa]^{\overline{\overline{a}}}; f(u) \text{ is a dense open set of } \mathbb{P}\right\} \in E_a$$

but then the set

$$A = \left\{f(u);\ u \in [V_\kappa]^{\overline{\overline{a}}} \wedge f(u) \text{ is a dense open set of } \mathbb{P}\right\}$$

has only size κ. By κ-distributivity of $\mathbb{P}$, $\bigcap A$ is still dense. Let $p \in A \cap G$, we have that $j(p) \in D \cap G'$. $\dashv$

Notice that this lemma alone does not give the the desired result since G itself might not be in $M[G']$. We want to combine this and the techniques developed in the measurable case to get the desired result. Sadly for the forcing we have in mind, using only strongness will not suffice. We will use the concept of A-strongness to bypass this problem.

3.2 Forcing two ideally strong cardinals

Lemma 3.4. *Let A be the class of all strong cardinal. Suppose $V \models$ "GCH, κ is an A-strong cardinal, $\delta > \kappa$ is the only strong cardinal above κ". Let $n : \mathrm{OR} \to \mathrm{OR}$ such that $n(\gamma) = \gamma^+$ and let γ_μ denote the smallest strong cardinal above μ. For γ strong, let $\mathbb{P}_\gamma$ be $\mathrm{col}(n(\gamma), < \mu_\gamma)$, the levy collaps of μ_γ to $n(\gamma)$ and let $\mathbb{P}$ be the easton support iteration of all $\mathbb{P}_\gamma$ for γ strong such that μ_γ exists. Let G be $\mathbb{P}$-generic over V. In $V[G]$, κ is strong.*

Proof. We first follow the same strategy as in the measurable case. For some set of ordinals I, let $\mathbb{P} \upharpoonright I$ be the easton forcing iteration of $\mathbb{P}_\gamma$ for all $\gamma \in I$. Let G be $\mathbb{P}$-generic over V, let $\lambda > \delta$ be a large enough regular cardinal with ${}^{\kappa}V_\lambda \subseteq V_\lambda$, we have to show that there is an embedding

$$\tilde{j} : V[G] \to \tilde{M}$$

with the property that $H_\lambda^{V[G]} \subseteq M$. Let E be an $\langle \kappa, V_\lambda \rangle$-extender witnessing that κ is A-λ-strong in V. We want to lift up the embedding $j : V \to M$ associated to E.

Let us recall the cardinal arithmetic setting. We have that

$$\lambda < \mathrm{card}^V(j(\kappa)) < \mathrm{card}^V(j(\delta)) < \mathrm{card}^V((2^{j(\delta)})^M) < \lambda^{+V}.$$

Moreover since we use V_λ to index the extender, the ultrapower is closed under κ-sequences. Notice that since E is a witness that κ is A-λ-strong, we have that $\mathbb{P} \subseteq j(\mathbb{P})$. Since G is $\mathbb{P}$-generic over V and $(H_\lambda)^V = (H_\lambda)^M$, we thus have that G is $\mathbb{P}^M \upharpoonright \kappa = \mathbb{P}$-generic over M.

If we can show that there is an $\tilde{G} \in V[G]$ such that $G * \tilde{G}$ is $j(\mathbb{P})$-generic over M and $j"G = \tilde{G} \cap \mathrm{ran}(j)$, we will be able to lift the embedding j to an embedding

$$\tilde{j} : V[G] \to M[G \times \tilde{G}]$$

Let $\sigma_{\mathbb{P}^*}$ be an $M^{\mathbb{P}}$-name for $j(\mathbb{P}) \upharpoonright [\delta, j(\kappa)[$ and $\mathbb{P}^* = \sigma_{\mathbb{P}^*}^G$. Let further $\sigma_{\mathbb{P}^{**}}$ be a $M^{\mathbb{P} * \mathbb{P}^*}$-name for $\mathbb{P}_{j(\kappa)}$.

We want to find G^*, a $\mathbb{P}^*$-generic filter over $M[G]$ and G^{**}, a $\mathbb{P}_{j(\kappa)}$-generic filter over $M[G \times G^*]$. That way using the factor lemma [Jec03, Lemma 21.8 pp. 396], we will have that $G \times G^* \times G^{**}$ is a $j(\mathbb{P})$-generic filter over M. In order to produce a $\mathbb{P}^*$-generic filter, we'd like to use Lemma 3.2, sadly we need a filter generic over $M[G]$ rather than just M. Let us argue why the proof still holds true.

Claim 1. *There is a filter $G^* \in V[G]$ that is $\mathbb{P}^*$-generic over $M[G]$.*

Proof. We want to run the very same argument as in Lemma 3.2. Let us first show that $M[G]$ is still closed under κ-sequences. Let τ be the name for a κ-sequence in $V[G]$. Without loss of generality, we can assume that τ is a nice name, that is, it is of the form

$$\tau = \{\langle\langle \eta, \xi\rangle, q\rangle;\ \eta < \kappa \wedge q \in A^\eta \wedge q \Vdash \tau(\eta) = \xi\},$$

where A^η is a maximal antichain. Since $A^\eta \in V_\lambda$, each A^η is in M. Since M is closed under κ-sequences. the sequence of all A^η is in M as well and thus τ is in M. Therefore $M[G]$ is closed under κ-sequences from $V[G]$. Every dense set of $\mathbb{P}^*$ in M is of the form $j(f)(a)^G$, for an $f : [V_\kappa]^{\overline{\overline{a}}} \to V_\kappa^{\mathbb{P}}$ and some $a \in V_\lambda^V = V_\lambda^M$, as j is the ultrapowermap generated by E. By GCH we can count in V all such f in a sequence of order type κ^+, $\langle f_\xi; \xi < \kappa^+ \rangle$. Also remark that V_λ has cardinality λ in $M[G]$ as well as in $V[G]$. Using the fact that for any given ξ, $j(f_\xi) \in M$, in $M[G]$ we can look at the set

$$X_\xi = \left\{ j(f_\xi)(a)^G;\ a \in V_\lambda \wedge j(f_\xi)(a) \text{ is a } \mathbb{P}\text{-name for a dense set in } \mathbb{P}^* \right\}.$$

Since $M[G]$ believes that the forcing iteration $\mathbb{P}^*$ is an iteration of levy collapses of strong cardinals above λ, $\mathbb{P}^*$ is λ-closed in M. Now define the sequence p_ξ for $\xi < \kappa^+$ as follows

i. p_0 be the empty condition,

ii. $p_{\xi+1}$ is a condition below p_ξ and below each element of X_ξ,

iii. if $\nu < \kappa^+$ is a limit ordinal, let p_ν be some condition below each p_ξ for $\xi < \nu$.

The successor steps works in $M[G]$ because X_ξ and p_ξ are both in $M[G]$ and $\mathbb{P}$ is λ-closed, the limit steps works because they are definable sequences in $V[G]$ of length at most κ, hence by the κ-closedness of $M[G]$ the sequences are also in $M[G]$, hence by the λ-closedness of $\mathbb{P}$ in $M[G]$, p_ν is definable in $M[G]$.

Now the sequence $\langle p_\xi; \xi < \kappa^+ \rangle \subseteq M$ is definable in $V[G]$, let $G^* \in V[G]$ be the filter generated by all this points. G^* is $\mathbb{P}^*$-generic over $M[G]$. $\dashv$

Let $\mathbb{P}^{**} = \sigma_{\mathbb{P}^{**}}^{G \times G^*}$. Setting G' and G'' sucht that $G' = G \cap H_\kappa$ and $G' \times G'' = G$, we can see that we are already able to lift j to some $j_1 : V[G'] \to M[G' * G^*]$. As in the last step, we won't be able to directly use the appropriate lemma, in this case Lemma 3.3. But with some small modification, the main idea of the lemma carries on in our situation.

Notice that $\mathbb{P}$ is an iteration of successor length. Let $\tau \in M$ be a name for an open dense set of $\mathbb{P}^{**}$. Hence there is a $a \in [V_\lambda]^{<\omega}$ and a $f : [V_\kappa]^{\overline{\overline{a}}} \to V$ such that $\tau = j(f)(a)$ and

$$X = \left\{ u \in [V_\kappa]^{\overline{\overline{a}}};\ f(u) \text{ is a } \mathbb{P} \restriction \kappa\text{-name for an open dense set in } \mathbb{P}_\kappa \right\} \in E_a$$

We have that $\left\{ (f(u))^{G'};\ u \in X \right\}$ is of cardinality κ hence the intersection of all such sets

$$\dot{D}^{G'} = \bigcap \left\{ (f(u))^{G'};\ u \in X \right\}$$

is still a dense set in $V[G']$, where $\dot{D}$ is a name such that there is a $q \in G'$ with

$$q \Vdash \text{“}\dot{D} = \bigcap\{(f(u));\ u \in X\} \text{ and } \dot{D} \text{ is a dense set”}.$$

Let $\sigma^{G'} \in \dot{D}^{G'} \cap G''$ and let $p \in G'$, $p < q$, with $p \Vdash$ “$\sigma \in \dot{D}$”.

We have that $p = j(p) \Vdash$ “$j(\sigma) \in \tau$”. Since $p \in G' \subseteq G * G^*$ it follows that $j(\sigma)^{G*G^*} \in \tau^{G*G^*}$. As the iteration has an easton support, it is is bounded below κ at stage larger or equal to κ. This shows that:

$$j(\sigma)^{G*G^*} = j''\sigma^G \in j''G''.$$

Thus G^{**}, the closure of $j''G''$ in $M[G * G^*]$, is a $\mathbb{P}^{**}$-generic filter over $M[G * G^*]$. By the factor lemma $G * G^* * G^{**}$ is $j(\mathbb{P})$-generic over M. Setting $\tilde{G} = G^* * G^{**}$, we get the desired result by lifting j using the classical definition:

$$j(\tau^G) = j(\tau)^{G*\tilde{G}}$$

for τ a $\mathbb{P}$-name in V. ⊣

Corollary 3.5. *Let A be the class of all strong cardinal. Suppose $V \vDash$ “GCH, κ is an A-strong cardinal, $\delta > \kappa$ is the only strong cardinal above κ”. Then for every successor cardinal μ below the least strong cardinal, there is a forcing $\mathbb{Q}$ such that, whenever G is $\mathbb{Q}$-generic over V, κ and δ are ideally strong in $V[G]$, $\mu^{+V[G]} = \kappa$ and $n(\kappa)^{+V[G]} = \lambda$.*

Proof. First apply $\mathbb{P} \restriction \kappa$, where $\mathbb{P}$ is the forcing defined in the previous lemma. From the point of view of λ this is a small forcing, hence if G' is $\mathbb{P} \restriction \kappa$-generic over V, λ is strong in $V[G']$. Now we can just do the levy collaps $\mathrm{col}(n(\kappa), \lambda)$. By results of the last section, λ is ideally strong in $V[G', G'']$, where G'' is $\mathrm{col}(n(\kappa), \lambda)$-generic over $V[G']$. By the factor lemma, this is the same as forcing in one time with $\mathbb{P}$ as defined in the last lemma. Let $G_1 = G' \times G''$. In $V[G_1]$, κ is still strong, hence we can force with $\mathrm{col}(\mu, < \kappa)$ for some regular cardinal μ. Let G_2 be $\mathrm{col}(\mu, < \kappa)$-generic over $V[G_1]$ and set $G = G_1 \times G_2$. In $V[G]$ κ is ideally strong. Let us show that λ remains ideally strong in $V[G]$.

Remark that $\mathrm{col}(\mu, < \kappa) \cap V[G_1]$ and $\mathrm{col}(\mu, < \kappa) \cap V$ are forcing equivalent. Hence we can first force with $\mathrm{col}(\mu, < \kappa) \cap V$ and then force with $\mathbb{P}$. In the first extension $V[G_2]$, λ remains strong hence by the previous theorem λ is ideally strong in $V[G_2, G_1]$. ⊣

3.3 Forcing many generically strong cardinals

Definition 3.6. We say that a cardinal $\kappa \in V$ is *generically strong* if for all $A \in V$ there is a forcing $\mathbb{P}$ such that, if G is $\mathbb{P}$-generic over V, in $V[G]$ there is a definable embedding $j : V \to M \subseteq V[G]$ with critical point κ and $A \in M$.

Obviously if κ is ideally strong it is generically strong.

Lemma 3.7. *Let κ, λ be two strong cardinals and μ, ν two successor cardinals such that $\mu < \kappa < \nu < \lambda$. Let $\mathbb{P} = \mathrm{col}(\mu, < \kappa) \times \mathrm{col}(\nu, < \lambda)$ and let G be $\mathbb{P}$-generic over V. In $V[G]$ κ and λ are generically strong.*

Proof. Let $\mu, \nu, \kappa, \lambda$ and G be as in the theorem. After forcing with $\mathrm{col}(\mu, < \kappa)$, λ remains strong, hence by Theorem 2.8, λ is an ideally strong cardinal in $V[G]$. Let $X \in V[G]$ be some set. We only have to show that there is a forcing $\mathbb{P}$, such that if H is $\mathbb{P}$-generic over $V[G]$, there is a definable embedding in $V[G, H]$,

$$\tilde{\jmath} : V[G] \to \tilde{M}$$

such that $G, X \in \tilde{M}$ and $\mathrm{cp}(\tilde{\jmath}) = \kappa$. Split G into G_κ $\mathrm{col}(\mu, < \kappa)$-generic over V and G^ν_λ, $\mathrm{col}(\nu, < \lambda) \cap V$-generic over $V[G_\kappa]$.

Let τ be a $\mathbb{P}$-name for X and let θ be a large enough regular cardinal such that $\{\tau\} \cup (2^\lambda)^+ \subseteq H_\theta$ and ${}^\kappa V_\theta \subseteq V_\theta$. Since κ is strong in V there is a $\langle \kappa, V_\theta \rangle$-extender, say E, such that $H_\theta \subseteq \mathrm{Ult}(V, E)$. Let

$$j : V \to M = \mathrm{Ult}(V, E)$$

be the associated ultrapowermap. We have that $\theta < j(\kappa) < \theta^{+V}$. By Lemma 3.3, we know that we can lift j to $\bar{\jmath}' : V[G^\nu_\lambda] \to M[G^j_\lambda]$, where

$$G^j_\lambda = \{q \in \mathrm{col}(j(\nu), j(\lambda)) \cap M;\ \exists p \in G^\nu_\lambda\ q < j(p)\}$$

is $\mathrm{col}(j(\nu), j(\lambda))$-generic over M.

Let G_{θ^+} be $\mathrm{col}(\mu, < \theta^+)$-generic over V, such that:

i. $G \in V[G_{\theta^+} \cap \mathrm{col}(\mu, < \lambda^{+V})]$,

ii. $G_\kappa = G_{\theta^+} \cap \mathrm{col}(\mu, < \kappa)$.

We can construct such an G_{θ^+}, because by [Fuc08, lemma 2.2] $\mathrm{col}(\mu, < \lambda) \times \mathrm{col}(\mu, \lambda)$ is forcing equivalent to $\mathrm{col}(\mu, \{\lambda\})$. Let $G_{\theta+1} = G_{\theta^+} \cap \mathrm{col}(\mu, < \theta + 1)$. We first want to create a $\mathrm{col}(\mu,]\theta, j(\kappa)[)^{M[G_{\theta+1}]}$-generic filter over M, G_1.

Claim 1. *There is a $G_{j(\kappa)} \in V[G_{\theta+1}]$ such that:*

i. *$G_{j(\kappa)}$ is* $\mathrm{col}(\mu, j(\kappa))$ *generic over* M,

ii. $G_{j(\kappa)} \cap (\mathrm{col}(\mu, < \theta))^V = G_{\theta+1} \cap (\mathrm{col}(\mu, < \theta))^V$,

iii. *we can lift j to some:*

$$j \subseteq \bar{j} : V[G_\kappa] \to M[G_{j(\kappa)}].$$

Proof. Remark that since $H_\theta^V = H_\theta^M$, we have that

$$\mathrm{col}(\mu, < \theta + 1) \cap M = \mathrm{col}(\mu, < \theta + 1) \cap V$$

As $M \subseteq V$, we have that $G_{\theta+1}$ is $\mathrm{col}(\mu, < \theta + 1)$-generic over M as well. Now look at $\mathrm{col}(\mu,]\theta, j(\kappa)[) \cap M$ in V; As M is κ-closed it is a $< \mu$-closed forcing in V. $\mathrm{col}(\mu,]\theta, j(\kappa)[) \cap M$ adds a surjective function from μ to θ. By [Fuc08, lemma 2.2] it is forcing equivalent to $\mathrm{col}(\mu, \{\theta\})$, hence we can define a $\mathrm{col}(\mu,]\theta, j(\kappa)[) \cap M$-generic filter G_1 over V from $G_{\theta^+} \cap \mathrm{col}(\mu, \{\theta + 1\})$. But since $M \subseteq V$, being a dense set is upward absolute between the two models, hence G_1 is also generic over M. Set

$$G_{j(\kappa)} = G_{\theta+1} \times G_1.$$

By the product lemma, $G_{j(\kappa)}$ is $\mathrm{col}(\mu, < j(\kappa)) \cap M$-generic over M. Remark that, as $j'' G_\kappa \subseteq G_{j(\kappa)}$, we can lift j to an embedding

$$\bar{j} : V[G_\kappa] \to M[G_{j(\kappa)}]. \qquad \dashv$$

Remark that since $\mathrm{col}(j(\nu), j(\lambda))$ is $< j(\nu)$ closed, $G_{j(\kappa)}$ is $\mathrm{col}(\mu, < j(\kappa) \cap M$-generic over $V[G_\lambda^j]$ as well. Hence by the product forcing theorem $G_{j(\kappa)} \times G_\lambda^j$ is $\mathrm{col}(\mu, < j(\kappa)) \cap M \times \mathrm{col}(j(\nu), < j(\lambda)) \cap M$-generic over M.

Let $\tilde{j} : V[G] \to M[G_{j(\kappa)} \times G_\lambda^j]$ be such that

$$\tilde{j}(\tau^G) = j(\tau)^{G_{j(\kappa)} \times G_\lambda^j},$$

where τ is a $V^{\mathbb{P}}$-name.

Claim 2. *$\tilde{j}$ is a fully elementary embedding that lifts j.*

Proof. Let φ be some formula such that $V[G] \vDash \varphi(\tau^G)$ for some $\mathbb{P}$-name τ. There is a $p \in G$ such that

$$p \Vdash \varphi(\tau)$$

Hence by the elementarity of j:

$$j(p) \Vdash \varphi(j(\tau))$$

But by construction $j(p) \in G_{j(\kappa)} \times G^j_\lambda$ and $j(\tau)^{G_{j(\kappa)} \times G^j_\lambda} = \tilde{\jmath}(\tau^G)$, hence

$$M[G_{j(\kappa)} \times G^j_\lambda] \vDash \varphi(\tilde{\jmath}(\tau^G)).$$

Let $x \in V$ then $x = \check{x}^G$ and thus $\tilde{\jmath}(x) = j(\check{x})^{G_{j(\kappa)} \times G^j_\lambda} = j(x)$, as $j(\check{x}) = \check{j(x)}$. Hence $j \subseteq \tilde{\jmath}$. ⊣

Hence we can lift j to

$$\tilde{\jmath} : V[G] \to M[G_{j(\kappa)} \times G^j_\lambda]$$

on the other hand $\tau, G \in M[G_{j(\kappa)} \times G^j_\lambda]$. As $\tilde{\jmath}$ is definable from j, G and $G_{j(\kappa)} \times G^j_\lambda$, it is definable in $V[G_{\theta^+}]$. Hence κ is generically strong in $V[G]$. ⊣

Notice that the proof actually showed:

Theorem 3.8. *Let κ be strong in V and $\mu < \kappa$ some cardinal. Let G be $\mathrm{col}(\mu, <\kappa)$-generic over V and $\mathbb{P}$ some $<\kappa^+$-closed forcing in V. Let H be $\mathbb{P}$ -generic over $V[G]$. Then κ is generically strong in $V[G, H]$.*

Proof. Let $\mathbb{P}$, G and H be as in the lemma. Let θ be some large cardinal, such that $\mathbb{P} \in H_\theta$ and ${}^\kappa V_\theta \subseteq V_\theta$. It suffices to prove that there is some embedding

$$\pi : V[G, H] \to M,$$

such that $H^V_\theta \subseteq M$. Let E be a $\langle \kappa, V_\theta \rangle$-extender and j the associated ultrapower. By Lemma 3.3, we can lift j to some

$$\bar{\jmath} : V[H] \to M[H^j],$$

where H^j is the M-closure of $j''H$ in M. The last proof showed that we can then lift j to some

$$\tilde{\jmath} : V[G, H] \to V[G_{j(\kappa)}, H^j],$$

where $G_{j(\kappa)}$ is some $\mathrm{col}(\mu, j(\kappa))$-generic filter over M such that $G = G_{j(\kappa)} \cap \mathrm{col}(\mu, \kappa)$ and $H \in M[G_{j(\kappa)}]$. ⊣

It is not hard to see that applying this theorem to the easton support forcing product of the levy collapse of strong cardinals, we get the following corollary:

Corollary 3.9. *A is a set of strong cardinals such that* $\mathrm{otp}(A) < \min(A)$*, and let $f : A \to \mathrm{OR}$ a function such that for all $\mu \in A$, $f(\mu)$ is a successor cardinal and for all $\mu < \nu \in A$ $\mu < f(\nu)$. Then there is a forcing $\mathbb{P}$ such that if G is $\mathbb{P}$-generic over V:*

i. every $f(\mu)$ is a successor cardinal, moreover $f(\mu)^{+V[G]} = \mu$

ii. every μ in A is generically strong in $V[G]$.

Proof. Let A, f be as in the theorem and $\mathbb{P}$ the easton support forcing product of all $\mathbb{Q}_\kappa$ for $\kappa \in A$, where $\mathbb{Q}_\kappa = \mathrm{col}(f(\kappa), < \kappa)$. That is $p \in \mathbb{P}$ if $p \in \Pi_{\kappa \in A}\mathbb{Q}_\kappa$ and for all limit point λ of A, the set of $i < \lambda$ such that $(p)_i \neq \mathbb{1}_{\mathbb{Q}_i}$ is bounded in λ. For every $\kappa \in A$ we can split the forcing $\mathbb{P}$ in three pieces $\mathbb{P}_\kappa$ the easton support product of all $\mathbb{Q}_i$ for $i \in A \cap \kappa$, $\mathbb{Q}_\kappa$ and $\mathbb{P}^\kappa$ the easton support product of all $\mathbb{Q}_i$ such that $i \in A \setminus \kappa + 1$. Notice that $\mathbb{P}^\kappa$ is κ-closed. For every filter G, $\mathbb{P}$-generic over V, let $G_\kappa = G \cap \mathbb{P}_\kappa$ and $G^\kappa = G \cap (\mathbb{Q}_\kappa \times \mathbb{P}^\kappa)$. $\mathbb{P}_\kappa$ is a small forcing, hence κ is strong in $V[G_\kappa]$, by Theorem 3.8 κ is generically strong in $V[G_\kappa, G^\kappa] = V[G]$. ⊣

Giving one concrete example of such a function f:

Corollary 3.10. *Suppose* ZFC+ *"there are ω strong cardinals" is consistent, then so is* ZFC+ *"every $\aleph_{2n+1}$ is generically strong for $n \in \omega$"*

3.4 The consistency strength of many generically strong cardinals

We have seen how to get many generically strong cardinals, starting with the same amount of strong cardinals. Let us now answer the reverse question, whether one gets the strong cardinals "back". Let Ω be some large measurable cardinal and μ_0 a $< \Omega$-complete ultrafilter on Ω. From now on we will work in V_Ω.

Theorem 3.11. *Suppose there is no inner model with a Woodin cardinal. Let κ be generically strong in V, then κ is strong in the core model.*

Proof. Let $K = K^V$ be the core model as defined in [Ste96]. We work towards contradiction.

Claim 1. *Suppose κ is not strong in K, there is a θ such that for every $\nu > \theta$ either $\mathrm{cp}(E^K_\nu) > \theta$ or $\mathrm{cp}(E^K_\nu) < \kappa$.*

Proof. Let θ be smallest cardinal strictly larger than the Mitchell order of κ. We claim that θ has already the desired properties. Suppose not and let F be an extender on the K-sequence with critical point $\lambda < \theta$ and index $\nu > \theta$. Let $\mathcal{M} = \mathrm{Ult}(K\|\nu, F)$ and j be the associated ultrapower map. We know that $K|\nu \models$ "κ is λ-strong", hence $\mathcal{M} \models$ "κ is $j(\lambda)$-strong". Let E be some extender of the $\mathcal{M}$ sequence with critical point κ and index larger than θ. Since θ is a cardinal, there must be cofinally many E-generators below θ. Let μ be such a generator. Then $E \restriction \mu+1$ has natural length $\mu+1$, hence by the initial segment condition either the completion of E is on the $\mathcal{M}$ sequence or it is one ultrapower away. Since $\mu+1$ is a successor ordinal, the second case can not occur.Thus, we have that there is some $\mu+1 < \gamma < \nu$ such that $E^{\mathcal{M}}_\gamma$ is the trivial completion of $E \restriction \mu+1$. Since $\gamma < \nu$ we have that $E^{\mathcal{M}}_\gamma = E^K_\gamma$ by coherency. Hence for every $\nu < \theta$ we can find a $\gamma > \nu$ such that E_γ has critical point κ and is on the K sequence, a contradiction to the definition of θ! $\dashv$

Let θ be as in the claim. Since κ is generically strong, there is a forcing $\mathbb{P}$ such that for every $\mathbb{P}$-generic G over V, there is an embedding in $V[G]$

$$j : V \to M$$

with $H^V_{\theta^+} \in M$. Let K^M be the core model as computed in M. Notice that $K^V = K^{V[G]}$, we will drop the superscript and call it K in what follows.

Claim 2. *K^M is a universal weasel in $V[G]$.*

Proof. The same proof as **??** shows that K^M is iterable in $V[G]$. The set of fixed point of j is a club set in $\{\alpha;\ \mathrm{cf}(\alpha) \neq \kappa\}$, but since $\mathbb{P}$ has the $(2^{\mathrm{card}(\mathbb{P})})^{+V}$-c.c. for stationary many successor of some fixed point α of j, we have that $\alpha^{+V} = \alpha^{+K} \leq j(\alpha)^{+K^M} \leq \alpha^{+M} \leq \alpha^{+V[G]}$. For all α larger than $(2^{\mathrm{card}(\mathbb{P})})^{+V}$, $\alpha^{+V} = \alpha^{+V[G]}$. Hence weak covering is true for some thick class in K^M, hence it is a universal weasel in $V[G]$. $\dashv$

We would like to coiterate K with K^M, but then the following diagram might not be commutative.

$$\begin{tikzcd}
K \arrow[d, "j"'] \arrow[dr, "\pi^{\mathcal{T}}_\Lambda"] & \\
K^M \arrow[r, "\pi^{\mathcal{Q}}_\Lambda"'] & Q
\end{tikzcd}$$

By slightly modifying the iterations, we can get a common iterate in a way that makes the triangle commutative. We will use a variation of the technique from the proof of Lemma 7.13 of [Ste96]. By [Ste96, Lemma 8.3] there is a universal weasel W such that $K|\theta \triangleleft W$ (in fact W witness that $K|\theta$ is A_0-sound), W has the hull property at all α and the definability property at all $\alpha < \theta$. By [Ste96, Lemma 8.2], W is a simple iterate of K, actually the iteration $\mathcal{T}_0$ from K to W is linear and only uses measures, that is extenders with only one generator. Finally if $\pi^{\mathcal{T}_0}_{0,\infty}$ is the iteration map, by applying j we get an iteration tree $j(\mathcal{T})$ on K^M such that the whole commutes as in the following diagram:

$$\begin{tikzcd} K \arrow[r, "\pi^{\mathcal{T}_0}_{0,\infty}"'] \arrow[d, "j"] & W \arrow[d, "j"] \\ K^M \arrow[r, "\pi^{\mathcal{T}^j_0}_{0,\infty}"'] & j(W) \end{tikzcd}$$

We have that

$$\mathrm{Def}(W) = \pi^{\mathcal{T}_0}_{0,\infty}{}''K$$

and the iteration is above θ. We can lift that iteration via j to get an linear iteration of K^M. Since the class of fixed points of j is thick in W, Ω is thick in $j(W)$ and

$$\mathrm{Def}(j(W)) = j''\,\mathrm{Def}(W).$$

Let us coiterate W and $j(W)$ in $V[G]$ and let $\mathcal{T}_1$ and $\mathcal{U}$ be the respective trees of the coiteration. Since both W and $j(W)$ are universal weasel in $V[G]$ there is no drop on both side of the iteration. The coiteration might not commute on the whole range, but it does commute on $\mathrm{ran}(\pi^{\mathcal{T}_0}_{0,\infty}) = \mathrm{Def}(W)$ since all the elements of $\mathrm{Def}(W)$ are definable with skolem terms and parameters in a thick class of fix points, see **??**. Hence if we set $\mathcal{T} = \mathcal{T}_0{}^\frown\mathcal{T}_1$ and $\mathcal{Q} = j(\mathcal{T}_0){}^\frown\mathcal{U}$

$$\begin{tikzcd} K \arrow[r, "\pi^{\mathcal{T}}"] \arrow[d, "j"] & W \arrow[d, "j"] \arrow[dr, "\pi^{\mathcal{T}_1}"] & \\ & & Q \\ K^M \arrow[r, "\pi^{j(\mathcal{T})}"'] & j(W) \arrow[ur, "\pi^{\mathcal{U}}"] & \end{tikzcd}$$

This shows that K and K^M iterate via $\langle \mathcal{T}, \mathcal{U} \rangle$ to a common model Q such that the iterations commute with j. Let $\pi^{\mathcal{T}}_{\Lambda} : K \to Q$ be the iteration map on the K side and $\pi^{\mathcal{Q}}_{\Lambda} : K^M \to Q$ the iteration map on the K^M side, where Λ is the length of the iteration.

Claim 3. *There is no $\mu \leq \kappa$ such that the coiteration uses extenders with critical point μ on both side of the coiteration.*

Proof. Suppose not an let E be the first extender with critical point $\mu \leq \kappa$ used on the W side and F the first extender with critical point μ used on the $j(W)$ side. Notice that μ has the same subsets in every model. Let Γ be a thick class of fixed points of $\pi^{\mathcal{T}}_{\Lambda}$ and $\pi^{\mathcal{Q}}_{\Lambda} \circ j$. Suppose $\mathrm{lh}(E) < \mathrm{lh}(F)$, and let $X \in E_a$. Since W has the hull and definability property at all $\alpha < \theta$, there are $\vec{\eta} \in \Gamma$ and a skolem term τ such that $X = \tau^W(\vec{\eta})$. Hence

$$X = \tau^{j(W)}(\vec{\eta}) \cap \kappa^{\overline{\overline{a}}}$$

Notice that we need to cut with κ just for the case $\mu = \kappa$. As $\pi^{\mathcal{T}}_{\Lambda}(X) = \tau^Q(\vec{\eta})$ and

$$\pi^{\mathcal{Q}}_{\Lambda}(X) = \tau^Q(\vec{\eta}) \cap \pi^{\mathcal{Q}}_{\Lambda} \circ j(\kappa^{\overline{\overline{a}}}).$$

Since $\mathrm{cp}(F) \leq \kappa$ and $a \in [\mathrm{lh}(F)]^{<\omega}$, $a \in [\pi^{\mathcal{Q}}_{\Lambda} \circ j(\kappa)]^{<\omega}$. Thus we have the following equivalence:

$$\begin{aligned}
X \in E_a &\iff a \in \pi^{\mathcal{T}}_{\Lambda}(X) \\
&\iff a \in \tau^Q(\vec{\eta}) \\
&\iff a \in \tau^Q(\vec{\eta}) \cap \pi^{\mathcal{Q}}_{\Lambda} \circ j(\kappa^{\overline{\overline{a}}}) \\
&\iff a \in \pi^{\mathcal{Q}}_{\Lambda}(X) \\
&\iff X \in F_a
\end{aligned}$$

Hence E and F are compatible, a contradiction to **??**! If $\mathrm{lh}(E) > \mathrm{lh}(F)$, we can argue the very same way. ⊣

Claim 4. $\mathrm{cp}(\pi^{\mathcal{T}}_{\Lambda}) = \kappa$ *and* $\mathrm{cp}(\pi^{\mathcal{Q}}_{\Lambda}) > \kappa$.

Proof. By construction the iteration $\mathcal{T}_0$ is above κ, we claim that $\mathcal{T}_1$ does not have critical points less than κ on the main branch. Suppose not, and let μ be the smallest ordinal such that there is an extender with critical point μ used in the coiteration. By commutativity, μ is the smallest on the $j(W)$ side as well. Hence both side would have use an extender with identical critical point less than κ a contradiction to the previous claim! Thus the critical point of $\pi^{\mathcal{T}}_{\Lambda}$ is at least κ. Since the diagram commutes and j has critical point κ, $\pi^{\mathcal{T}}_{\Lambda}$ must have critical point κ as well. By the previous claim, this implies that $\mathrm{cp}(\pi^{\mathcal{Q}}_{\Lambda}) > \kappa$. ⊣

The last claim shows that $\mathcal{P}(\kappa) \cap K = \mathcal{P}(\kappa) \cap K^M$, hence $\kappa^{+K} = \kappa^{+K^M}$. Since $K|\theta \in M$, we can coiterate $K|\theta$ with K^M in M. The coiteration coincide with the coiteration of K and K^M in V. Let Δ be the length of the coiteration of $K\|\theta$ with K^M.

Claim 5.

$$\pi^{\mathcal{T}}_{0,\Lambda} \restriction K\|\theta = \pi^{\mathcal{T}}_{0,\Delta} \restriction K\|\theta,$$

that is the main branch of the coiteration of $K\|\theta$ with K^M is an initial segment of the main branch of the coiteration of K with K^M.

Proof. Suppose not, then there is an extender E used on the main branch of the K side of the coiteration with index higher than θ such that $\mathrm{cp}(E) < \theta$. But by the properties of θ, this implies that $\mathrm{cp}(E) < \kappa$. As E is on the main branch, we would have $\mathrm{cp}(\pi^{\mathcal{T}}_{\Lambda}) < \kappa$ a contradiction to the previous claim! $\dashv$

This shows that $\pi^{\mathcal{T}}_{\Lambda} \restriction K|\theta \in M$. Hence the last model of the iteration $Q|\pi^{\mathcal{Q}}_{\Lambda}(j(\theta))$ is in M as well and we can coiterate $Q|\pi^{\mathcal{Q}}_{\Lambda}(j(\theta))$ with K^M in M. Since $Q|\pi^{\mathcal{Q}}_{\Lambda}(j(\theta))$ is an iterate of K^M, it does not move in the coiteration and the K^M side is simply the normal iteration to $Q|\pi^{\mathcal{Q}}_{\Lambda}(j(\theta))$. Hence we have that $j \restriction (K^M|j(\theta)) \in M$. Since the diagram commute, we can deduce $j \restriction \mathcal{P}(\kappa) \cap K$ by

$$j(x) = y \iff (\pi^{\mathcal{T}}_{\Lambda} \restriction K|\theta)(x) = (\pi^{\mathcal{Q}}_{\Lambda} \restriction K^M|j(\theta))(y).$$

Let $\alpha < \theta$ and F be the extender of length α derived from $j \restriction \mathcal{P}(\kappa) \cap K$. F coheres with K^M. We want to study the iterability of the phalanx $\langle K^M, \mathrm{Ult}(K^M, F), \mathrm{lh}(F)\rangle$.

Claim 6. *The phalanx $\langle K^M, \mathrm{Ult}(K^M, F), \mathrm{lh}(F)\rangle$ is iterable.*

Proof. The aim of the proof is to show that there is an embedding from $\mathrm{Ult}(K^M, F)$ to some Q^*, where Q^* is an iterate of K^M beyond $j(\mathrm{lh}(F))$. Let us first construct Q^* and then show that we can embed $\mathrm{Ult}(K^M, F)$ in it. Let $\mathcal{T}^j$ be the iteration on K^M copied from $\mathcal{T}$ via j. We claim that at each step we can factorize by taking an ultrapower with F:

$$\begin{tikzcd}
K \arrow[r, "\pi^{\mathcal{T}}_{0,\xi}"] \arrow[dr, "i^K_F"] \arrow[dd, "j"'] & \cdots \arrow[rr, "\pi^{\mathcal{T}}_{\eta,\Lambda}"] & & Q \arrow[dd, "j_\Lambda"] \arrow[dr, "i^Q_F"] & \\
& \mathrm{Ult}(K^M, F) \arrow[dl, "k_0"] & \cdots & & \mathrm{Ult}(Q, F) \arrow[dl, "k_\Lambda"] \\
K^M \arrow[r, "\pi^{\mathcal{T}^j}_{0,\xi}"] & \cdots \arrow[rr, "\pi^{\mathcal{T}^j}_{\eta,\Lambda}"] & & Q^* &
\end{tikzcd}$$

Figure 1: Copying $\mathcal{T}$

The $j_\xi : \mathcal{M}^{\mathcal{T}}_\xi \to \mathcal{M}^{\mathcal{T}^j}_\xi$'s are the usual copy maps, hence we have that whenever $\eta \leq \xi < \mathrm{lh}(\mathcal{T})$,

$$j_\xi \restriction \mathrm{lh}(E^{\mathcal{T}}_\eta) = j_\eta \restriction \mathrm{lh}(E^{\mathcal{T}}_\eta).$$

By the previous claim we know that $\mathcal{T}$ is above κ. Moreover there are no truncations in $\mathcal{T}$ and thus in $\mathcal{T}^j$. Hence for every $X \in \mathcal{P}(\kappa) \cap K$,

$$\pi^{\mathcal{T}}_{0,\xi}(X) \cap \kappa = X.$$

Since $j(\kappa) \geqslant \mathrm{lh}(F)$, the iteration $\mathcal{T}^j$ is above $\mathrm{lh}(F)$, hence if $a \in [\mathrm{lh}(F)]^{<\omega}$ $\pi^{\mathcal{T}^j}_{\xi,\eta}(a) = a$. Using the commutativity of the diagram:

$$\begin{tikzcd}
K \arrow[r, "\pi^{\mathcal{T}}_{0,\xi}"'] \arrow[d, "j"] & \mathcal{M}^{\mathcal{T}}_\xi \arrow[d, "j_\xi"] \\
K^M \arrow[r, "\pi^{\mathcal{T}^j}_{0,\xi}"'] & \mathcal{M}^{\mathcal{T}^j}_\xi
\end{tikzcd}$$

we have that:

$$j_\xi(\pi^{\mathcal{T}}_{0,\xi}(X)) = \pi^{\mathcal{T}^j}_{0,\xi}(j(X)).$$

Thus for $a \in [\mathrm{lh}(F)]^{<\omega}$ and $X \in \mathcal{P}([\kappa]^{\overline{\overline{a}}}) \cap K$:

$$a \in j_\xi(X) \iff a \in j_\xi(\pi^{\mathcal{T}}_{0,\xi}(X)) \iff \pi^{\mathcal{T}^j}_{0,\xi}(a) \in \pi^{\mathcal{T}^j}_{0,\xi}(j(X)) \iff a \in j(X)$$

Hence the $\langle \kappa, \mathrm{lh}(F) \rangle$-extender derived by j_ξ is nothing else than F and thus we can factorize j_ξ by $i^{\mathcal{M}^{\mathcal{T}}_\xi}_F$ with some map k_ξ such that the diagram below commutes:

$$\begin{tikzcd}
\mathcal{M}^{\mathcal{T}}_\xi \arrow[dr, "i^{\mathcal{M}^{\mathcal{T}}_\xi}_F"] \arrow[dd, "j_\xi"] & \\
& \mathrm{Ult}(\mathcal{M}^{\mathcal{T}}_\xi, F) \arrow[dl, "k_\xi"] \\
\mathcal{M}^{\mathcal{T}^j}_\xi &
\end{tikzcd}$$

Let $i^{K^M}_F : K^M \to \mathrm{Ult}(K^M, F)$ and $i^Q_F : Q \to \mathrm{Ult}(Q, F)$ be the ultrapower maps. Then defining $k : \mathrm{Ult}(K^M, F) \to \mathrm{Ult}(Q, F)$ such that

$$i^{K^M}_F(f)(a) \mapsto i^Q_F(\pi^{\mathcal{U}}_\Lambda(f) \restriction \kappa)(a),$$

where $a \in [\mathrm{lh}(F)]^{<\omega}$ and $f : \kappa^{\overline{\overline{a}}} \to K^M$, $f \in K^M$. Let us show that this map is an embedding. Let φ be a formula.

$$\begin{aligned}
\mathrm{Ult}(Q,F) \models \varphi(k(i_F^{K^M}(f)(a))) &\iff \mathrm{Ult}(Q,F) \models \varphi(i_F^Q(\pi_\Lambda^{\mathcal{Q}}(f) \upharpoonright \kappa)(a)) \\
&\iff \left\{u; Q \models \varphi(\pi_\Lambda^{\mathcal{U}}(f)(u))\right\} \cap \kappa \in F_a \\
&\iff \pi_\Lambda^{\mathcal{Q}}(\left\{u; K^M \models \varphi(f(u))\right\}) \cap \kappa \in F_a) \\
&\iff \left\{u; K^M \models \varphi(f(u))\right\} \in F_a \\
&\iff \mathrm{Ult}(K^M,F) \models \varphi(i_F^{K^M}(f)(a))
\end{aligned}$$

The first equivalence holds by definition of k, the third because $\mathrm{cp}(\pi_\Lambda^{\mathcal{Q}}) \geqslant \kappa$, the second and fourth is Łoś theorem for ultrapower. Putting everything together we get the following diagram:

$$\begin{tikzcd}
 & K \arrow[rr, "j"] \arrow[dl, "j"'] \arrow[d, "i_F^K"] \arrow[ddr, "\pi_\Lambda^{\mathcal{T}}"] & & K^M \arrow[d, "i_F^{K^M}"] \arrow[ddl, "\pi_\Lambda^{\mathcal{Q}}"'] \\
K^M \arrow[ddr, "\pi_\Lambda^{\mathcal{T}^j}"] & \mathrm{Ult}(K,F) \arrow[l, "k_0"'] & & \mathrm{Ult}(K^M,F) \arrow[ddl, "k"] \\
 & & Q \arrow[dl, "j_\Lambda"'] \arrow[d, "i_F^Q"] & \\
 & Q^* & \mathrm{Ult}(Q,F) \arrow[l, "k_\Lambda"] &
\end{tikzcd}$$

Figure 2: The complete diagram

Hence we can embed $\mathrm{Ult}(K^M,F)$ into Q^* by $k_\Lambda \circ k$. Since $k_\Lambda \circ k$ has critical point strictly larger than $\mathrm{lh}(F)$, the map

$$\langle \mathrm{id}, k_\Lambda \circ k\rangle : \langle K^M, \mathrm{Ult}(K^M,F), \mathrm{lh}(F)\rangle \to \langle K^M, Q^*, \mathrm{lh}(F)\rangle,$$

is an embedding as well. Moreover since $\mathcal{T}$ was above κ, we have that $\mathcal{T}^j$ the iteration from K^M to Q^* is above $j(\kappa) > \mathrm{lh}(F)$. Thus we can embed $\langle K^M, Q^*, \mathrm{lh}(F)\rangle$the following way:

$$(\pi^{\mathcal{T}^j}, \mathrm{id}) : \langle K^M, Q^*, \mathrm{lh}(F)\rangle \to \langle Q^*, Q^*, \mathrm{lh}(F)\rangle.$$

$\langle Q^*, Q^*, \mathrm{lh}(F)\rangle$ is clearly iterable since Q^* is an iterate of an universal weasel. This finishes the proof of the claim. $\dashv$

By [Ste96, Lemma 8.6 p. 77] this is, in fact equivalent to F being on the K^M sequence. Hence every initial segment of $j \upharpoonright \mathcal{P}(\kappa) \cap K$ is on the K^M sequence. But this implies that κ is Shelah in K^M, a contradiction! $\dashv$

Using Theorem 3.11 this gives us an immediate consistency strength result:

Theorem 3.12. *For $i \leq \omega$ the following two theories are equiconsistent:*

i. ZFC+ *"there are α generically strong cardinals, where α is less than the least generically strong cardinal"*

ii. ZFC+ *"there are α strong cardinals, where α is less than the least strong cardinal"*

4 Supercompactness

In this section we want to show that we can apply some of the forcing techniques developed to force generically strong cardinals from strong cardinals to supercompact cardinals.

Definition 4.1. Let κ be a cardinal and γ some ordinal. κ is called γ-*supercompact* if and only if there is an embedding $j : V \to M$ such that ${}^{\gamma}M \cap V \subseteq M$. κ is called *supercompact* if it is γ-supercompact for all γ.

Remark 4.2. Let κ be γ-supercompact and $j : V \to M$ an embedding witnessing the γ-supercompactness. For any cardinal $\nu < \gamma$, $j''\nu \in M$.

Theorem 4.3. *Suppose* ZFC+ *"there exist ω many supercompact cardinals" is consistent, then so is* ZFC+ *"each $\aleph_{n+1}$ is generically strong".*

Proof. Let $\kappa_0 = \omega$ and $\langle \kappa_{n+1};\ n \in \omega \rangle$ be a monotone enumeration of all supercompact cardinals. Let $\mathbb{P}$ be the easton support forcing iteration of $\operatorname{col}(\kappa_n, < \kappa_{n+1})$ for $n \in \omega$. We want to show that if G is $\mathbb{P}$-generic over V then in $V[G]$ every $\aleph_{n+1}$ is generically strong. Let $\kappa = \kappa_{n+1}$ be supercompact in V and $A \in V[G]$ be some subset of the ordinals. Let θ regular be bigger than $\sup(A)^{\overline{\overline{\mathbb{P}}}}$ and 2^μ, where $\mu = (2^{\sup\{\kappa_i;\ i<\omega\}})^+$. Let $j : V \to M$ be the embedding witnessing the θ-supercompactness of κ. Let G^θ be $\operatorname{col}(\omega, < \theta^{++})$-generic over V such that $G \in V[G^\theta]$.

It suffices to construct a $\tilde{G}$ with the properties that: $j''G \subseteq \tilde{G}$ and $\tilde{G}$ is $j(\mathbb{P})$-generic over M. We split the forcing $j(\mathbb{P})$ in three parts:[4]

$$\mathbb{P}_n = \prod_{i<n} \mathrm{col}(\kappa_i, \ < \kappa_{i+1}), \ \mathbb{Q}_n^j = \mathrm{col}(\kappa_n, < j(\kappa_{n+1})),$$

and finally

$$\mathbb{P}^{j,n} = \prod_{n<i<\omega} \mathrm{col}(j(\kappa_i), < j(\kappa_{i+1})).$$

We will choose generics over V for $\mathbb{P}_0$ and $\mathrm{col}(\kappa_n, < \mu)$, this is equivalent to choosing generics over M since $H_\theta^V \subseteq M$. We will construct the generic for $\mathbb{P}^{j,n}$ defining some master condition. Similarly we can split $\mathbb{P}$ in three forcings $\mathbb{P} = \mathbb{P}_n * \mathbb{Q}_n * \mathbb{P}^n$:

$$\mathbb{P}_n = \prod_{i<n} \mathrm{col}(\kappa_i, < \kappa_{i+1}), \mathbb{Q}_n = \mathrm{col}(\kappa_n, < \kappa)$$

and

$$\mathbb{P}^n = \prod_{n<i<\omega} \mathrm{col}(\kappa_i, < \kappa_{i+1}).$$

Set $G_n = G \cap \mathbb{P}_n$. Looking at $\mathbb{Q}^* = \mathrm{col}(\kappa_n, \sup\{\kappa_n;\ n < \omega\})$, by [Fuc08, lemma 2.2] we have that $(\mathbb{Q}_n * \mathbb{P}^n) \times \mathbb{Q}^*$ and $\mathbb{Q}^*$ are forcing equivalent, hence there is a filter G^*, $\mathbb{Q}^*$-generic over M, such that $G \in M[G_n \times G^*]$. Notice that G^* is $\mathbb{Q}^*$-generic over V as well. Using the general theory about Levy collapse, as found in [Kan03, p.127 ff], there is a $\mathrm{col}(\kappa_n, < \mu)$-generic filter over M, say G_1, that is also generic over V with $G^* \in M[G_n \times G_1]$.

Hence there is a filter $G_n \times G_1$, $\mathbb{P}_n * \mathrm{col}(\kappa_n, < \mu)$-generic over V and M, such that $G \in M[G_n, G_1]$. Let H^* be $\mathrm{col}(\kappa_n, [\mu, j(\kappa)[) \cap M[G_n]$-generic over $M[G_n, G_1]$ and set $H_n = G_1 \times H^*$, by the product lemma H_n is $col(\kappa_n, j(\kappa)) \cap M[G_n]$-generic over $M[G_n]$. Notice that we can choose $H^n \in V[G^\theta]$, as all forcings we saw so far are in H_{θ^+} and hence are in a countable model in $V[G^\theta]$.

We now want to construct a generic filter G^n, $\mathbb{P}^{j,n}$-generic over $M[G_n, H_n]$, such that $j''(G \restriction \mathbb{P}^n) \subseteq G^n$. Remember that

$$\mathbb{P}^{j,n} = \prod_{n<i<\omega} \mathrm{col}(j(\kappa_i), < j(\kappa_{i+1})).$$

and $\mathbb{P}^{j,n} = j(\mathbb{P}^n)$.

Let us now work in $M[G_n, H_n]$. Since j was witnessing the θ compactness of κ we have that $j''\kappa_i \in M$ for all i, moreover G is in $M[G_n, H_n]$.

[4] in a slight abuse of notation, we use the symbol $\prod$ to denote the easton support product

Hence can compute $q_i = j''(G \restriction \mathrm{col}(\kappa_i, \kappa_{i+1}))$ in $M[G_n, H_n]$. As q_i has size κ_i in $M[G_n, H_n]$, it is a condition of $\mathrm{col}(j(\kappa_i), < j(\kappa_{i+1}))$ for $i > n$. Hence $\dot{q} = \langle \check{q}_i ; i < \omega \rangle$ is a condition of the forcing $\mathbb{P}^{j,n}$. Let G^n be $\mathbb{P}^{j,n}$-generic over $M[G_n, H_n]$ with $\dot{q} \in G_2$. As we have seen, we can lift j to $\tilde{\jmath} : V[G] \to M[G_n, H_n, G^n]$.

A was in $V[G]$ hence, by choice of θ, there is a nice name τ with $\tau \in H_\theta$, thus $\tau \in M$ and $A \in M[G_n, H_n, G^n]$. Remark again that $\mathcal{P}(\mathbb{P}^{j,n}) \cap M[G_n, H_n]$ is countable in $V[G^\theta]$ hence we can choose $G^n \in V[G^\theta]$, thus we can define the embedding $\tilde{\jmath}$ inside $V[G^\theta]$ and κ is generically strong. $\dashv$

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