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Definable Hamel bases and $\mathsf{AC}_\omega(\mathbb{R})$

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Abstract. There is a model of ZF with a lightface Δ^1_3 definable Hamel bases in which $\mathsf{AC}_\omega(\mathbb{R})$ fails.

1. Introduction. Answering a question from [PP, p. 433] it was shown in [BS$^+$] that there is a Hamel basis in the Cohen–Halpern–Lévy model. In this paper we show that in a variant of this model, there is a projective, in fact a lightface Δ^1_3, Hamel basis.

Throughout this paper, by a *Hamel basis* we always mean a basis for $\mathbb{R}$, construed as a vector space over $\mathbb{Q}$. We denote by E_0 the *Vitali equivalence relation*, xE_0y iff $x-y \in \mathbb{Q}$ for $x, y \in \mathbb{R}$. We also write $[x]_{E_0} = \{y \colon yE_0x\}$ for the E_0-equivalence class of x. A transversal for the set of all E_0-equivalence classes picks exactly one member from each $[x]_{E_0}$. The range of any such transversal is also called a *Vitali set*. If we identify $\mathbb{R}$ with the Cantor space ${}^\omega 2$, then xE_0y iff $\{n \colon x(n) \neq y(n)\}$ is finite.

A set $\Lambda \subset \mathbb{R}$ is a *Luzin set* iff Λ is uncountable but $\Lambda \cap M$ is at most countable for every meager set $M \subset \mathbb{R}$. A set $S \subset \mathbb{R}$ is a *Sierpiński set* iff S is uncountable but $S \cap N$ is at most countable for every null set $N \subset \mathbb{R}$ ("null" in the sense of Lebesgue measure). A set $B \subset \mathbb{R}$ is a *Bernstein set* iff $B \cap P \neq \emptyset \neq P \setminus B$ for every perfect set $P \subset \mathbb{R}$. A *Burstin basis* is a Hamel basis which is also a Bernstein set. It is easy to see that $B \subset \mathbb{R}$ is a Burstin basis iff B is a Hamel basis and $B \cap P \neq \emptyset$ for every perfect $P \subset \mathbb{R}$ (see e.g. [BC$^+$, Proposition 2.5]).

A set $m \subset \mathbb{R} \times \mathbb{R}$ is called a *Mazurkiewicz set* iff $\mathrm{Card}(m \cap \ell) = 2$ for every straight line $\ell \subset \mathbb{R} \times \mathbb{R}$.

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By $\mathsf{AC}_\omega(\mathbb{R})$ we mean the statement that for all sequences $(A_n : n < \omega)$ such that $\emptyset \neq A_n \subset \mathbb{R}$ for all $n < \omega$ there is some choice function $f : \omega \to \mathbb{R}$, i.e., $f(n) \in A_n$ for all $n < \omega$.

D. Pincus and K. Prikry [PP] study the Cohen–Halpern–Lévy model H. The model H is obtained by adding a countable set of Cohen reals (say over L) without adding their enumeration; H does not satisfy $\mathsf{AC}_\omega(\mathbb{R})$. It is shown in [PP] that there is a Luzin set in H, so that in ZF, the existence of a Luzin set does not even imply $\mathsf{AC}_\omega(\mathbb{R})$. [BS$^+$, Theorems 1.7 and 2.1] show that in H there is a Bernstein set as well as a Hamel basis. As in ZF the existence of a Hamel basis implies the existence of a Vitali set, the latter also re-proves Feferman's result (see [PP]) that there is a Vitali set in H.

Therefore, in ZF the conjunction of statements (1), (3), and (5) below (which in ZF implies (4)) does not yield $\mathsf{AC}_\omega(\mathbb{R})$.

(1) There is a Luzin set.
(2) There is a Sierpiński set.
(3) There is a Bernstein set.
(4) There is a Vitali set.
(5) There is a Hamel basis.
(6) There is a Burstin basis.
(7) There is a Mazurkiewicz set.

(2) is false in H [BS$^+$, Lemma 1.6]. We neither know if (6) is true in H, nor do we know if (7) is true in H. We aim to prove that in ZF, the conjunction of *all* of these statements does not imply $\mathsf{AC}_\omega(\mathbb{R})$, even if the respective sets are required to be projective.

The Luzin set constructed in [PP, Theorem on p. 429] is Δ^1_2. In ZFC, there is no analytic Hamel basis (see [Sie1], [Sie2], [Jo]), but by a theorem of A. Miller [Mi, Theorem 9.26], in L there is a coanalytic Hamel basis; see also e.g. [Sch1, Corollary 2 and Lemma 4]. On the other hand, it can be verified that the model from [BS$^+$] does not have a projective Vitali set ([1]). For the convenience of the reader as well as to motivate what is to come, we shall sketch the proof of this at the beginning of the first section (see Lemma 2.1).

The papers [BC$^+$] and [BS] produce models of ZF plus DC plus (6) and of ZF plus DC plus (7), respectively. By another theorem of A. Miller [Mi, Theorem 7.21], in L there is a coanalytic Mazurkiewicz set. It is not known if there is a Mazurkiewicz set which is Borel.

The result of the current paper is the following.

THEOREM 1.1. *There is a model of* ZF *plus* $\neg\mathsf{AC}_\omega(\mathbb{R})$ *in which the following hold true.*

([1]) To display our ignorance: we do not know if the model from [BC$^+$] has a definable Hamel basis.

(a) *There is a lightface* Δ^1_2 *Luzin set.*
(b) *There is a lightface* Δ^1_2 *Sierpiński set.*
(c) *There is a lightface* Δ^1_3 *Bernstein set.*
(d) *There is a lightface* Δ^1_3 *Hamel basis.*

2. Jensen's perfect set forcing revisited. In what follows, we shall mostly think of reals as elements of the Cantor space ${}^\omega 2$. We shall need a variant of the Cohen–Halpern–Lévy model.

Getting a definable Hamel basis in the absence of $\mathsf{AC}_\omega(\mathbb{R})$ forces us to indeed work with a model which is different from the original Cohen–Halpern–Lévy model. This follows from the following folklore result which we include here as a motivation. Recall [BS$^+$, Lemma 1.1] that a Hamel basis trivially produces a Vitali set.

Recall that the original Cohen–Halpern–Lévy model is produced as follows (see [PP], and also [BS$^+$, p. 3567]). Let g be $\mathbb{C}(\omega)$-generic over L (2), and let A denote the countable set of Cohen reals which g adds. Then

$$H = \mathsf{HOD}^{L[g]}_{A\cup\{A\}}. \tag{2.1}$$

The model H has a Hamel basis which in H is definable from the *set* A of Cohen reals, see the proof in [BS$^+$, Section 2]. On the other hand, H does not have a Hamel basis which in H is definable without the parameter A, as the following lemma tells us.

LEMMA 2.1. *The Cohen–Halpern–Lévy model* H *from* (2.1) *does not have a Vitali set which is definable in* H *from ordinals and reals.*

Proof. Let g be $\mathbb{C}(\omega)$-generic over L, let A denote the countable set of Cohen reals which g adds, and let H be defined as in (2.1).

Suppose the conclusion of Lemma 2.1 is wrong. By minimizing the ordinal parameters, we may fix $a \in [A]^{<\omega}$ such that in H there is a Vitali set which is definable just from a, say via the formula $\varphi(-,a)$. Let $c \in A \setminus a$, and say $n<\omega$ and $s \in {}^n 2$ are such that

$$H \models \varphi(s^\frown c{\restriction}[n,\omega), a). \tag{2.2}$$

Let $\dot c$, $\dot a$, $\dot A$, and $\dot g$ be canonical $\mathbb{C}(\omega)$-names for c, a, A, and g, respectively, so that $\dot A^g = A$, and $\dot g^g = g$, and pick $p \in g$ such that

$$p \Vdash^{\mathbb{C}(\omega)}_L \text{“}\mathsf{HOD}^{L[\dot g]}_{\dot A\cup\{\dot A\}} \models \varphi(\check s^\frown \dot c{\restriction}[\check n,\omega), \dot a).\text{”} \tag{2.3}$$

Let g^* be $\mathbb{C}(\omega)$-generic over L with $p \in g^*$ such that g^* is identical with g except that g^* incorporates a finite nontrivial variant of g only in the coordinate of $\mathbb{C}(\omega)$ which gives rise to c in such a way that $\dot c^{g^*}{\restriction}[n,\omega) \neq c{\restriction}[n,\omega)$,

(2) $\mathbb{C}(\omega)$ denotes the finite support product of ω copies of Cohen forcing.

but $\dot{c}^{g^*}$ is E_0-equivalent to c. We have $L[g^*] = L[g]$, $\mathsf{HOD}^{L[g^*]}_{\dot{A}^{g^*}\cup\{\dot{A}^{g^*}\}} = H$, and $\dot{a}^{g^*} = a$, and (2.3) then yields

$$H \models \varphi(s^\frown \dot{c}^{g^*}\restriction[n,\omega), a). \tag{2.4}$$

But (2.2) and (2.4) contradict the fact that $\varphi(-,a)$ defines a Vitali set in $L[g]$. ∎

The same argument shows that the model from [BC⁺] does not have a Vitali set which is definable from ordinal and real parameters.

In order to construct our model, we now need to introduce a variant of Jensen's subforcing of Sacks forcing [Je] (see also [KL, Definition 6.1]), which we shall call $\mathbb{P}$. The reason why we cannot work with Jensen's forcing directly is that it does not seem to have the Sacks property (see e.g. [BC⁺, Definition 2.15]).

By way of notation, if $\mathbb{Q}$ is a forcing and $N > 0$ is any ordinal, then $\mathbb{Q}(N)$ denotes the finite support product of N copies of $\mathbb{Q}$, ordered component-wise. In this paper, we shall only consider $\mathbb{Q}(N)$ for $N \le \omega$. If α is a limit ordinal, then $<_{J_\alpha}$ denotes the canonical well-ordering of J_α [Sch2, Definition 5.14 and p. 79] (3), and $<_L = \bigcup\{<_{J_\alpha} : \alpha \text{ is a limit ordinal}\}$.

Let us work in L until further notice. Let us first define a sequence $((\alpha_\xi,\beta_\xi) : \xi < \omega_1)$ of pairs of countable ordinals as follows: $\alpha_\xi =$ the least $\alpha > \sup(\{\beta_{\bar\xi} : \bar\xi < \xi\})$ such that $J_\alpha \models \mathsf{ZFC}^-$ (4), and $\beta_\xi =$ the least $\beta > \alpha_\xi$ such that $\rho_\omega(J_\beta) = \omega$ (see [Sch2, Definition 11.22]; $\rho_\omega(J_\beta) = \omega$ is equivalent to $\mathcal{P}(\omega) \cap J_{\beta+\omega} \not\subset J_\beta$).

The sequence $((\alpha_\xi,\beta_\xi) : \xi < \omega_1)$ is well-defined and $((\alpha_{\bar\xi},\beta_{\bar\xi}) : \bar\xi < \xi)$ is a hereditarily countable element of J_{β_ξ} for every $\xi < \omega_1$ by [Sch2, Lemma 11.53].

We shall also make use of the sequence $(f_\xi : \xi < \omega_1)$ which is just ${}^\omega J_{\omega_1}$, enumerated according to the order of constructibility. That is, for each $\xi < \omega_1$, $\bar f_\xi$ is the $<_L$-least f such that $f \in ({}^\omega J_{\omega_1} \cap J_{\omega_1}) \setminus \{\bar f_{\bar\xi} : \bar\xi < \xi\}$. (By acceptability [Sch2, Lemma 11.53], ${}^\omega J_{\omega_1} \cap J_{\omega_1} = {}^\omega J_{\omega_1} \cap L$.) Then if π denotes the Gödel pairing function [Sch2, p. 35], we let $f_{\pi((\xi_1,\xi_2))} = \bar f_{\xi_1}$. Each J_{α_ξ} is closed under Gödel pairing and its inverse; we have $f_\xi \in J_{\alpha_\xi}$ for all $\xi < \omega_1$, and for each $f \in {}^\omega J_{\omega_1} \cap J_{\omega_1}$ the set of ξ such that $f = f_\xi$ is cofinal in ω_1.

Let us then define $(\mathbb{P}_\xi, \mathbb{Q}_\xi : \xi \le \omega_1)$. Each $\mathbb{P}_\xi$ will consist of perfect trees $T \subset {}^{<\omega}2$ such that if $T \in \mathbb{P}_\xi$ and $s \in T$, then $T_s = \{t \in T : t \subset s \vee s \subset t\} \in \mathbb{P}_\xi$ as well (5). Each $\mathbb{P}_\xi$ will be construed as a p.o. by stipulating $T \le T'$ (T "is

(3) The reader unfamiliar with the J-hierarchy may read L_α instead of J_α.

(4) Here, ZFC^- denotes ZFC without the power set axiom. Every J_α satisfies the strong form of AC according to which every set is the surjective image of some ordinal. In the absence of $V = L$, one has to be careful about how to formulate ZFC^- [GHJ].

(5) We denote by $x \subset y$ the fact that x is a (not necessarily proper) subset of y.

stronger than" T') iff $T \subset T'$. We will have $\mathbb{P}_\xi \in J_{\alpha_\xi}$ and $\mathbb{P}_{\bar\xi} \subset \mathbb{P}_\xi$ whenever $\bar\xi \leq \xi \leq \omega_1$.

To begin, let $\mathbb{P}_0$ be the set of all basic clopen sets $U_s = \{t \in {}^{<\omega}2 \colon t \subset s \vee s \subset t\}$, where $s \in {}^{\omega}2$. If $\lambda \leq \omega_1$ is a limit ordinal, then $\mathbb{P}_\lambda = \bigcup\{\mathbb{P}_\xi \colon \xi < \lambda\}$.

Now fix $\xi < \omega_1$, and suppose that $\mathbb{P}_\xi$ has already been defined. We shall define $\mathbb{Q}_\xi$ and $\mathbb{P}_{\xi+1}$.

Let $g_\xi \in {}^{\omega}J_{\alpha_\xi}$ be the following ω-sequence. If there is some $N < \omega$ such that f_ξ is an ω-sequence of subsets of $\mathbb{P}_\xi(N)$, each of which is predense in $\mathbb{P}_\xi(N)$, then for each $n < \omega$ let $g_\xi(n)$ be the open dense set

$$\{(T_1, \dots, T_N) \in \mathbb{P}_\xi(N) \colon \exists (T'_1, \dots, T'_N) \in f_\xi(n)\ (T_1, \dots, T_N) \leq (T'_1, \dots, T'_N)\},$$

and write $N_\xi = N$. Otherwise we just set $g_\xi(n) = \mathbb{P}_\xi(1)$ for each $n < \omega$, and write $N_\xi = 1$. Let d_ξ be the $<_{J_{\beta_\xi+\omega}}$-least $d \in {}^{\omega\times\omega}(\bigcup_{N<\omega} \mathcal{P}(\mathbb{P}_\xi(N)) \cap J_{\beta_\xi}) \cap J_{\beta_\xi+\omega}$ such that

(i) for each $(n, N) \in \omega \times \omega$, $d(n, N)$ is an open dense subset of $\mathbb{P}_\xi(N)$ which exists in J_{β_ξ},
(ii) for each $N < \omega$ and each open dense subset D of $\mathbb{P}_\xi(N)$ which exists in J_{β_ξ} there is some $n < \omega$ with $d(n, N) \subset D$,
(iii) $d(n, N_\xi) \subset g_\xi(n)$ for each $n < \omega$, and
(iv) $d(n+1, N) \subset d(n, N)$ for each $(n, N) \in \omega \times \omega$.

Let us now look at the collection of all systems $(T^m_s \colon m < \omega,\ s \in {}^{<\omega}2)$ with the following properties:

(a) $T^m_s \in \mathbb{P}_\xi$ for all m, s,
(b) for each $T \in \mathbb{P}_\xi$ there are infinitely many $m < \omega$ with $T^m_\emptyset = T$,
(c) $T^m_t \leq T^m_s$ for all m, $t \supset s$,
(d) $\mathrm{stem}(T^m_{s^\frown 0})$ and $\mathrm{stem}(T^m_{s^\frown 1})$ are incompatible elements of T^m_s for all m, s,
(e) if $(m, s) \neq (m', s')$, where $m, m' < n$ and $\mathrm{lh}(s) = \mathrm{lh}(s') = n + 1$ for some n, then $\mathrm{stem}(T^m_s)$ and $\mathrm{stem}(T^{m'}_{s'})$ are incompatible, and
(f) for all $N \leq n < \omega$ and all pairwise different $(m_1, s_1), \dots, (m_N, s_N)$ with $m_1, \dots, m_N < n$ and $s_1, \dots, s_N \in {}^{n+1}2$,
$$(T^{m_1}_{s_1}, \dots, T^{m_N}_{s_N}) \in d_\xi(n, N).$$

It is easy to work in $J_{\beta_\xi+\omega}$ and construct initial segments $(T^m_s \colon m < \omega,\ s \in {}^{<\omega}2,\ \mathrm{lh}(s) \leq n)$ of such a system by induction on $n < \omega$. Notice that (f) formulates a constraint only for $m_1, \dots, m_N < \mathrm{lh}(s_1) - 1 = \dots = \mathrm{lh}(s_N) - 1$, and writing $n = \mathrm{lh}(s_1) - 1$, there are $\sum_{N=1}^{n} \frac{(n\cdot 2^{n+1})!}{(n\cdot 2^{n+1}-N)!}$ (i.e., finitely many) such constraints.

We let $(T^m_{s,\xi} \colon m<\omega,\ s\in {}^{<\omega}2)$ be the $<_{\beta_\xi+\omega}$-least such system $(T^m_s \colon m<\omega,\ s \in {}^{<\omega}2)$. For every $m < \omega$, $s \in {}^{<\omega}2$, we let

$$A^m_{s,\xi} = \bigcap_{n \geq \mathrm{lh}(s)} \bigcup_{\substack{t \supset s \\ \mathrm{lh}(t)=n}} T^m_{t,\xi} = \{\mathrm{stem}(T^m_{t,\xi}) \restriction k \colon t \supset s,\ k < \omega\}.$$

Of course, $A^m_{s,\xi} \le T^m_{s,\xi}$. Notice that (e) implies that

(2.5) $\quad A^m_{s,\xi} \cap A^{m'}_{s',\xi}$ is finite, unless $m = m'$ and $s \subset s'$ or $s' \subset s$.

(2.5) will imply that $A^m_{s,\xi}$ and $A^{m'}_{s',\xi}$ will be incompatible in every $\mathbb{P}_\eta$, $\eta > \xi$, unless $m = m'$ and $s \subset s'$ or $s' \subset s$. If $s \subset s'$, then $A^m_{s',\xi} = (A^m_{s,\xi})_{s'}$.

We set $\mathbb{Q}_\xi = \{A^m_{s,\xi} \colon m < \omega,\ s \in {}^{<\omega}2\}$. Finally, we set $\mathbb{P}_{\xi+1} = \mathbb{P}_\xi \cup \mathbb{Q}_\xi$.

If $T \in \mathbb{Q}_\xi$ and $s \in T$, then $T_s \in \mathbb{Q}_\xi$, so that inductively if $T \in \mathbb{P}_\xi$ and $s \in T$, then $T_s \in \mathbb{P}_\xi$ for all $\xi \le \omega_1$.

This defines $(\mathbb{P}_\xi, \mathbb{Q}_\xi \colon \xi \le \omega_1)$. We shall also write $\mathbb{P} = \mathbb{P}_{\omega_1}$. Let us now work towards showing the Sealing Lemma 2.3 and that for each $N < \omega$, $\mathbb{P}(N)$ has the c.c.c. (see Lemma 2.4).

Lemma 2.2. *Let $N < \omega$ and $\xi < \omega_1$. Then*

$$D = \{(T_1, \dots, T_N) \in \mathbb{Q}_\xi(N) \colon \operatorname{stem}(T_i) \perp \operatorname{stem}(T_j) \text{ for } i \ne j\}$$

is dense in $\mathbb{P}_{\xi+1}(N)$ [6].

Proof. Let $(T_1, \dots, T_N) \in \mathbb{P}_{\xi+1}(N)$. For $i \in \{1, \dots, N\}$ such that $T_i \in \mathbb{P}_\xi$ pick some $m_i < \omega$ such that $T_i = T^{m_i}_{\emptyset,\xi}$, and write $s_i = \emptyset$. This is possible by (b). If $i \in \{1, \dots, N\}$ is such that $T_i \in \mathbb{Q}_\xi$, then say $T_i = A^{m_i}_{s_i,\xi}$. Now pick $n > \max(\{m_1, \dots, m_N\})$ and $t_1 \supset s_1, \dots, t_N \supset s_N$ such that $\operatorname{lh}(t_1) = \dots = \operatorname{lh}(t_N) = n + 1$ and the (m_i, t_i) are pairwise different.

Then by (e) the finite sequences $\operatorname{stem}(T^{m_i}_{t_i,\xi})$ are pairwise incompatible, so that by $A^{m_i}_{t_i,\xi} \le T^{m_i}_{t_i,\xi}$, the $A^{m_i}_{t_i,\xi}$ are pairwise incompatible. But then $(A^{m_1}_{t_i,\xi}, \dots, A^{m_N}_{t_N,\xi}) \in D$ and $(A^{m_1}_{t_i,\xi}, \dots, A^{m_N}_{t_N,\xi}) \le (T_1, \dots, T_N)$. ■

Lemma 2.3 (Sealing). *Let $N < \omega$ and $\xi < \omega_1$. If $D \in J_{\beta_\xi}$ is predense in $\mathbb{P}_\xi(N)$, then D is predense in all $\mathbb{P}_\eta(N)$, for all $\eta \ge \xi$ and $\eta \le \omega_1$.*

Proof (by induction on η). The cases $\eta = \xi$ and η being a limit ordinal are trivial. Suppose $\eta \ge \xi$, $\eta < \omega_1$, and D is predense in $\mathbb{P}_\eta(N)$. Write $D' = \{(T_1, \dots, T_N) \in \mathbb{P}_\eta(N) \colon \exists (T'_1, \dots, T'_N) \in D\, (T_1, \dots, T_N) \le (T'_1, \dots, T'_N)\}$. As $\beta_\xi \le \beta_\eta$, we have $D' \in J_{\beta_\eta}$, and by (ii) and (iv) there is some $n_0 < \omega$ with $d_\eta(n, N) \subset D'$ for every $n > n_0$.

To show that D' (and hence D) is predense in $\mathbb{P}_{\eta+1}(N)$, by Lemma 2.2 it suffices to show that for all $(T_1, \dots, T_N) \in \mathbb{Q}_\eta(N)$ there is some $(T'_1, \dots, T'_N) \in \mathbb{Q}_\eta(N)$ such that $(T'_1, \dots, T'_N) \le (T_1, \dots, T_N)$, and $(T'_1, \dots, T'_N)$ is below some element of D'.

So let $(A^{m_1}_{s_1,\eta}, \dots, A^{m_N}_{s_N,\eta}) \in \mathbb{Q}_\eta(N)$ be arbitrary. Let

$$n > \max(\{n_0, N-1, m_1, \dots, m_N, \operatorname{lh}(s_1), \dots, \operatorname{lh}(s_N)\}),$$

and let $t_1 \supset s_1, \dots, t_N \supset s_N$ be such that $\operatorname{lh}(t_1) = \dots = \operatorname{lh}(t_N) = n + 1$. By increasing n further if necessary, we may certainly assume that $t_1, \dots, t_N$

[6] Here, $\operatorname{stem}(T_i) \perp \operatorname{stem}(T_j)$ means that the two stems are incompatible.

are picked in such a way that $(m_1, t_1), \ldots, (m_N, t_N)$ are pairwise different. Then

$$(T^{m_1}_{t_1,\eta}, \ldots, T^{m_N}_{t_N,\eta}) \in d_\eta(n, N) \subset D'$$

by (f). But

$$(A^{m_1}_{t_1,\eta}, \ldots, A^{m_N}_{t_N,\eta}) \leq (T^{m_1}_{t_1,\eta}, \ldots, T^{m_N}_{t_N,\eta}),$$

and also

$$(A^{m_1}_{t_1,\eta}, \ldots, A^{m_N}_{t_N,\eta}) \leq (A^{m_1}_{s_1,\eta}, \ldots, A^{m_N}_{s_N,\eta}),$$

which means that $(A^{m_1}_{s_1,\eta}, \ldots, A^{m_N}_{s_N,\eta})$ is compatible with an element of D'. ∎

Corollary 2.1. *Let $N < \omega$ and $\xi < \omega_1$. Then*

$$\{(T_1, \ldots, T_N) \in \mathbb{Q}_\xi(N) \colon \operatorname{stem}(T_i) \perp \operatorname{stem}(T_j) \textit{ for } i \neq j\}$$

is predense in $\mathbb{P}(N)$.

Lemma 2.4. *Let $N < \omega$. Then $\mathbb{P}(N)$ has the c.c.c.*

Proof. Let $A \subset \mathbb{P}(N)$ be a maximal antichain, $A \in L$. Let $j \colon J_\beta \to J_{\omega_2}$ be elementary and such that $\beta < \omega_1$ and $\{\mathbb{P}, A\} \subset \operatorname{ran}(j)$. Write $\xi = \operatorname{crit}(j)$. We have $j^{-1}(\mathbb{P}(N)) = \mathbb{P}(N) \cap J_\xi = \mathbb{P}_\xi(N)$, and $j^{-1}(A) = A \cap J_\xi = A \cap \mathbb{P}_\xi(N) \in J_\beta$ is a maximal antichain in $\mathbb{P}_\xi(N)$. Moreover, $\beta_\xi > \beta$, so that by Lemma 2.1, $A \cap \mathbb{P}_\xi(N)$ is predense in $\mathbb{P}(N)$. This means that $A = A \cap \mathbb{P}_\xi$ is countable. ∎

Lemma 2.5. *Let $N < \omega$. Then $(c_1, \ldots, c_N) \in {}^N({}^\omega 2)$ is $\mathbb{P}(N)$-generic over L iff for all $\xi < \omega_1$ there is an injection $t \colon \{1, \ldots, N\} \to \mathbb{Q}_\xi$ such that $c_i \in [t(i)]$ for all $i \in \{1, \ldots, N\}$.*

Proof. "⇒": This readily follows from Corollary 2.1.

"⇐": Let $A \subset \mathbb{P}(N)$ be a maximal antichain, $A \in L$. By Lemma 2.4, we may certainly pick some $\xi < \omega_1$ with $A \subset \mathbb{P}_\xi(N)$ and $A \in J_{\alpha_\xi}$. Say n_0 is such that $d_\xi(n, N) \subset \{(T_1, \ldots, T_N) \in \mathbb{P}_\xi \colon \exists (T'_1, \ldots, T'_N) \in A \ (T_1, \ldots, T_N) \leq (T'_1, \ldots, T'_N)\}$ for all $n \geq n_0$. By our hypothesis, we may pick pairwise different $(m_1, s_1), \ldots, (m_N, s_N)$ with $\operatorname{lh}(s_1) = \cdots = \operatorname{lh}(s_N) = n + 1$ for some $n \geq n_0$ and $c_i \in [T^{m_i}_{s_i}, \xi]$ for all $i \in \{1, \ldots, N\}$. But then $(T^{m_1}_{s_i,\xi}, \ldots, T^{m_N}_{s_N})$ is below an element of A, which means that the generic filter given by $(c_1, \ldots, c_N)$ meets A. ∎

Corollary 2.2. *Let $N < \omega$, and let $(c_1, \ldots, c_N) \in {}^N({}^\omega 2)$ be $\mathbb{P}(N)$-generic over L. If $x \in L[(c_1, \ldots, c_N)]$ is $\mathbb{P}$-generic over L, then $x \in \{c_1, \ldots, c_N\}$.*

Proof. Let $x \in L[(c_1, \ldots, c_N)]$ be $\mathbb{P}$-generic over L with $x \notin \{c_1, \ldots, c_N\}$. By Lemma 2.5 "⇒", for each $\xi < \omega_1$ there is then an injection $t_\xi \colon \{1, \ldots, N\} \to \mathbb{Q}_\xi$ such that $c_i \in [t_\xi(i)]$ for all $\xi < \omega_1$ and $i \in \{1, \ldots, N\}$. Again by Lemma 2.5 "⇒", for each $\xi < \omega_1$ there is some $t(\xi) \in \mathbb{Q}_\xi$ such that $x \in [t(\xi)]$. As $T \in \mathbb{Q}_\xi$ and $s \in T$ implies $T_s \in \mathbb{Q}_\xi$, we may in fact assume that for each $\xi < \omega_1$, $t(\xi) \notin \{t_\xi(1), \ldots, t_\xi(N)\}$.

But then $(c_1, \dots, c_N, x) \in {}^{N+1}({}^\omega 2)$ is $\mathbb{P}(N+1)$-generic over L by Lemma 2.5 "$\Leftarrow$". In particular, x is $\mathbb{P}$-generic over $L[c_1, \dots, c_N]$ and hence $x \notin L[(c_1, \dots, c_N)]$, a contradiction. ∎

Corollary 2.3. *Let $N < \omega$, and let $(c_1, \dots, c_N) \in {}^N({}^\omega 2)$ be $\mathbb{P}(N)$-generic over L. Then inside $L[(c_1, \dots, c_N)]$, $\{c_1, \dots, c_N\}$ is a lightface Π^1_2 set.*

Proof. Let $\varphi(x)$ express that for all $\xi < \omega_1$ there is some $T \in \mathbb{Q}_\xi$ such that $x \in [T]$. The formula $\varphi(x)$ may be written in a lightface Π^1_2 fashion, and it defines $\{c_1, \dots, c_N\}$ inside $L[(c_1, \dots, c_N)]$. ∎

Lemma 2.6 (Sacks property). *Let $N < \omega$, and let g be $\mathbb{P}(N)$-generic over L. For each $f \colon \omega \to \omega$ with $f \in L[g]$, there is some $h \in L$ with domain ω such that for each $n < \omega$, $f(n) \in h(n)$ and* (7) $\mathrm{Card}(h(n)) \le 2^{(n+1)^2}$.

Proof. Let $\tau \in L^{\mathbb{P}(N)}$ with $\tau^g = f$. Let $(A_n \colon n < \omega) \in L$ be such that for each n, A_n is a maximal antichain of $\vec{T} \in \mathbb{P}(N)$ such that for some $m < \omega$, $\vec{T} \Vdash \tau(\check{n}) = \check{m}$. We may pick some $\xi < \omega_1$ such that $\bigcup\{A_n \colon n < \omega\} \subset \mathbb{P}_\xi(N)$ and $(A_n \colon n < \omega) = f_\xi$.

By Lemma 2.5, there are pairwise different $(m_1, s_1), \dots, (m_N, s_N)$ such that

$$(A^{m_1}_{s_1,\xi}, \dots, A^{m_N}_{s_N,\xi}) \in g.$$

Let

$$n > \max(\{N-1, m_1, \dots, m_N, \mathrm{lh}(s_1), \dots, \mathrm{lh}(s_N)\}).$$

If $t_1 \supset s_1, \dots, t_N \supset t_N$ are such that $\mathrm{lh}(t_1) = \dots = \mathrm{lh}(t_N) = n+1$, then $(T^{m_1}_{t_1,\xi}, \dots, T^{m_N}_{t_N,\xi}) \in d_\xi(n, N) \subset A_n$, so that also

$$\exists m < \omega\ (T^{m_1}_{t_1,\xi}, \dots, T^{m_N}_{t_N,\xi}) \Vdash \tau(\check{n}) = \check{m}.$$

Therefore, if we let

$$h(n) = \{m < \omega \colon \exists t_1 \supset s_1, \dots, \exists t_N \supset t_N\, (\mathrm{lh}(t_1) = \dots = \mathrm{lh}(t_N) = n+1 \wedge \\ (T^{m_1}_{t_1,\xi}, \dots, T^{m_N}_{t_N,\xi}) \Vdash \tau(\check{n}) = \check{m})\},$$

then $(A^{m_1}_{s_1,\xi}, \dots, A^{m_N}_{s_N,\xi}) \Vdash \tau(\check{n}) \in (h(n))^\vee$, hence $f(n) \in h(n)$, and $\mathrm{Card}(h(n)) = (2^{n+1})^N \le 2^{(n+1)^2}$ for all but finitely many n. ∎

3. The variant of the Cohen–Helpern–Lévy model. Let us force with $\mathbb{P}(\omega)$ over L, and let g be a generic filter. Let c_n, $n < \omega$, denote the Jensen reals which g adds. Let $A = \{c_n \colon n < \omega\}$. The model

$$H = H(L) = \mathsf{HOD}^{L[g]}_{A \cup \{A\}} \tag{3.1}$$

(7) In what follows, the only thing that will matter is that the bound on $\mathrm{Card}(h(n))$ only depends on n and not on the particular f.

of all sets which inside $L[g]$ are hereditarily definable from parameters in $\mathrm{OR} \cup A \cup \{A\}$ is the variant of the Cohen–Halpern–Lévy model (over L) we shall work with. For the case of Jensen's original forcing this model was first considered in [E].

For any finite $a \subset A$, we write $L[a]$ for the model constructed from the finitely many reals in a.

LEMMA 3.1. *Inside* H, A *is a* (*lightface*) Π^1_2 *set.*

Proof. Let $\varphi(-)$ be the Π^1_2 formula from the proof of Lemma 2.3. If $H \models \varphi(x)$, $x \in L[a]$, $a \in [A]^{<\omega}$, then $L[a] \models \varphi(x)$ by Shoenfield, so $x \in a \subset A$. On the other hand, if $c \in A$, then $L[c] \models \varphi(c)$ and hence $H \models \varphi(c)$ again by Shoenfield. ■

If we fix some Gödelization of formulae (or some enumeration of all the rud functions, resp.) at the outset, each $L[a]$, $a \in [A]^{<\omega}$, comes with a unique canonical global well-ordering $<_a$, by which we mean the one which is induced by the *natural* order of the elements of a and the fixed Gödelization device in the usual fashion. The assignment $a \mapsto <_a$, $a \in [A]^{<\omega}$, is hence in H ([8]). This is a crucial fact.

Let us fix a bijection

$$e\colon \omega \to \omega \times \omega, \tag{3.2}$$

and let us write $((n)_0, (n)_1) = e(n)$.

We shall also make use the following (cf. [BS$^+$, Lemma 1.2]).

LEMMA 3.2.

(1) *Let* $a \in [A]^{<\omega}$ *and* $X \subset L[a]$, $X \in H$, *say* $X \in \mathsf{HOD}^{L[g]}_{b \cup \{A\}}$, *where* $b \supseteq a$, $b \in [A]^{<\omega}$. *Then* $X \in L[b]$.
(2) *There is no well-ordering of the reals in* H.
(3) A *has no countable subset in* H.
(4) $[A]^{<\omega}$ *has no countable subset in* H.

Proof sketch. (1) Every permutation $\pi\colon \omega \to \omega$ induces an automorphism e_π of $\mathbb{P}(\omega)$ by sending p to q, where $q(\pi(n)) = p(n)$ for all $n < \omega$. It is clear that no (automorphism of names induced by) e_π moves the canonical name for A, call it $\dot{A}$. Let us also write $\dot{c}_n$ for the canonical name for c_n, $n < \omega$. Now if a and b are as in the statement of (1), say $b = \{c_{n_1}, \ldots, c_{n_k}\}$, if $p, q \in \mathbb{P}(\omega)$, if $\pi \restriction \{n_1, \ldots, n_k\} = \mathrm{id}$, $p \restriction \{n_1, \ldots, n_k\}$ is compatible with $q \restriction \{n_1, \ldots, n_k\}$, and $\mathrm{supp}(\pi(p)) \cap \mathrm{supp}(q) \subseteq \{n_1, \ldots, n_k\}$, if $x \in L$, if $\alpha_1, \ldots, \alpha_m$ are ordinals,

---

([8]) More precisely, the ternary relation consisting of all (a, x, y) such that $x <_a y$ is definable over H.

and if φ is a formula, then

$$p \Vdash_L^{\mathbb{P}(\omega)} \varphi(\check{x}, \check{\alpha}_1, \ldots, \check{\alpha}_m, \dot{c}_{n_1}, \ldots, \dot{c}_{n_k}, \dot{A}) \iff e_{\pi(p)} \Vdash_L^{\mathbb{P}(\omega)} \varphi(\check{x}, \check{\alpha}_1, \ldots, \check{\alpha}_m, \dot{c}_{n_1}, \ldots, \dot{c}_{n_k}, \dot{A})$$

and $e_\pi(p)$ is compatible with q, so that the statement $\varphi(\check{x}, \check{\alpha}_1, \ldots, \check{\alpha}_m, \dot{c}_{n_1}, \ldots, \dot{c}_{n_k}, \dot{A})$ will be decided by conditions $p \in \mathbb{P}(\omega)$ with $\operatorname{supp}(p) \subseteq \{n_1, \ldots, n_k\}$. But every set in $L[b]$ is coded by a set of ordinals, so if X is as in (1), this shows that $X \in L[b]$.

(2) Every real is a subset of L. Hence by (1), if $L[g]$ had a well-ordering of the reals in $\mathsf{HOD}^{L[g]}_{a \cup \{A\}}$, some $a \in [A]^{<\omega}$, then every real of H would be in $L[a]$, which is nonsense.

(3) Assume that $f\colon \omega \to A$ is injective, $f \in H$. Let $x \in {}^\omega\omega$ be defined by $x(n) = f((n)_0)((n)_1)$, so that $x \in H$. By (1), $x \in L[a]$ for some $a \in [A]^{<\omega}$. But then $\operatorname{ran}(f) \subset L[a]$, which is nonsense, as there is some $n < \omega$ such that $c_n \in \operatorname{ran}(f) \setminus a$.

(4) This readily follows from (3). $\blacksquare_{\text{Lemma 3.2}}$

Let us recall another standard fact.

$$\text{(3.3)} \qquad \text{If } a, b \in [A]^{<\omega}, \text{ then } L[a] \cap L[b] = L[a \cap b].$$

To see this, let us assume without loss of generality that $a \setminus b \neq \emptyset \neq b \setminus a$, and say $a \setminus b = \{c_n \colon n \in I\}$ and $b \setminus a = \{c_n \colon n \in J\}$, where I and J are non-empty disjoint finite subsets of ω. Then $a \setminus b$ and $b \setminus a$ are mutually $\mathbb{P}(I)$- and $\mathbb{P}(J)$-generic over $L[a \cap b]$. But then $L[a] \cap L[b] = L[a \cap b][a \setminus b] \cap L[a \cap b][b \setminus a] = L[a \cap b]$ (cf. [Sch2, Problem 6.12]).

For any $a \in [A]^{<\omega}$, we write $\mathbb{R}_a = \mathbb{R} \cap L[a]$ and $\mathbb{R}_a^+ = \mathbb{R}_a \setminus \bigcup\{\mathbb{R}_b \colon b \subsetneq a\}$. Then $(\mathbb{R}_a^+ \colon a \in [A]^{<\omega})$ is a partition of $\mathbb{R}$: by Lemma 3.2(1),

$$\text{(3.4)} \qquad \mathbb{R} \cap H = \bigcup\{\mathbb{R}_a^+ \colon a \in [A]^{<\omega}\},$$

and $\mathbb{R}_a \cap \mathbb{R}_b = \mathbb{R}_{a \cap b}$ by (3.3), so that

$$\text{(3.5)} \qquad \mathbb{R}_a^+ \cap \mathbb{R}_b^+ = \emptyset \quad \text{for distinct } a, b \in [A]^{<\omega}.$$

For $x \in \mathbb{R}$, we shall also write $a(x)$ for the unique $a \in [A]^{<\omega}$ such that $x \in \mathbb{R}_a^+$, and we shall write $\#(x) = \operatorname{Card}(a(x))$.

Adrian Mathias showed that in the original Cohen–Halpern–Lévy model there is a definable function which assigns to each x an ordering $<_x$ such that $<_x$ is a well-ordering iff x can be well-ordered (cf. [Ma, p. 182]). The following is a special simple case of this, adapted to the current model H.

Lemma 3.3 (A. Mathias). *In H, the union of countably many countable sets of reals is countable.*

Proof. Let us work inside H. Let $(A_n \colon n < \omega)$ be such that for each $n < \omega$, $A_n \subset \mathbb{R}$ and there exists some surjection $f\colon \omega \to A_n$. For each such pair n, f let $y_{n,f} \in {}^\omega\omega$ be such that $y_{n,f}(m) = f((m)_0)((m)_1)$. If $a \in [A]^{<\omega}$ and $y_{n,f} \in \mathbb{R}_a$, then $A_n \in L[a]$. By (3.3), for each n there is a unique $a_n \in [A]^{<\omega}$ such that $A_n \in L[a_n]$ and $b \supset a_n$ for each $b \in [A]^{<\omega}$ such that $A_n \in L[b]$. Notice that A_n is also countable in $L[a_n]$.

Using the function $n \mapsto a_n$, an easy recursion yields a surjection $g\colon \omega \to \bigcup\{a_n : n < \omega\}$: first enumerate the finitely many elements of a_0 according to their natural order, then enumerate the finitely many elements of a_1 according to their natural order, etc. As A has no countable subset, $\bigcup\{a_n : n < \omega\}$ must be finite, say $a = \bigcup\{a_n : n < \omega\} \in [A]^{<\omega}$. But then $\{A_n : n < \omega\} \subset L[a]$. (We do not claim $(A_n : n < \omega) \in L[a]$.)

For each $n < \omega$, we may now let f_n the $<_a$-least surjection $f\colon \omega \to A_n$. Then $f(n) = f_{(n)_0}((n)_1)$ for $n < \omega$ defines a surjection from ω onto $\bigcup\{A_n : n < \omega\}$, as desired. ■Lemma 3.3

The following is not true in the original Cohen–Halpern–Lévy model. Its proof exploits the Sacks property of Lemma 2.6 (cf. [JMS, Theorem 11]) and [GQ, Lemma 31]. We give a full proof here for the reader's convenience.

LEMMA 3.4.

(1) *Let $M \in H$ be a null set in H. Then there is a G_δ null set M' with $M' \supset M$ whose code is in L.*
(2) *Let $M \in H$ be a meager set in H. Then there is an F_σ meager set M' with $M' \supset M$ whose code is in L.*

Proof. (1) Let $M \in H$ be a null set in H.

Let us work in H. Let $(\epsilon_n : n < \omega)$ be any sequence of positive reals. Let $\bigcup_{s\in X} U_s \supset M$, where $X \subset {}^{<\omega}2$ and $\mu(\bigcup\{U_s : s \in X\}) \le \epsilon_0$ (9). Let $e\colon \omega \to X$ be onto. Let $(k_n : n < \omega)$ be defined by $k_n =$ the smallest k (strictly greater than k_{n-1} if $n > 0$) such that $\mu(\bigcup\{U_s : s \in e"\omega \setminus k\}) \le \epsilon_n$. Write $k_{-1} = 0$. Then $\mu(\bigcup\{U_s : s \in e"[k_{n-1}, k_n)\}) \le \epsilon_n$ for every $n < \omega$.

Now fix $\epsilon > 0$. Let

$$\epsilon_n = \frac{\epsilon}{2^{n^2+3n+2}},$$

and let $(k_n : n < \omega)$ and $e\colon \omega \to {}^{<\omega}2$ be such that $\bigcup_{s\in X} U_s \supset M$ and $\mu(\bigcup\{U_s : s \in e"[k_{n-1}, k_n)\}) \le \epsilon_n$ for every $n < \omega$. By Lemma 3.2(1), we may now apply Lemma 2.6 inside $L[a]$ for some $a \in [A]^{<\omega}$ such that $\{e, (k_n : n<\omega)\} \subset L[a]$ and find a function $h \in L$ with domain ω such that for each $n < \omega$, $h(n)$ is a finite union U_n of basic open sets such that $\{U_s : s \in e"[k_{n-1}, k_n)\} \subset U_n$ and $\mu(U_n) \le \epsilon/2^{n+1}$. But then $\mathcal{O} = \bigcup\{O_n : n < \omega\} \supset M$ is open, $\mathcal{O}$ is coded in L (i.e., there is $Y \in L$, $Y \subset {}^{<\omega}2$, with $\mathcal{O} = \bigcup\{U_s : s \in Y\}$), and $\mu(\mathcal{O}) \le \epsilon$.

(9) Here, μ denotes Lebesgue measure.

Hence for every $n < \omega$ we may let $\mathcal{O}_n$ be the unique open set with $\mathcal{O}_n \supset M$, $\mu(\mathcal{O}_n) \le \frac{1}{n+1}$, which has a code in L, and whose code in L is $<_L$-least among all the codes giving such a set. Then $\bigcap\{\mathcal{O}_n : n < \omega\}$ is a G_δ null set with code in L and which covers M.

(2) Let $M \in H$ be a meager set in H, say $M = \bigcup\{N_n : n < \omega\}$, where each N_n is nowhere dense.

Let us again work in H. It is easy to verify that a set $P \subset {}^\omega 2$ is nowhere dense iff there is some $z \in {}^\omega 2$ and some strictly increasing $(k_n : n < \omega)$ such that for all $n < \omega$,

$$\{x \in {}^\omega 2 : x\restriction[k_n, k_{n+1}) = z\restriction[k_n, k_{n+1})\} \cap P = \emptyset. \tag{3.6}$$

Look at $f : \omega \to \omega$, where $f(m) = k_{n+1}$ for the least n with $m \le k_n$. By Lemma 3.2(1), we may first apply Lemma 2.6 inside $L[a]$ for some $a \in [A]^{<\omega}$ such that $f \in L[a]$ and get a strictly increasing function $h : \omega \to \omega$, $h \in L$, such that $h(m) \ge f(m)$ for all $m < \omega$. Write $\ell_0 = 0$ and $\ell_{n+1} = h(\ell_n)$, so that for each n there is some n' with

$$\ell_n \le k_{n'} < k_{n'+1} \le \ell_{n+1}, \tag{3.7}$$

so that

$$\{x \in {}^\omega 2 : x\restriction[l_n, l_{n+1}) = z\restriction[k_n, k_{n+1})\} \cap P = \emptyset. \tag{3.8}$$

Define $e : \omega \to \omega$ by $e(n) = \sum_{q=0}^{n} 2^{(q+1)^2}$. By Lemma 3.2(1), we may apply Lemma 2.6 once more, this time inside $L[a]$ for some $a \in [A]^{<\omega}$ such that $z \in L[a]$ and get some $n \mapsto (z_i^n : i \le 2^{(n+1)^2})$ inside L such that for all n and i, $z_i^n : e(n) \to 2$, and for all n there is some i with $z\restriction e(n) = z_i^n$. From this we get some $z' : \omega \to \omega$, $z' \in L$, such that for all $n > 0$ there is some n' with $e(n-1) \le n' < e(n)$ and $z'\restriction[\ell_{n'}, \ell_{n'+1}) = z\restriction[\ell_{n'}, \ell_{n'+1})$. But then, writing

$$D = \{x \in {}^\omega 2 : \exists n\ x\restriction[\ell_{e(n)}, \ell_{e(n+1)}) = z'\restriction[\ell_{e(n)}, \ell_{e(n+1)})\}, \tag{3.9}$$

we find that D is coded by $(\ell_n : n < \omega)$ and z' which are both in L; D is open and dense; and $D \cap P = \emptyset$.

Hence for every $n < \omega$ we may let $\mathcal{O}_n$ be the unique open dense set with $\mathcal{O}_n \cap N_n = \emptyset$ which has a code in L, and whose code in L is $<_L$-least among all the codes giving such a set. Then $\bigcup\{{}^\omega 2 \setminus \mathcal{O}_n : n < \omega\}$ is an F_σ meager set with code in L and which covers M. ■

Corollary 3.1. *In H, there is a lightface Δ^1_2 Sierpiński set as well as a lightface Δ^1_2 Luzin set.*

Proof. There is a lightface Δ^1_2 Luzin set in L. By Lemma 3.4(2), any such set is still a Luzin set in H. The same is true with "Luzin" replaced by "Sierpiński" and Lemma 3.4(2) replaced by Lemma 3.4(1). ■

Lemma 3.5. *In H, there is a lightface Δ^1_3 Bernstein set.*

Proof. In this proof, let us think of reals as elements of the Cantor space ${}^\omega 2$. Let us work in H.

Recalling that $\#(x) = \mathrm{Card}(a(x))$, we let

$$B = \{x \in \mathbb{R} \colon \exists \text{ even } n\ (2^n < \#(x) \le 2^{n+1})\},$$
$$B' = \{x \in \mathbb{R} \colon \exists \text{ odd } n\ (2^n < \#(x) \le 2^{n+1})\}.$$

Obviously, $B \cap B' = \emptyset$.

Let $P \subset \mathbb{R}$ be perfect. We aim to see that $P \cap B \ne \emptyset \ne P \cap B'$.

Say $P = [T] = \{x \in {}^\omega 2 \colon \forall n\ x{\restriction}n \in T\}$, where $T \subseteq {}^{<\omega}2$ is a perfect tree. Modulo some fixed natural bijection ${}^{<\omega}2 \leftrightarrow \omega$, we may identify T with a real. By (3.4), we may pick some $a \in [A]^{<\omega}$ such that $T \in L[a]$. Say $\mathrm{Card}(a) < 2^n$, where n is even.

Let $b \in [A]^{2^{n+1}}$, $b \supset a$, and let $x \in \mathbb{R}^+_b$. In particular, $\#(x) = 2^{n+1}$. It is easy to work in $L[b]$ and construct some $z \in [T]$ such that $x \le_T z \oplus T$, ([10]) e.g., arrange that if $z{\restriction}m$ is the kth splitting node of T along z, where $k \le m < \omega$, then $z(m) = 0$ if $x(k) = 0$ and $z(m) = 1$ if $x(k) = 1$.

If we had $\#(z) \le 2^n$, then $\#(z \oplus T) \le \#(z) + \#(T) < 2^n + 2^n = 2^{n+1}$, so that $\#(x) < 2^{n+1}$ since $x \le_T z \oplus T$, a contradiction. Hence $\#(z) > 2^n$. By $z \in L[b]$, $\#(z) \le 2^{n+1}$. Therefore, $z \in P \cap B$.

The same argument shows that $P \cap B' \ne \emptyset$. Thus B (and also B') is a Bernstein set.

We infer that $x \in B$ iff

$$\exists a \in [A]^{<\omega}\ \exists \text{ even } n\ \exists J_\alpha[a]$$
$$(x \in J_\alpha[a] \wedge 2^n < \mathrm{Card}(a) \le 2^{n+1} \wedge \forall b \subsetneq a\ \forall J_\beta[b]\ x \notin J_\beta[b]),$$

which is true iff

$$\forall a \in [A]^{<\omega}\ \forall J_\alpha[a]\ (x \in J_\alpha[a] \wedge \forall b \subsetneq a'\ \forall J_\beta[b]\ x \notin J_\beta[b]$$
$$\rightarrow \exists \text{ even } n\ 2^n < \mathrm{Card}(a) \le 2^{n+1}).$$

By Lemma 3.1, this shows that B is lightface Δ^1_3. ■

Recall that for any $a \in [A]^{<\omega}$, we write $\mathbb{R}_a = \mathbb{R} \cap L[a]$. Let us now also write $\mathbb{R}_{<a} = \mathrm{span}(\bigcup\{\mathbb{R}_b \colon b \subsetneq a\})$, and $\mathbb{R}^*_a = \mathbb{R}_a \setminus \mathbb{R}_{<a}$. In particular, $\mathbb{R}_{<\emptyset} = \{0\}$ by our above convention that $\mathrm{span}(\emptyset) = \{0\}$, and $\mathbb{R}^*_\emptyset = (\mathbb{R} \cap L) \setminus \{0\}$.

The proof of Claim 3.2 below will show that

$$\mathbb{R} \cap H = \mathrm{span}\Big(\bigcup\{\mathbb{R}^*_a \colon a \in [A]^{<\omega}\}\Big). \tag{3.10}$$

Also, we have $\mathbb{R}^*_a \subset \mathbb{R}^+_a$, so that by (3.5),

$$\mathbb{R}^*_a \cap \mathbb{R}^*_b = \emptyset \quad \text{for distinct } a, b \in [A]^{<\omega}. \tag{3.11}$$

([10]) Here, $(x \oplus y)(2n) = x(n)$ and $(x \oplus y)(2n+1) = y(n)$, $n < \omega$.

Lemma 3.6. *In H, there is a Δ^1_3 Hamel basis.*

Proof. We call $X \subset \mathbb{R}^*_a$ *linearly independent over* $\mathbb{R}_{<a}$ iff whenever

$$\sum_{n=1}^{m} q_n \cdot x_n \in \mathbb{R}_{<a},$$

where $m \in \mathbb{N}$, $m \geq 1$, and $q_n \in \mathbb{Q}$ and $x_n \in X$ for all n, $1 \leq n \leq m$, then $q_1 = \cdots = q_m = 0$. In other words, $X \subset \mathbb{R}^*_a$ is linearly independent over $\mathbb{R}_{<a}$ iff

$$\operatorname{span}(X) \cap \mathbb{R}_{<a} = \{0\}.$$

We call $X \subset \mathbb{R}^*_a$ *maximal linearly independent over* $\mathbb{R}_{<a}$ iff X is linearly independent over $\mathbb{R}_{<a}$ and no $Y \supsetneq X$, $Y \subset \mathbb{R}^*_a$ remains linearly independent over $\mathbb{R}_{<a}$. In particular, $X \subset \mathbb{R}^*_\emptyset = (\mathbb{R} \cap L) \setminus \{0\}$ is linearly independent over $\mathbb{R}_{<\emptyset} = \{0\}$ iff X is a Hamel basis for $\mathbb{R} \cap L$.

For any $a \in [A]^{<\omega}$, we let $b_a = \{x^a_i \colon i < \theta^a\}$, for some $\theta^a \leq \omega_1$, be the unique set such that

(i) for each $i < \theta^a$, x^a_i is the $<_a$-least $x \in \mathbb{R}^*_a$ such that $\{x^a_j \colon j < i\} \cup \{x\}$ is linearly independent over $\mathbb{R}_{<a}$, and
(ii) b_a is maximal linearly independent over $\mathbb{R}_{<a}$.

By the above crucial fact, the function $a \mapsto b_a$ is well-defined and *exists inside* H. In particular,

$$B = \bigcup \{b_a \colon a \in [A]^{<\omega}\}$$

is an element of H.

We claim that B is a Hamel basis for the reals of H, which will be established by Claims 3.2 and 3.3.

Claim 3.2. $\mathbb{R} \cap H \subset \operatorname{span}(B)$.

Proof of Claim 3.2. Assume not, and let $n < \omega$ be the least size of some $a \in [A]^{<\omega}$ such that $\mathbb{R}^*_a \setminus \operatorname{span}(B) \neq \emptyset$. Pick $x \in \mathbb{R}^*_a \setminus \operatorname{span}(B) \neq \emptyset$, where $\operatorname{Card}(a) = n$.

We must have $n > 0$, as $b_\emptyset$ is a Hamel basis for the reals of L. Then, by the maximality of b_a, while b_a is linearly independent over $\mathbb{R}_{<a}$, $b_a \cup \{x\}$ cannot be linearly independent over $\mathbb{R}_{<a}$. This means that there are $q \in \mathbb{Q}$, $q \neq 0$, $m \in \mathbb{N}$, $m \geq 1$, and $q_n \in \mathbb{Q} \setminus \{0\}$ and $x_n \in b_a$ for all n, $1 \leq n \leq m$, such that

$$z = q \cdot x + \sum_{n=1}^{m} q_n \cdot x_n \in \mathbb{R}_{<a}.$$

By the definition of $\mathbb{R}_{<a}$ and the minimality of n, we then find that $z \in \operatorname{span}(\bigcup\{b_c \colon c \subsetneq a\})$, which clearly implies that $x \in \operatorname{span}(\bigcup\{b_c \colon c \subseteq a\}) \subset \operatorname{span}(B)$, a contradiction. $\blacksquare_{\text{Claim 3.2}}$

Claim 3.3. *B is linearly independent.*

Proof of Claim 3.3. Assume not. This means that there are $1 \le k < \omega$, $a_i \in [A]^{<\omega}$ pairwise different, $m_i \in \mathbb{N}$, $m_i \ge 1$ for $1 \le i \le k$, and $q^i_n \in \mathbb{Q}\setminus\{0\}$ and $x^i_n \in b_{a_i}$ for all i and n with $1 \le i \le k$ and $1 \le n \le m_i$ such that

$$\sum_{n=1}^{m_1} q^1_n \cdot x^1_n + \cdots + \sum_{n=1}^{m_k} q^k_n \cdot x^k_n = 0. \tag{3.12}$$

By the properties of b_{a_i}, $\sum_{n=1}^{m_i} q^i_n \cdot x^i_n \in \mathbb{R}^*_{a_i}$, so that (3.12) buys us that there are $z_i \in \mathbb{R}^*_{a_i}$, $z_i \neq 0$, $1 \le i \le k$, such that

$$z_1 + \cdots + z_k = 0. \tag{3.13}$$

There must be some i such that there is no j with $a_j \supsetneq a_i$, which implies that $a_j \cap a_i \subsetneq a_i$ for all $j \neq i$. Let us assume without loss of generality that $a_j \cap a_1 \subsetneq a_1$ for all j, $1 < j \le k$.

Let $a_1 = \{c_\ell \colon \ell \in I\}$, where $I \in [\omega]^{<\omega}$, and let $a_j \cap a_1 = \{c_\ell \colon \ell \in I_j\}$, where $I_j \subsetneq I$ for $1 < j \le l$.

In what follows, a *nice name* τ for a real is a name of the form

$$\tau = \bigcup_{n,m<\omega} \{(n,m)^\vee\} \times A_{n,m}, \tag{3.14}$$

where each $A_{n,m}$ is a maximal antichain of conditions of the forcing in question deciding that $\tau(\check{n}) = \check{m}$.

We know that z_1 is $\mathbb{P}(I)$-generic over L, so that we may pick a nice name $\tau_1 \in L^{\mathbb{P}(I)}$ for z_1 with $(\tau_1)^{g\restriction I} = z_1$. Similarly, for $1 < j \le k$, z_j is $\mathbb{P}(I_j)$-generic over $L[g\restriction(\omega \setminus I)]$, so that we may pick a nice name $\tau_j \in L[g\restriction(\omega \setminus I)]^{\mathbb{P}(I_j)}$ for z_j with $(\tau_j)^{g\restriction I_j} = z_j$. We may construe each τ_j, $1 < j \le k$, as a name in $L[g\restriction(\omega \setminus I)]^{\mathbb{P}(I)}$ by replacing each $p\colon I_j \to \mathbb{P}$ in an antichain as in (3.14) by $p'\colon I \to \mathbb{P}$, where $p'(\ell) = p(\ell)$ for $\ell \in I_j$ and $p'(\ell) = \emptyset$ otherwise. Let $p \in g\restriction I$ be such that

$$p \Vdash^{\mathbb{P}(I)}_{L[g\restriction(\omega\setminus I)]} \tau_1 + \tau_2 + \cdots + \tau_k = 0.$$

We now find that inside $L[g\restriction(\omega \setminus I)]$, there are nice $\mathbb{P}(I)$-names τ'_j, $1 < j \le k$ (namely, τ_j, $1 < j \le k$), such that still inside $L[g\restriction(\omega \setminus I)]$,

(1) $p \Vdash^{\mathbb{P}(I)} \tau_1 + \tau'_2 + \cdots + \tau'_k = 0$, and
(2) for all j, $1 < j \le k$, and for all p in one of the antichains of the nice name τ'_j, $\operatorname{supp}(p) \subseteq I_j$.

By Lemma 2.4, the nice names $\tau_1, \tau'_2, \ldots, \tau'_k$ may be coded by reals, and both (1) and (2) are arithmetic in such real codes for $\tau_1, \tau'_2, \ldots, \tau'_k$, so that by $\tau_1 \in L^{\mathbb{P}(I)}$ and Σ^1_1-absoluteness between L and $L[g\restriction(\omega \setminus I)]$ there are inside L nice $\mathbb{P}(I)$-names τ'_j, $1 < j \le k$, such that in L, (1) and (2) hold true. Writing $z'_j = (\tau'_j)^{g\restriction I}$, we see by (2) that $z'_j \in \mathbb{R}_{I_j}$ for $1 < j \le k$, and $z_1 + z'_2 + \ldots + z'_k = 0$ by (1). But then $z_1 \in \mathbb{R}^*_I \cap \mathbb{R}_{<I}$, which is absurd. $\blacksquare_{\text{Claim 3.3}}$

We now infer that $x \in B$ iff

$$\begin{aligned}\exists a \in [A]^{<\omega}\ \exists J_\alpha[a]\ \exists (x_i \colon i \le \theta) \in J_\alpha[a]\ \exists X \subset \theta+1 \ &(\text{the } x_i \text{ enumerate}\\ &\text{the first } \theta+1 \text{ reals in } J_\alpha[a] \text{ according to } <_a \wedge\, \theta \in X \wedge x = x_\theta \wedge\\ &\forall i \in \theta \setminus X\ \exists J_\beta[a]\ J_\beta[a] \models Y_i \text{ is not linearly independent over } \mathbb{R}_{<a} \wedge\\ &\forall i \in X\ \forall J_\beta[a]\ J_\beta[a] \models Y_i \text{ is linearly independent over } \mathbb{R}_{<a}),\end{aligned}$$

where $\{x_j \colon j \in X \cap i\} \cup \{x_i\}$.

Using Lemma 3.1, "$x \in B$" can thus be written in a lightface Σ^1_3 fashion. But also $x \in B$ iff

$$\begin{aligned}\forall a \in [A]^{<\omega}\ \forall J_\alpha[a]\ \forall (x_i \colon i \le \theta) \in J_\alpha[a]\ \forall X \subset \theta+1 \ \big(&(\text{the } x_i \text{ enumerate}\\ &\text{the first } \theta+1 \text{ reals in } J_\alpha[a] \text{ according to } <_a \wedge\, x = x_\theta \wedge\\ &\forall i \in (\theta+1) \setminus X\ \exists J_\beta[a]\, J_\beta[a] \models Y_i \text{ is not linearly independent over } \mathbb{R}_{<a} \wedge\\ &\forall i \in X\ \forall J_\beta[a]\ J_\beta[a] \models Y_i \text{ is linearly independent over } \mathbb{R}_{<a}) \to \theta \in X\big).\end{aligned}$$

Again by Lemma 3.1, "$x \in B$" can thus be written in a lightface Π^1_3 fashion. We have shown that B is lightface Δ^1_3. ∎

4. Open questions. We finish by stating some open problems.

(1) Is there a model of ZF plus $\neg\mathsf{AC}_\omega(\mathbb{R})$ where there are sets as in (a)–(d) of Theorem 1.1 of lower projective complexity?

(2) Does the model H from (3.1) on p. 8 have a Burstin basis? An affirmative answer along the lines of the argument from [BC+] would require us to show that

$$\mathbb{R}_{<a} \in (s^0)^{L[a]} \quad \text{for all } a \in [A]^{<\omega}, \tag{4.1}$$

where s^0 denotes the Marczewski ideal. We do not know if (4.1) is true; we do not even know if

$$\mathbb{R}^*_a \neq \emptyset \quad \text{for all } a \in [A]^{<\omega}. \tag{4.2}$$

L. Wu and L. Yu have recently shown that (4.2) is true for $\mathrm{Card}(a) = 2$, but it is not known if it holds for $\mathrm{Card}(a) = 3$. The second author has shown that if A is a countable set of Cohen reals over L (or, for that matter, any countable set of dominating reals over L), then (4.2) is true for $a \in [A]^{<\omega}$ of arbitrary size, i.e., (4.2) holds true for $\mathbb{R}^*_a$ as defined in [BS+].

(3) Does the model H from (3.1) have a Mazurkiewicz set?

We may force with the forcings from [BC+] and [BS] to add a Burstin basis and a Mazurkiewicz set, respectively, over H (without adding any reals), but then those sets will not be definable in that extension.

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References

[BS] M. Beriashvili and R. Schindler, *Mazurkiewicz sets*, https://ivv5hpp.uni-muenster.de/u/rds/mazurkiewicz_sets.pdf

[BS+] M. Beriashvili, R. Schindler, L. Wu and L. Yu, *Hamel bases and well-ordering the continuum*, Proc. Amer. Math. Soc. 146 (2018), 3565–3573.

[BC+] J. Brendle, F. Castiblanco, R. Schindler, L. Wu and L. Yu, *A model with everything except for a well-ordering of the reals*, submitted.

[E] A. Enayat, *On the Leibniz–Mycielski axiom in set theory*, Fund. Math. 181 (2004), 215–231.

[GQ] St. Geschke and S. Quickert, *On Sacks forcing and the Sacks property*, in: Classical and New Paradigms of Computation and Their Complexity Hierarchies, B. Löwe et al. (eds.), Kluwer, Dordrecht, 2004, 95–139.

[GHJ] V. Gitman, J. Hamkins, and T. Johnstone, *What is the theory* ZFC *without power set?*, Math. Logic Quart. 62 (2016), 391–406.

[Je] R. B. Jensen, *Definable sets of minimal degree*, in: Mathematical Logic and Foundations of Set Theory, North-Holland, Amsterdam, 1970, 122–128.

[Jo] F. B. Jones, *Measure and other properties of a Hamel basis*, Bull. Amer. Math. Soc. 48 (1942), 472–481.

[JMS] H. Judah, A. W. Miller, and S. Shelah, *Sacks forcing, Laver forcing, and Martin's Axiom*, http://www.math.wisc.edu/∼miller/res/unpub/perf.pdf (this preprint has little in common with a paper by the same authors and with the same title in Arch. Math. Logic 31 (1992)).

[KL] V. Kanovei and V. Lyubetsky, *A countable definable set of reals containing no definable elements*, Math. Notes 102 (2017), 338–349.

[Ma] A. Mathias, *The order extension principle*, in: Axiomatic Set Theory, Proc. Sympos. Pure Math. 13, Part II, Amer. Math. Soc., Providence, RI, 1974, 179–183.

[Mi] A. Miller, *Infinite combinatorics and definability*, Ann. Pure Appl. Logic 41 (1989), 179–203.

[PP] D. Pincus and K. Prikry, *Luzin sets and well ordering the continuum*, Proc. Amer. Math. Soc. 49 (1975), 429–435.

[Sch1] R. Schindler, *Projective sets and large cardinals*, talks at Kiel University, 2002; https://ivv5hpp.uni-muenster.de/u/rds/talks_kiel.pdf.

[Sch2] R. Schindler, *Set Theory: Exploring Independence and Truth*, Springer, 2012.

[Sie1] W. Sierpiński, *Sur la question de la mesurabilité de la base de M. Hamel*, Fund. Math. 1 (1920), 105–111.

[Sie2] W. Sierpiński, *La base de M. Hamel et la propriété de Baire*, Publ. Math. Univ. Belgrade (1935), 220–224.

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