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Bernstein sets with a large continuum but no axiom of choice *)

Let us force over L. Let $\kappa \geq \omega_1$ be regular, and let G be $\mathbb{C}(\kappa)$-generic over L. For $i < \kappa$, let $c_i \in {}^\omega\omega$ be defined by $c_i(n) = m$ iff $\exists p \in G \ p(i)(n) = m$, i.e., c_i is the i^{th} Cohen real added by G.

Let $I \subset \kappa$, $I \in L$, $\overline{\overline{I}} \leq \aleph_0$. Let us write $o(I)$ for the order type of I. Let $\alpha \geq o(I)$ be least s.t. $L_{\alpha+1} \models$ "α is countable," and write $\alpha(I) = \alpha$. Let $f : \omega \leftrightarrow L_\alpha$ be the $L_{\alpha+1}$-least bijection, and let $E \subset \omega^2$ be such that $(\omega, E) \overset{f}{\cong} (L_\alpha ; \in)$. Let $\pi : o(I) \leftrightarrow I$ be the order isomorphism, and let $g \subset \omega$ be the set of all $n < \omega$ s.t. there are $i < o(I)$ and $k, m < \omega$ with $f(n) = (i, k, m)$ and $p(\pi(i))(k) = m$ for some $p \in G$ (i.e., $c_{\pi(i)}(k) = m$). Hence E codes L_α, and g codes

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$(c_{\pi(i)} : i < o(I))$ relative to E.

Let us define $z^I : \omega \longrightarrow \omega$ by

$$z^I(2\ell) = \begin{cases} 1 & \text{iff} \quad (\ell)_0 \; E \; (\ell)_1 \\ 0 & \text{iff} \quad (\ell)_0 \not{E} \; (\ell)_1 \end{cases}$$ and

$$z^I(2\ell+1) = \begin{cases} 1 & \text{iff} \quad \ell \in g \\ 0 & \text{iff} \quad \ell \notin g \end{cases},$$

where $((\ell)_0, (\ell)_1) = e(\ell)$ for some canonical $e : \omega \leftrightarrow \omega \times \omega$.

We may thus compute $L_\alpha[(\bar{c}_i : i < o(I))]$ from z^I, where $\bar{c}_i = c_{\pi(i)}$.

<u>Claim 1.</u> For every $x \in {}^\omega\omega \cap L[G]$ there is some $I \in [\kappa]^{\aleph_0} \cap L$ such that $x \leq_T z^I$.

Proof: If $x \in {}^\omega\omega \cap L[G]$, then $x \in L[(c_i : i \in I)]$ for some $I \in [\kappa]^{\aleph_0} \cap L$, as $\mathbb{C}(\kappa)$ has the c.c.c. The rest is easy. $\dashv$

For $I \in [\kappa]^{\aleph_0} \cap L$, let us write d^I for the E_0-degree of z^I, i.e.,
$d^I = \{ x \in {}^\omega\omega : \exists n_0 \, \forall n \geq n_0 \; x(n) = z^I(n) \}$.

Let us consider the model

$$N = \mathrm{HOD}^{L[G]}_{({}^{\omega}\omega \cap L[G]) \cup \{(d^I,\, o(I)) : I \in [\kappa]^{\aleph_0} \cap L\}}.$$

N thus not only knows the collection of all d^I but also the function $d^I \mapsto o(I)$.

<u>Claim 2.</u> $\mathrm{cf}(\leq_T) = \kappa$ in $L[G]$.

Proof: If $X \in [{}^{\omega}\omega]^{<\kappa} \cap L[G]$, then, as $\mathbb{C}(\kappa)$ has the c.c.c., there is some $J \in [\kappa]^{<\kappa} \cap L$ with $X \subset L[(c_i : i \in J)]$. But then if $i \notin J$, $c_i \notin L[(c_i : i \in J)]$, hence $c_i \not\leq_T x$ for all $x \in X$. So X can't be cofinal in $\leq_T \cap L[G]$.

On the other hand, $\overline{\overline{[\kappa]^{\aleph_0} \cap L}} = \kappa$ in L and $\{z_I : I \in [\kappa]^{\aleph_0} \cap L\}$ is cofinal in $\leq_T \cap L[G]$ by Claim 1. $\dashv$

This immediately implies:

<u>Claim 3.</u> In N, there is no $\gamma < \kappa$ s.t. $\mathrm{ran}(f)$ is $\leq_T$-cofinal for some $f : \gamma \longrightarrow E_0$-degrees, but $\{d^I : I \in [\kappa]^{\aleph_0} \cap L\}$

is $\leq_T$-cofinal.

The proofs on pp. 3 – 7 of [BeSch 1] show that:

<u>Claim 4.</u> In N, there is no well-ordering of the reals.

Proof sketch: Let $p, q \in \mathbb{C}(\kappa)$. Let $p^* \leq p$ and $q^* \leq q$ be such that $p^*(i)(n)\downarrow$ iff $q^*(i)(n)\downarrow$ for all $i < \kappa$ and $n < \omega$. (Of course, the set of all i, n with $p^*(i)(n)\downarrow$ is finite.) Define $\pi^{p,q} : \mathbb{C}(\kappa) \longrightarrow \mathbb{C}(\kappa)$ as follows.

$\pi^{p,q}(r) = r$ unless $r \parallel p$ or $r \parallel q$, in which case

$$\pi^{p,q}(r)(i)(n) = \begin{cases} q(i)(n) & \text{if } r(i)(n)\downarrow \text{ and } r(i)(n) = p(i)(n) \\ p(i)(n) & \text{if } r(i)(n)\downarrow \text{ and } r(i)(n) = q(i)(n) \end{cases}.$$

Let $\dot{d}^I \in L^{\mathbb{C}(\kappa)}$ be a canonical name for d^I, $I \in [\kappa]^{\aleph_0} \cap L$. It is easy to see that $\tilde{\pi}^{p,q}(\dot{d}^I) = \dot{d}^I$, where $\tilde{\pi}^{p,q} : L^{\mathbb{C}(\kappa)} \longrightarrow L^{\mathbb{C}(\kappa)}$ is the embedding induced by $\pi^{p,q}$. This gives that the canonical name for $(d^I, o(I))$ is also fixed by any $\tilde{\pi}^{p,q}$, and hence if τ is the canonical name for $((d^I, o(I)) : I \in [\kappa]^{\aleph_0} \cap L)$, then $\tilde{\pi}^{p,q}(\tau) = \tau$ for all p, q. In other words, τ is homogeneous with respect to $\mathbb{C}(\kappa)$.

The same argument shows that if $X \in [\kappa]^{\aleph_0} \cap L$, then $((d^I, o(I)) : I \in [\kappa]^{\aleph_0} \cap L)$ has a name in $L[G \restriction X]^{\mathbb{C}(\kappa \setminus X)}$ which is homogeneous with respect to $\mathbb{C}(\kappa \setminus X)$. The arguments in [Sch, pp. 117f.] then show that ${}^\omega\omega \cap N = {}^\omega\omega \cap L[G]$ cannot be well-ordered in N. $\dashv$

Let us now write $\mathcal{D}$ for the collection of all $(d^I, o(I))$, $I \in [\kappa]^{\aleph_0} \cap L$. N can't see the function $d^I \mapsto I$, but it does see $d^I \mapsto o(I)$.

<u>Claim</u> 5. There is a Bernstein set in N.

Proof: Inside N, let us define

$$B = \{ x \in {}^\omega\omega : \text{ there is some } (d, \alpha) \in \mathcal{D} \text{ such that } \alpha \text{ is even}^*, \ x \leq_T z \text{ for all / some } z \in d, \text{ but}$$

there is no $(\bar{d},\bar{\alpha}) \in \mathcal{D}$ with $\bar{\alpha} < \alpha$ and $x \leq_T z$ for all / some $z \in \bar{d}\}$, and

$B' = \{ x \in {}^\omega\omega$: there is some $(d,\alpha) \in \mathcal{D}$ such that α is odd*, $x \leq_T z$ for all / some $z \in d$, but there is no $(\bar{d},\bar{\alpha}) \in \mathcal{D}$ with $\bar{\alpha} < \alpha$ and $x \leq_T z$ for all / some $z \in \bar{d}\}$.

Here, "even*" and "odd*" refer to the following. Let $C \subset \omega_1$ be the club of all α s.t. if $x, y \subset \alpha$, $\mathrm{otp}(x) < \alpha$, $\mathrm{otp}(y) < \alpha$, then $\mathrm{otp}(x \cup y) < \alpha$, and let $h : \omega_1 \longrightarrow C$ be the monotone enumeration. α is even* iff there is some even β with $\alpha = h(\beta)$, and α is odd* iff there is some odd β with $\alpha = h(\beta)$.

Let us verify that $B \cap [T] \neq \emptyset$ for every perfect tree T on ω. Let T be given, say $T \leq_T z^I$, $I \in [\kappa]^{\aleph_0} \cap L$. Pick $J \supset I$, $J \in [\kappa]^{\aleph_0} \cap L$ such that $o(I) < o(J) = \alpha$ and α is even*. Notice $z^I \leq_T z^J$.

Let $b \in [T]$ be such that

$$T \oplus b =_T z^J .$$

In particular, $b \leq_T z^J$, so that $(d^J, \alpha) \in \mathcal{D}$ is such that $b \leq_T z$ for all/some $z \in d^J$.
If we had some $(d, \bar{\alpha}) \in \mathcal{D}$ with $\bar{\alpha} < \alpha$ and $b \leq_T z$ for all/some $z \in d$, say $d = d^K$, $K \in [a]^{\aleph_0} \cap L$, $o(K) = \bar{\alpha}$, then $T \oplus b \leq_T z^{I \cup K}$. However $o(I \cup K) < \alpha$, as $o(I) < \alpha$ and $o(K) < \alpha$. But then $z^J \leq_T z^{I \cup K}$ gives that $c_i \leq_T z^{I \cup K}$ for some $i \notin I \cup K$.
This is a contradiction! $\dashv$

[BeSch1] Beriashvili, Schindler, "Bernstein sets don't give Vitali sets"

[Sch] Schindler, Set theory: exploring independence and truth, Springer-Verlag.