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The extender algebra for mice without Woodin cardinals

Let δ be an inaccessible cardinal. We consider a language $\mathcal{L}$ intended to describe a subset A of δ. The atomic formulae of $\mathcal{L}$ are "$\check{\xi} \in \dot{a}$" for $\xi < \delta$ and a fixed predicate symbol $\dot{a}$, and we get all the $\mathcal{L}$-formulae by closing under $\neg$ and conjunctions $\bigwedge$ of length $< \delta$.

If $\mathcal{E}$ is a collection of extenders $E \in V_\delta$, then we may associate to $\mathcal{E}$ the following set of axioms:

$$(A_E) \qquad \bigvee i_E(\vec{\varphi}) \restriction lh(E) \longrightarrow \bigvee \vec{\varphi}$$

for $E \in \mathcal{E}$, $i_E : V \longrightarrow ult(V; E)$ the ultrapower embedding, $lh(E) =$ the length of E, $\vec{\varphi} = (\varphi_i : i < crit(E)) \in V_{crit(E)+1}$ a sequence of $\mathcal{L}$-formulae. For our purposes, we may identify the length of E with its strength, i.e., with the largest α s.t. $V_\alpha \subset ult(V; E)$.

The partial order $\leq$ on $\mathcal{L}$ given by

$$\varphi \leq \psi \text{ iff } \{ (A_E) : E \in \mathcal{E} \} \vdash \varphi \rightarrow \psi$$

is called the extender algebra. Here, "$\vdash$" means "provability." Details on this may be found e.g. in [FSTTIMTG]. In what follows, we often confuse $\mathcal{L}$ with $\mathcal{L} \setminus \{\varphi : \{(A_E) : E \in \mathcal{E}\} \cup \{\varphi\} \vdash 0 = 1\}$.

Let M be a countable active mouse s.t. if κ denotes the critical point of M's top measure, then $M \models$ "δ is a Woodin cardinal." Let $A \subset \delta$, e.g. $V_\delta = L_\delta[A]$. It is standard to produce an iteration tree $\mathcal{I}$ on M of length $\delta + 1$ s.t. δ is the critical point of the top measure of $M_\delta^{\mathcal{I}}$ and for every (total) extender E from the sequence of $M_\delta^{\mathcal{I}} | \delta$ and for all $\vec{\varphi} = (\varphi_i : i < crit(E)) \in V_{crit(E)+1}^{M_\delta^{\mathcal{I}}}$, $A \models (A_E)$. As a consequence, A (or rather $\{ \varphi \in \mathcal{L}^{M_\delta^{\mathcal{I}}} : A \models \varphi \}$) is $M_\delta^{\mathcal{I}}$-generic for the extender algebra of $M_\delta^{\mathcal{I}}$ given by the extenders from the sequence of $M_\delta^{\mathcal{I}} | \delta$.

The picture changes if we start out with a smaller mouse M.

Still, let δ be inaccessible and $E \subset \delta$, e.g. $V_\delta = L_\delta[A]$. Let M be any countable active mouse, and let κ be the critical point of M's top extender. We may still easily produce an iteration tree $\mathcal{T}$ on M of length $\delta+1$ s.t. if $i: M \longrightarrow \mathcal{M}^{\mathcal{T}}_\delta$ denotes the iteration embedding, then

(a) $\delta = i(\kappa)$, and

(b) if E is a (total) extender from the sequence of $\mathcal{M}^{\mathcal{T}}_\delta | \delta$ and if $\vec{\varphi} = (\varphi_i : i < \mathrm{crit}(E)) \in V^{\mathcal{M}^{\mathcal{T}}_\delta}_{\mathrm{crit}(E)+1}$ is a sequence of formulae of $\mathcal{L}^{\mathcal{M}^{\mathcal{T}}_\delta}$, then $A \models (\mathbb{A}_E)$.

As an immediate consequence,

$$G_A = \{\varphi \in \mathcal{L}^{\mathcal{M}^{\mathcal{T}}_\delta} : A \models \varphi\}$$

has nonempty intersection with any $\Gamma \in \mathcal{M}^{\mathcal{T}}_\delta | \delta$ s.t. $\Gamma \subset \mathcal{L}^{\mathcal{M}^{\mathcal{T}}_\delta}$ and $\mathcal{M}^{\mathcal{T}}_\delta \models$ "Γ is a maximal antichain": if Γ were a counterexample, then

$\bigwedge_{\varphi \in \Gamma} \neg \varphi \in \mathcal{L}$ and $\Gamma \cup \{\bigwedge_{\varphi \in \Gamma} \neg \varphi\}$ would also be an antichain.

If in addition $M \models$ "κ is a Woodin cardinal," then we are back to the situation on p.2, as we may then prove that if $\Gamma \in M_\delta^I$, $\Gamma \subset \mathcal{L}^{M_\delta^I}$, $M_\delta^I \models$ "Γ is a maximal antichain," then $\Gamma \in M_\delta^I | \delta$. But we may now easily generalize this as follows.

Let $T \subset V_\delta$. $\lambda < \delta$ is called <u>T-strong up to δ</u> iff f.a. $\alpha < \delta$ there is some extender $E \in V_\delta$ with $\mathrm{crit}(E) = \lambda$ and s.t. $V_\alpha \subset \mathrm{ult}(V; E)$ and $i_E(T) \cap V_\alpha = T \cap V_\alpha$. Let $T^* \subset V_{\delta+1}$. δ is called T^*-<u>Woodin</u> iff for all $T \in T^*$ there is some $\kappa < \delta$ which is T-strong up to δ.

Hence δ is Woodin iff δ is $V_{\delta+1}$-Woodin.

We call δ <u>definably Woodin</u> iff δ is $\mathrm{Def}(V_\delta)$-Woodin, where $\mathrm{Def}(V_\delta)$ denotes the collection of subsets of V_δ which are definable over $(V_\delta; \in)$

via a first order formula and parameters from V_δ.

The above considerations produced the following.

<u>Lemma</u> 1. If $M \models$ "κ is definably Woodin," then A (or rather G_A) is $\mathrm{Def}(\mathcal{M}_\delta^{\mathcal{J}}|\delta)$-generic over $\mathcal{M}_\delta^{\mathcal{J}}$.

This is a reformulation of Theorem 2 of [CTU] which clarifies the connection of the forcing Q as being defined on p. 15 of [CTU] with the extender algebra. The mouse mm ("mighty mouse") as being defined on p. 7 of [CTU] may in our framework be characterized as follows.

<u>Lemma</u> 2. mm is the least active mouse M s.t. if κ denotes the critical point of M's top extender, then $M \models$ "κ is definably Woodin."

Lemma 2 will immediately follow from Lemma 3. In order to state this, let us streamline the definitions of "(nicely) β, n – strong" as follows.

Let us call $\lambda < \delta$ $\delta, 1$ – strong* iff λ is $\emptyset$ – strong up to δ, and let us call $\lambda < \delta$ $\delta, n+1$ – strong* iff λ is $\{\eta < \delta : \eta$ is δ, n – strong$^*\}$ – strong up to δ.

<u>Lemma</u> 3. Let $n \geq 1$. If $n > 1$, then let us assume that $\{\eta < \delta : \eta$ is $\delta, (n-1)$ – strong$^*\}$ is unbounded in δ.
Let $\lambda < \delta$ be δ, n – strong*. Then λ is A – strong up to δ for all A which are Σ_n definable over $(V_\delta ; \in)$ from a parameter in V_λ.

Proof by induction on $n \geq 1$. $n = 1$ ✓

$n > 1$: Fix $\lambda < \delta$ δ, n – strong*, A Σ_n definable over $(V_\delta ; \in)$ from $z \in V_\lambda$, say

$$x \in A \text{ iff } V_\delta \models \exists y \underbrace{\varphi(y, x, z)}_{\Pi_{n-1}} .$$

Let $\alpha < \delta$ be a limit cardinal, $\lambda < \alpha$, s.t. $\{\eta < \alpha : \eta$ is $\delta, (n-1)$-strong$^*\}$ is unbounded in α, and let $i : V \to ult(V; E)$ be s.t. $crit(E) = \lambda$, $V_\alpha \subset ult(V; E)$, and for all $\eta < \alpha$, η is $\delta, (n-1)$-strong* iff $ult(V; E) \models$ "η is $\delta, (n-1)$-strong*."

Let $x \in A \cap V_\alpha$. Let $rk_E(x) < \eta < \alpha$ be $\delta, (n-1)$-strong. Let $y_0 \in V_\delta$ be s.t. $V_\delta \models \varphi(y_0, x, z)$. By the induction hypothesis, there is a suff. strong extender F with $crit(F) = \eta$ s.t. if $j : V \to ult(V; F)$, then $y_0 \in ult(V; F)$, $ult(V_\delta; F) \models \varphi(y_0, x, z)$; and $rk(y_0) < j(\eta)$. By elementarity, there is then some $y_1 \in V_\eta \subset V_\alpha$ s.t. $V_\delta \models \varphi(y_1, x, z)$. But then $ult(V_\delta; E) \models \varphi(y_1, x, z)$ by the inductive hypothesis, so that $ult(V_\delta; E) \models \exists y\, \varphi(y, x, z)$.

This argument shows that $A \cap V_\alpha = i(A) \cap V_\alpha$.

$\dashv$ (Lemma 3)

References.

[CTU] S. Friedman, Capturing the universe, preprint, Aug 28, 2020.

[FSTTIMTG] R. Schindler, From set theoretic to inner model theoretic geology, to appear in "Research trends in Contemp. logic" (A. Daghighi, ed.).