19.~~8~~7. 2010 9:00 John Steel

Q: Con(PFA) $\Rightarrow$ Con(Supercompact)
⋮

Structure so far:

Belongs to models of AD^+

strong hyps $\xrightarrow{\text{core model theory}}$ $\left(\begin{array}{c}\text{Absolutely definable IS}\end{array} \longleftrightarrow \begin{array}{c}\text{Absolutely definable scales}\end{array}\right)$

Descriptive Inner ~~m~~ model theory

<u>Derived model theorem</u> (Steel's paper)

<u>Def</u> Homogeneity systems, $S_{\vec{\mu}}$, Hom^X_κ

<u>Def</u> κ-absolute complementing trees, κ-~~weak~~ UB

<u>Def</u> κ-weakly homogeneous set

<u>Facts</u> (0) ~~Hom~~ Hom^X_κ sets are determined

(1) If δ is Woodin, then:

A is δ^+-weakly homo $\Rightarrow \neg A$ is $<\delta$-homogeneous

(2) If λ is a limit of Woodin cardinals then $\mathrm{Hom}_{<\lambda}$ is closed under $\exists^{\mathbb{R}}$ and $\neg$, and also under $\forall^{\mathbb{R}}$ and $\leq_W$.

<u>Exercise</u> $\exists \gamma < \lambda$: $\mathrm{Hom}_\gamma = \mathrm{Hom}_{<\lambda}$

(3) Let δ be Woodin then

A is δ^+-UB $\Rightarrow$ A is $<\delta$-whom

Is λ is a limit of Woodin cardinals then

$\mathrm{Hom}_{<\lambda} = \mathrm{UB}_{<\lambda} = \mathrm{Whom}_{<\lambda}$

<u>Tree production lemma</u> Let δ be Woodin, $\varphi(x,v)$ be a formula, a be a set. Suppose

(Generic absoluteness) (1) For a $<\delta$-generic G and H that is $\leq\delta$-generic over $V[G]$ and $x \in \mathbb{R}$:
$V[G] \models \varphi[x,a] \iff V[G,H] \models \varphi[x,a]$

(2) (Stationary tower correctness)
For G that is $\mathbb{Q}_{<\delta}$-generic and $\sigma: V \to M = Ult(V,G)$ and $x \in \mathbb{R}^{V[G]}$:
$V[G] \models \varphi[x,a]$ iff $M \models \varphi[x, \sigma(a)]$

Then there are $<\delta$ absolute complements T, U s.t. $p[T] = \{x \mid \varphi(x,a)\}$ in all $V[g]$ when g is $<\delta$-generic

<u>Corollary</u> Every $\mathrm{Hom}_{<\lambda}$ set has a $\mathrm{Hom}_{<\lambda}$ scale for λ a limit of Woodins.

<u>Derived model theorem</u> Let λ be a limit of Woodins. Let G be V-generic / $Col(\omega, <\lambda)$. Put

$$\mathbb{R}^*_G = \bigcup_{\alpha<\lambda} \mathbb{R}^{V[G\restriction\alpha]}$$

$$\mathrm{Hom}^*_G = \{p[T] \cap \mathbb{R}^*_G \mid (\exists \alpha<\lambda)\ V[G\restriction\alpha] \models T \text{ is } <\lambda \text{ absolutely complementing}\}$$

$$A_G = \{A \subseteq \mathbb{R}^*_G \mid A \in V(\mathbb{R}^*) \text{ and } L(A, \mathbb{R}^*) \models AD^+\}$$

Then

(1) $L(A_G, \mathbb{R}^*) \models AD^+$

(2) $\mathrm{Hom}^*_G = \{A \subseteq \mathbb{R}^* \mid A \text{ is Suslin co-Suslin in } L(A_G, \mathbb{R}^*)\}$

(3) Σ^2_1 reflects to Suslin co-Suslin:
$L(A, \mathbb{R}^*_G) \models M_{\varphi} \prec_{\Sigma_1} V$ for φ = the collection of all Suslin co-Suslin sets

<u>Def</u> For $\Delta \subseteq \mathcal{P}(\mathbb{R})$: ~~M_Δ~~

$M_\Delta = \bigcup \{ m \mid m$ is transitive and $\exists E, F \in \Delta$ s.t. $(\mathbb{R}/E, F/E) \cong (m, \in) \}$

Use when $A \in \Delta \Rightarrow L_\alpha(A, \mathbb{R}) \cap \mathcal{P}(\mathbb{R}) \subseteq \Delta$ for α = the least admissible over $A, \mathbb{R}$.

$M_\Delta \models (\forall m)(\exists f)(f : \mathbb{R} \xrightarrow{\text{onto}} m)$

<u>Proof</u> Set $HC^* = HC(V^{\mathbb{R}^*})$ for $A \in \mathrm{Hom}_{<\lambda}$,

$A^* = p[T] \cap \mathbb{R}^*$ for any and all $<\lambda$-a.c. T s.t. $A = p[T]$. (Exercise: independent of T)

(So $\mathrm{Hom}^* = \{ A^* \mid A \in V[h]$ some $<\lambda$-generic h)

<u>Lemma</u> $(HC, \in, A)^V \prec (HC^*, \in, A^*)$

<u>Cor</u> Every Hom^*-set has a Hom^*-scale

Thus: If $\mathrm{Hom}^* \subseteq A_G$ then $\mathcal{G}^{V(\mathbb{R}^*)} \subseteq \mathrm{Hom}^* \subseteq \mathcal{G}^{L(\mathbb{R}^*, \mathrm{Hom}^*)}$

$\subseteq \mathcal{G}^{L(A_G, \mathbb{R}^*)} \subseteq \mathcal{G}^{V(\mathbb{R}^*)}$

Here is the use of the assumption

<u>Lemma</u> $\mathrm{Hom}^* \subseteq A_G$.

Let $A \in \mathrm{Hom}_{<\lambda}^{V[g]}$. We want: $L(A^*, \mathbb{R}^*) \models AD^+$.

Show: if $\exists B \in L(A^*, \mathbb{R}^*)\ (HC^*, \in, A^*, B^*) \models \varphi[B, A^*]$

then there is $B \in \mathrm{Hom}_{<\lambda}^{V[g]}$ s.t.

$(HC, \in, A, B)^{V[g]} \models \varphi(B, A)$

so $(HC^*, \in, A^*, B^*) \models \varphi[B^*, A^*]$.

So: "Σ^2_1 reflects to Suslin co-Suslin in $L(A_G, \mathbb{R}^*)$", i.e.

$M_{\mathcal{G}} \prec_{\Sigma_1} M_{P(\mathbb{R})} \prec_{\Sigma_1}$ in $L(A^*, \mathbb{R}^*)$

Easy because Θ is regular in $L(A^*, \mathbb{R}^*)$.

Proof Take $\varphi = \emptyset$. Let γ_0 be least s.t.

$$L_{\gamma_0}(A^*, \mathbb{R}^*) \models ZF^- + (\exists B)(HC, \in, A^*, B) \models \varphi$$

Pick $B \in OD_{z_0}$ some $z_0 \in \mathbb{R}^*$ and the least sequence of ordinals possible, i.e.

$$x \in B \text{ iff } L_{\gamma_0}(A^*, \mathbb{R}^*) \models \Theta[x, z_0, A^*]$$

Let $A^* = p[T]$ for some absolute complement (T, U) in $\bar{V}$.

$\varphi(x, T, z_0) \equiv$ " $x \in \mathbb{R}$ and for γ least s.t.

$$L_\gamma(p[T], \mathbb{R}) \models (\exists B)(HC, \in, B, p[T]) \models \varphi$$

$$L_\gamma \models \Theta[x, p[T], z_0]$$

(a) φ is Stationary tower ~~absolute~~ correct.

Lemma Let λ be a limit of Woodins.

(a). Let $j : V \longrightarrow M = Ult(V, H)$ where H is $\mathbb{Q}_{<\delta}$-generic, $\delta < \lambda$ Woodin. Then $j(Hom^V_{<\lambda}) \supseteq Hom^{V[H]}_{<\lambda}$ as a $\leq_w$ init segment.

(b) Let $j : V \longrightarrow M_\infty$ with $M_\infty = \lim_\alpha Ult(V, H \restriction \alpha)$

Then

$$j(Hom^V_{<\lambda}) \supseteq Hom^*$$

as a $\leq_w$ - init segment. When $\mathbb{R}^* = \mathbb{R}^{M_\infty}$

Now take mech. from the proof of the DMT.

Def C_Γ. Let $\Gamma \subseteq \mathcal{P}(\mathbb{R})$ be ω-parametrized, closed under $\forall^{\mathbb{R}}$, $\exists^\omega$, $\forall^\omega$, recursive substitution and PWO(Γ). Assume also $\exists^{\mathbb{R}}\Gamma \subseteq \Gamma$.

(We say Γ is "inductive like": Example: $\Gamma = \Sigma_1^{J_\alpha(\mathbb{R})}$ when $J_\alpha(\mathbb{R}) \models KP$.)

$\mathfrak{M}_\Gamma = (M_A, \in, A)$ where

if $\varphi : U \to \kappa$ is a Γ-norm on U = universal Γ-set
$A(x, \beta)$ iff $x = u$ and $\varphi(x) = \beta$

$\Gamma = \Sigma_1^{\mathfrak{M}_\Gamma}$, also $\mathfrak{M}_\Gamma = (J^A_{\underset{\sim}{\delta}_\Gamma}(\mathbb{R}), \in, A)$.

δ_Γ = sup of pwo's in $\underset{\sim}{\Delta}$.

Assume Scale(Γ). Let
$T = T_{\vec{\varphi}}$ where $\vec{\varphi}$ is a Γ-scale on universal Γ-set
(so $p[T] = U$)

We put for $x, y \in \mathbb{R}$

$x \in C_\Gamma(y)$ iff $(\exists \alpha < \delta_\Gamma)(x$ is OD from y over $(J^A_\alpha(\mathbb{R}), \in, \overbrace{\ }^{A \cap J^A_\alpha(\mathbb{R})}))$

Note: "$x \in C_\Gamma(y)$" is $\Sigma_1^{\mathfrak{M}_\Gamma}$ so it's Γ.

Assume $M_\Gamma \models AD$. Then $\forall y\ C_\Gamma(y)$ is countable.

<u>Lemma</u> $C_\Gamma(y) = \mathbb{R} \cap L[T_{\vec{\varphi}}, y]$.
<u>Pf</u> $\supseteq$: Use $\langle T_{\vec{\varphi}} \restriction \omega \times \alpha \mid \alpha < \delta_T \rangle$ is $\Sigma_1^{\mathfrak{M}_\Gamma}$
$\subseteq$: By Mansfield-Solovay.

Let $<_y$ be the canonical wo on $C_\Gamma(y)$ ... $\Sigma_1^{\mathfrak{M}_\Gamma}$
$R(x, y, z) \equiv x <_y z$ is in Γ

$I(z, y)$ iff $z = (w, u)$ where $u \in WO$ and
$(w)_i = |i|_u$ - the real in $C_\Gamma(y)$

I is $\Sigma_1^{M_\Gamma}$, hence Γ.

Now assume AD$^+$. Suppose also not all sets of reals are Suslin. (Eg $V = L(A, \mathbb{R})$.)
Let κ = largest Suslin cardinal.
$S(\kappa) = \{p[T] \mid T$ on $\omega \times \kappa\}$ = the class of κ-Suslin sets.
$S(\kappa) = \underset{\sim}{\Gamma}$ for Γ lightface inductive like with Scale(Γ).

Let $T = T_{\vec{\varphi}}$ when $\vec{\varphi}$ is a Γ-scale on the universal Γ-set.
Let
$T^* = \mathrm{Ult}(T, \mathcal{D})$ when $\mathcal{D}$ is the Martin measure.

<u>Theorem</u>: $V = L(T^*, \mathbb{R})$
<u>Proof</u> Force over V with "pointed Sacks forcing".
& <u>Conditions</u>: Trees on ω which are
- perfect
- pointed: $(\forall x \in [T])(T \leq_T x)$

$\{[x]_T \mid x \in [T]\} = \{d \mid [T] \in d\}$ ($\leq_T$)
<u>Martin</u> if $\forall y \exists x \geq_T y$ $x \in A$ then there is a perfect pointed tree T s.t. $[T] \subseteq A$.

$\mathbb{P}_{\text{sacks}} = \{T \mid T$ perfect pointed$\}, \geq\}$

Let g be V-generic / $\mathbb{P}_{sacks}$. For $A \subseteq \mathbb{R}$, $A \in V$

$$A \in \mathcal{U}_{x_g} \text{ iff } (\exists T \in g)([T]^V \subseteq A)$$

Martin $\Rightarrow$ $\mathcal{U}_{x_g}$ is an UF over $\mathcal{P}(\mathbb{R}) \cap V$

Consider

$$\prod_{x \in \mathbb{R}^V} L[T, x] \,/\, \mathcal{U}_{x_g}$$

We have Łoś' theorem. Then

$$x_g = [\mathrm{id}].$$

If $f : \mathbb{R} \to On$, $f \in V$ and $T \in \mathbb{P}_{sacks}$ then
$\exists U \leq_{\mathbb{P}} T$ s.t. $\forall x, y \in [U]$
$$x \equiv_T y \Rightarrow f(x) = f(y)$$

So f "is" a function on $\{d \in \mathcal{D} \mid U \leq d\}$.
f used in $\mathrm{Ult}(V, \text{Martin})$. So
$$[f]_U = [g]_{\text{martin}}$$
So $\quad [c_T]_U = T^*.$

Thm $\quad AD^+ \Rightarrow \mathrm{Ult}(V, \text{martin})$ is WF. ... To come.

Fact Let $A \subseteq \mathbb{R}$ be in V. Then for a $\underbrace{T\text{-cone}}_{}$
of x: $\quad A \cap L[T, x] \in L[T, x]$.
Need here κ = largest Suslin.

$x \leq_T y$ iff $x \in L[T, y]$

Let $A^* = [c_A]_U$. Then $A = A^* \cap \mathbb{R}^V$.
So $A^* \in L[T^*, x_g]$.
So $A \in L(T^*, \mathbb{R}^V)[x_g]$
But this is true of any x_g, so $A \in L(T^*, \mathbb{R}^V)$

In the situation above:
Let $z(i) = 0$ iff $\{i\}^{x_g} \in A^*$. Then
$z \in$ "$C_\Gamma(x_g)$" $= \mathbb{R} \cap L[T^*, x_g]$ and
$A = \{\{i\}^{x_g} \mid i \in z$ and $\{i\}^{x_g} \in \mathbb{R}^V\}$.

Working in $L(T^*, \mathbb{R})$, say
(p, β) codes $A \subseteq \mathbb{R}$ (relative to T^*) iff
$p \Vdash$ there is a β-th real $z = z_\beta^{T^*, x_g}$ in $<_{x_g}^{T^*}$
$_{\mathbb{P}_{\text{Back}}}$ "$C_\Gamma(x_g)$ as determined by T^*".
then $\Leftrightarrow$ and $A = \{\{i\}^{x_g} \mid \{i\}^{x_g} \in V$ and $z(i) = 0\}$
$y \in C_\Gamma(x)$ iff $(x_0, y, x) \in p[T]$
In particular: take $x = x_g$.

$(ZF + DC)$
Theorem (Woodin) Let $M, N \models AD^+$ and
$\mathbb{R} \subseteq M \cap N$. Suppose
$\mathcal{P}(\mathbb{R})^M \nsubseteq \mathcal{P}(\mathbb{R})^N$ and $\mathcal{P}(\mathbb{R})^M \nsubseteq \mathcal{P}(\mathbb{R})^N$
Then if $\Gamma = \mathcal{P}(\mathbb{R}) \cap M \cap N$
$L(\Gamma, \mathbb{R}) \models$ All sets are Suslin

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Work throughout in ZF + DC + AD.

We describe the structure of the Suslin cardinals and scaled pointclasses under AD.

Def κ-Suslin sets, ~~[illegible]~~

Def Suslin cardinals.

Def Semiscale

Fact $A = p[T]$ for a $T \subseteq \omega \times \alpha$ iff A admits an α-semiscale.

Def Scales.

Lemma TFAE

1. A is κ-Suslin
2. A admits a κ-scale
3. A admits a κ-scale
4. A admits a very good κ-scale.

Lemma Let κ be a Suslin cardinal.

1. There is a strictly increasing sequence of κ-Suslin sets
2. If $cf(\kappa) > \omega$ then may take A_α is $<\kappa$-Suslin.

Lemma For every $\kappa^{\geq\omega}$: $S(\kappa)$ is closed under ~~[illegible]~~ $\exists^{\mathbb{R}}$, $\wedge_\omega$, $\vee^\omega$

Theorem (Kechris) If κ is a Suslin cardinal then each A in $S(\kappa)$ is non-selfdual

Def Γ-norm, Γ-scale

Def pwo(Γ), scale(Γ)

Pointclasses

A pointclass Γ is a collection $\Gamma \subseteq \mathcal{P}(\mathbb{R})$ closed under $\leq_W$.

Lemma Wadge's lemma, Wadge degrees

Theorem (Martin-Monk) $\leq_W$ is well-founded

$o(\Gamma) = \sup\{o(A) \mid A \in \Gamma\}$

<u>Def</u> Self-dual Wadge degrees.

Nonselfdual pointclasses correspond to nonselfdual Wadge degrees. Can occur in 4 different ways.

<u>Thm</u> (Steel, van Wesep)

<u>Def</u> Sep(Γ)

<u>Theorem</u> For nonselfdual Γ: exactly one of $\Gamma, \check{\Gamma}$ holds

<u>Fact</u> For any nonselfdual Γ : pwo(Γ) $\Rightarrow \neg$sep(Γ)
(Need Γ closed under $\vee$)

<u>Def</u> A nonselfdual Γ is a <u>Lévy</u> pointclass iff
Γ is closed under $\exists^{\mathbb{R}}$ or $\forall^{\mathbb{R}}$
Σ^1_α, $\alpha < \Theta$ enumerate Lévy pointclasses closed under $\exists^{\mathbb{R}}$
Π^1_α, $\alpha < \Theta$ —"— $\forall^{\mathbb{R}}$

Projective-like hierarchies: Steel: Closure properties of pointclasses (Cabal)

<u>Def</u> $\delta(\Gamma)$ = sup of lengths of all pwo's in Γ.

<u>Fact</u> If Γ is selfdual, closed under $\exists^{\mathbb{R}}$ then $o(\Gamma) = \cancel{\ }\ \delta(\Gamma)$

<u>Def</u> (Steel) For a pointclass Γ:

$$\Lambda(\Gamma) = \bigcup \{ \Lambda(\Gamma) \mid \Lambda \text{ is selfdual closed under } \exists^{\mathbb{R}}, \wedge \}$$

So $\Lambda(\Gamma)$ is selfdual, closed under $\exists^{\mathbb{R}}$, $\wedge$ and
$\lambda = o(\Lambda(\Gamma))$ is a limit ordinal.

<u>Type 1</u> $cf(\lambda) = \omega$ $\quad$ B is self-dual and $\exists^{\mathbb{R}} B = \bigvee_\omega \Lambda$

Build projective hierarchy over it

<u>Types 2 & 3</u> $cf(\lambda) > \omega$, $o(B) = \lambda$ $(B, \mathbb{R} - B)$ is nonselfdual.
Assume they are not closed under quantifiers.
Let Σ^λ_{-1} be the side with the separation property
and Π^λ_{-1} the other side. Build $\Sigma^\lambda_n, \Pi^\lambda_n$.

(Type 3 means Σ^λ_{-1} closed under $\wedge$, type 2 not.)

<u>Type 4</u> $cof(\lambda) > \omega$ and both $B, \mathbb{R} - B$ closed under quantifiers. Let $\Gamma = \Gamma(\lambda)$ be the side with the separation property. Let $\Sigma^\lambda_1 = \Gamma \vee \breve{\Gamma}$, $\Pi^\lambda_1 = \Gamma \& \breve{\Gamma}$
Apply quantifiers to build $\Sigma^\lambda_n, \Pi^\lambda_n$.

The classes Π^λ_{-1} are called Steel pointclasses.

The $\Sigma^\lambda_n, \Pi^\lambda_n$ $n \geq -1$ ~~a~~ enumerate all Lévy pointclasses.

Prewellordering property

Type 1 $A = \bigcup_{n\in\omega} A_n$ each $A_n \in \Delta$. Let
$\varphi(x) = \mu k\,(x \in A_k)$. Then $x <^* y$ iff $(\exists k)(x \in A_k \,\&\, y \notin A_k)$
Similarly for $\leq^*$. Propagate by periodicity.

Types 2 & 3 Then $\Gamma = \Pi^{\lambda}_{-1}$ has the pwo. By AD$^+$ there is some ρ s.t. Δ is not closed under ρ-unions.
Let ρ be least possible. Then $\rho = cf(\lambda)$.
$(\rho \geq cf(\lambda)$ by Coding Lemma$)$
$(\rho \leq cf(\lambda)$ by computing with cofinalities$)$

Following Steel:

$\Gamma^* = \{ \bigcup_{\alpha<\rho} A_\alpha \mid (\forall\alpha)(A_\alpha \in \Delta) \;\&\; \{A_\alpha\}_{\alpha<\rho}$ is Σ^1_1-bounded$\}$

Claim $\Gamma = \Gamma^*$

Type 4 We have $\Gamma = \Gamma^* = \exists^{\mathbb{R}}\Gamma = \bigcup_\lambda \Delta$. Note: $\rho = \lambda$.
$A = B \cap (\mathbb{R} - D)$ are bounded union.

Useful Fact

Lemma (Martin) Let Γ be a non-selfdual ptclass closed under $\forall^{\mathbb{R}}$, $\vee$ and assume pwo(Γ). Then δ is closed under $< \delta(\Delta)$ unions and intersections.

Wellordered unions

Theorem Suppose Γ is non-selfdual, pwo(Γ) and $\exists^{\mathbb{R}}\Gamma \subseteq \Gamma$.
Then Γ is closed under well-ordered unions.

Case 1 Γ closed under $\bigwedge^{\omega}, \bigvee^{\omega}$ but not $\forall^{\mathbb{R}}$. Let

$\delta_1 = \sup\{|\prec| \mid \prec \in \Gamma$ & $\prec$ wellfounded$\}$

$\delta_2 = \sup\{|\prec| \mid \prec \in \exists^{\mathbb{R}}\check{\Gamma}$ & $\prec$ wellfounded$\}$

So $\delta_1 \leq \delta_2$. Let

ρ = least s.t. $\bigcup_{\alpha<\rho} A_\alpha \notin \Gamma$, each $A_\alpha \in \Gamma$. Then ρ $>\omega$ regular

Easily $\delta_1 < \delta_2$ (use closure of Γ under $\wedge^{\omega}$). We must have $\bigcup_\rho \Gamma \supseteq \exists^{\mathbb{R}}\check{\Gamma}$. Then $\rho \geq \delta_2$: Otherwise: if ρ' = the least s.t. $\Delta_1 = \Delta(\exists^{\mathbb{R}}\check{\Gamma})$ not closed under w.o.u.

then $\bigcup_{\rho'} \Delta_1 = \exists^{\mathbb{R}}\check{\Gamma} \Rightarrow$ pwo$(\exists^{\mathbb{R}}\check{\Gamma})$. ↯

So $\delta_1 < \delta_2 < \rho$.

Let $\prec$ be wf + $|\prec| \geq \delta_1$, $\prec \in \exists^{\mathbb{R}}\check{\Gamma}$. Write

$\prec = \bigcup_{\alpha<\rho} A_\alpha$, each $A_\alpha \in \Gamma$.

For $x \in \mathrm{dom}(\prec)$ let $\xi(x)$ = the eventual (stabilises - using the fact that ρ is regular.) rank of x in A_α.

This is an order-preserving map of $\prec$ into δ_1. ↯.

Rem λ a limit of Woodins. $j : V \to M = Ult(V, G)$
G generic for $\mathbb{Q}_\delta$ (δ Woodin, $\delta < \lambda$). Then ~~...~~ it is not necessarily true that: $j(\mathrm{Hom}_{<\lambda}) = \mathrm{Hom}_\lambda^{V[G]}$
If "$x \in \mathbb{R} \Rightarrow x \in C_{(\Sigma^2_1)\mathrm{Hom}_{<\lambda}}$" holds in V, ~~Res~~ $x \notin (C_{(\Sigma^2_1), \mathrm{Hom}_{<\lambda}})^{V[G]}$, o.w. letting
let $x \in \mathbb{R}^M - \mathbb{R}^V$, $M \models x \in C_{(\Sigma^2_1)\mathrm{Hom}_{<\lambda}}$ but
(This is true in M_ω. B is a $\mathrm{Hom}^{M_\omega}_{<\lambda}$ - IS for some $M_\omega | \alpha$ with $j : V \to M$ as above.
$x \in \mathbb{R}^M - \mathbb{R}^V$
$M \models \{x\}$ is $\Sigma_1^{2\,\mathrm{Hom}_{<\lambda}}(\{x\})$)

B witness $x \in \left(C_{(\Sigma^2_1)\mathrm{Hom}_{<\lambda}}\right)^{V[G]}$:

Since $(HC, \in, B)^{V[G]} \prec (HC^*, \in, B^*)$
we get $x \in OD^{\mathrm{Hom}^*}$ (this is definable in a symmetric way) so $x \in V$. ↯

~~Examples~~
Exercise g, h are $<\lambda$-generic. If
$V[g][h] \models \exists B \in \mathrm{Hom}_{<\lambda}\ (HC, \in, B) \models \varphi[x]$, $x \in V[g]$
then $V[g] \models \exists B \in \mathrm{Hom}_{<\lambda}\ (HC, \in, B) \models \varphi[x]$

(ZF+DC)
<u>Theorem</u> If $M, N \models AD^+$ and $\mathcal{P}(\mathbb{R})^M \not\subseteq N$, $\mathcal{P}(\mathbb{R})^N \not\subseteq N$ and $\mathbb{R}^M = \mathbb{R} = \mathbb{R}^N$. Then letting

$$\Gamma_0 = \mathcal{P}(\mathbb{R})^M \cap \mathcal{P}(\mathbb{R})^N$$

$L(\Gamma_0, \mathbb{R}) \models$ all sets are Suslin.

<u>Proof</u> If not, let ...

κ = the largest Suslin card of $L(\Gamma_0, \mathbb{R})$

Let $S(\kappa) = \underset{\sim}{\Gamma}$ for some Γ which is inductive-like; also Scale (Γ).

Let U be a universal Γ-set. Let

$T = T_{\vec{\varphi}}$ for a Γ scale φ on U

Let A, B be sets of reals that witness the divergence, i.e.

$|A|_W = \Theta^{L(\Gamma_0, \mathbb{R})}$ in M, $A \notin N$

$|B|_W = \Theta^{L(\Gamma_0, \mathbb{R})}$ in N, $B \notin M$

Then $L(A, \mathbb{R}) \models \Theta^{L(\Gamma_0, \mathbb{R})}$ is not Suslin

$L(B, \mathbb{R}) \models$ –"–

Why: $\overset{AD \Rightarrow}{AD^+} \Rightarrow$ the next Suslin cardinal after κ is ω-cofinal

But if $\langle A_n \mid n \in \omega \rangle$ is Wadge-cofinal in Γ then $\langle A_n \mid n \in \omega \rangle \in M \cap N$ ($A_n \leq A$ via σ_n and "$\langle \sigma_n \mid n \in \omega \rangle \in \mathbb{R}$")

Let $T^*_A = (\prod_{\mathcal{D}} T / \text{martin})^{L(A, \mathbb{R})}$

$T^*_B = (\prod_{\mathcal{D}} T / \text{martin})^{L(B, \mathbb{R})}$

Then A has a code relative to T^*_A

B –"– T^*_B

We need: on a Turing cone: of x:

$A \cap L[T, x] \in L[T, x]$

<u>Def</u> $\mathrm{ENV}(T) = \{ A \mid A \cap L[T, x] \in L[T, x] \}$

Recall: (p, β) codes A relative to T^*_A iff
$p \Vdash^{\mathbb{P}_{sacks}}_{L(A,\mathbb{R})}$ for $z = \beta^{th}$ real in $<^{T^*_A}_{x_g}$ we have:

$$\text{if } \{i\}^{x_g} \in \mathbb{R}^V \text{ then } z(i) = 0 \text{ iff } \{i\}^{x_g} \in A$$

We showed that there is a (p, β)-coding A relative to T^*_A.
Now recall:

$x <_{c_p(x)} y$ iff $(c, x, y) \in p[T]$. So

$y <^{T^*_A}_{x_g} z$ iff $(c_0, y, z) \in p[T^*_A]$.

Can also get γ s.t.
(p, γ) codes B relative to T^*_B in $L(B, \mathbb{R})$.
We may assume $\beta \leq \gamma$. (the same p!) So
$p \Vdash^{\mathbb{P}_{sacks}}_{L(B,\mathbb{R})} <^{T^*_B}_{x_g}$ has a β-th real

<u>Claim</u> $L(B, \mathbb{R}) \models x \in A \iff \exists q \leq p \ \exists i$
$$q \Vdash^{\mathbb{P}_{sacks}} z_\beta^{<^{T^*}_{x_g}}(i) = 0 \ \& \ \{i\}^{x_g} = x$$
so $A \in L(B, \mathbb{R})$, contradiction.
Why: otherwise we could find $i, q \leq p$ s.t.
$$q \Vdash^{\mathbb{P}_{sacks}}_{L(\mathbb{R},A)} z_\beta^{<^{T^*_A}_{x_g}}(i) = 0$$
and
$$q \Vdash^{\mathbb{P}_{sacks}}_{L(\mathbb{R},B)} z_\beta^{<^{T^*_B}_{x_g}}(i) = 1$$

Let $\pi : H \to V$ where H countable transitive
with everything relevant in $\mathrm{rng}(\pi)$.
$$\pi(S^*) = T^*_A \ , \ \pi(U^*) = T^*_B$$

Then $S^* \longrightarrow T$
$U^* \longrightarrow T$

Since Martin's measure is countably complete.
Here we are using full DC.

Let G be $\mathbb{P}_{sacks}$ generic / H with $g \in V$. So $g \in HC$.
$x_g \in \mathbb{R} = \mathbb{R}^M = \mathbb{R}^N$

Then
$z_{\bar\beta}^{<_{x_g}^{S^*}} = z_{\bar\beta}^{<_{x_g}^{U^*}}$ when $\bar\beta = \pi^{-1}(\beta)$.

Why: Let $I(w, z_{\bar\beta}^{<_{x_g}^{S^*}}, x_g)$ "S^*" in $L(\bar A, \mathbb{R})^H[x_g]$

Then $I(w, z_{\bar\beta}^{<_{x_g}^{S^*}}, x_g)$ holds in V, and $S^* \to T$.
so $<_{x_g}^{S^*}$ and $<_{x_g}^{U^*}$ are compatible. $\boxtimes$

Now go back to:
- λ a limit of Woodins
- G is $Col(\omega, <\lambda)$ - generic / V
- $\mathbb{R}^*_G$, Hom^*_G,
- $\mathcal{A}_G = \{ A \in \mathbb{R}^*_G \mid A \in V(\mathbb{R}^*)$ and $L(\mathbb{R}^*, A) \models AD^+ \}$

<u>Lemma</u> Let $A, B \in \mathcal{A}_G$. Then $A \leq_W B$ or $B \leq_W \mathbb{R} - A$

<u>Proof</u> Wma $A \notin Hom^*$. (But for all we know
$Hom^* \notin L(A, \mathbb{R}^*)$)

Let κ = the largest Suslin in $L(A, \mathbb{R}^*)$
Γ inductive like, Scale (Γ), $\underset{\sim}{\Gamma} = S(\kappa)$
U universal Γ-set, $U = p[S]$, S on $\omega \times \kappa$
Let $\varphi_0 : U \longrightarrow \kappa$ be a Γ-norm on U

Let $p[T_0] = \{(x,y) \mid x \le_{\varphi_0} y\}$ $\quad T_0$ on $\omega \times \kappa$

and T_1 on $\omega \times \kappa$ for $y \in U$

$p[(T_1)_y] = \{x \mid x \not\le_{\varphi_0} y\}$

Wma $T_0, T_1 \in V$. (in $V(\mathbb{R}^*)$ so in some $V[h]$)

<u>Claim</u> $p[T_0]^V$ is a pwo of $p[S]^V$ and each proper init of $p[T_0]^V$ is $\mathrm{Hom}^V_{<\lambda}$.

$((T_1)_y, T_0)$ give $<\lambda$ absolute complements.

<u>Claim</u> $U \notin \mathrm{Hom}^*$.

<u>Pf</u> If so, have $C \in \mathrm{Hom}^*$ coding a scale on U.

why? → So $L(C, \mathbb{R}^*) \models AD^+$. Since κ is the largest Suslin of $L(A, \mathbb{R}^*)$: $C \notin L(A, \mathbb{R}^*)$.

On the other hand: if $A \in L(C, \mathbb{R}^*)$ then $A \le_w C$, so $A \in \mathrm{Hom}^*$, as C is.

So $L(A, \mathbb{R}^*)$ and $L(C, \mathbb{R}^*)$ diverge past U, but $L(A, \mathbb{R}^*)$ has no more Suslin cardinals, so contradiction - if we had DC in $V(\mathbb{R}^*)$.

To fix this

$j : V \to M_\infty$ $\qquad \mathbb{R}^* = \mathbb{R}^{M_\infty}$

$M_\infty = \lim_\alpha \mathrm{Ult}(V, H \restriction \alpha)$

↳ sufficiently well-founded.

$C = (\bar{C})^*$, $\bar{C} \in V$, $j(\bar{C}) = C$. (As $C \in \mathrm{Hom}_{<\lambda}$)

But $A \in M_\infty$!

Let (p, β) code A relative to $S^* = \prod_{\mathcal{D}} S$ / martin

$S^* \to j(S^*)$. Then

$<^{S^*}_{x_g}$ is an initial segment of $<^{j(S^*)}_{x_g}$

for x_g $\mathbb{P}_{\text{acts}}$ generic over "everything".

(p,β) codes A relative to $j(S^*)$ in M_∞. So $A \in M_\infty \models ZFC$.
Now apply "no divergent models" in M_∞.
Thus $U \notin \mathrm{Hom}^*$. $\boxtimes$.

<u>Exercise</u> Every $\mathrm{Hom}^V_{<\lambda}$ set is $\leq_W$ some proper IS of $p[T_0]^V$.

Now as $U \notin \mathrm{Hom}^*$, the tree production lemma fails for
$\varphi(x,S) \equiv x \in p[S]$.
This means that the stationary tower correctness fails.
We have $j: V \to M \subseteq \mathrm{Ult}(V,H)$ $\quad H$ generic on $\mathbb{P}_{<\delta}$, $\delta < \lambda$.
$x \in p[j(S)] - p[S]$

Now:
$$\begin{tikzcd} V \arrow[r, "i"] \arrow[rr, bend right, "j"'] & M \arrow[r, "k"] & M_\infty \end{tikzcd}$$
$\leftarrow$ sufficiently wf
$\mathbb{R}^{M_\infty} = \mathbb{R}^*$

Using T_1 get: $p[T_0] \cap (\mathbb{R}^*)^2$ is a proper IS of
$p[j(T_0)] \cap (\mathbb{R}^*)^2$ as $x \in p[j(S)]$.
use $T_0 \to j(T_0)$, suborder
$T_1 \to j(T_1)$ no new points below $\mathrm{fld}(p[T_0])$.

Then $U \in M_\infty$ and $U \in j(\mathrm{Hom})$

Also $A \in M_\infty$ using $S^*_A \to j(S^*_A)$ as before
So $A \in j(\mathrm{Hom}_{<\lambda})$ using divergent models Thm in M_∞.

Similarly: $B \in j(\mathrm{Hom}_{<\lambda})$ for the same j.

But then $B \notin \mathrm{Hom}^*$ as ow $B \in j(\mathrm{Hom}_{<\lambda}^{\,\text{wma}})$

A similar argument to that above shows: $B \in M_\infty$.

U_B = universal set at largest Suslin of $L(B, \mathbb{R}^*)$

$U_B \in j(\mathrm{Hom}^V_{<\lambda})$ so $B \in j(\mathrm{Hom}^V_{<\lambda})$

So $A \leq_W B$ or $B \leq_W \mathbb{R} - A$ via some $\sigma \in \mathbb{R}^*$. $\boxtimes$

<u>Lemma</u> Either Hom^* is closed under # or
$(\exists A \in \mathrm{Hom}^*)\ L(A, \mathbb{R}^*) = L(A_G, \mathbb{R}^*)$

<u>Pf</u> Exercise. $\boxtimes$

So wma: Hom^* is closed under #.

<u>Corollary</u> $\mathrm{Hom}_{<\lambda}$ is closed under #.

<u>Pf</u> Exercise, "not quite trivial".

<u>Case 1</u> $\mathrm{Hom}^* = A_G$ (the case in the old DMT paper)

<u>Case 2.</u> Otherwise for $A \in A_G - \mathrm{Hom}^*$ we have
$\mathrm{Hom}^* \subseteq L(A, \mathbb{R})$. ← Claim 1

Let κ = the largest Suslin of $L(A, \mathbb{R}^*)$

Then κ = "Θ^{Hom^*}"

Let $\underset{\sim}{\Gamma} = S(\kappa)$, Γ inductive like, T tree on $\omega \times \kappa$

$p[T]$ = universal Γ-set

$T^*_\alpha = (T^*)^{L(A, \mathbb{R})}$ for any/all A with $|A|_W = \alpha$

$T^* = \bigoplus T^*_\alpha$

$T, T^* \in V[h]$ some $H \in HC^*$

$$\begin{tikzcd} & M \arrow[d] \\ j : V[h] \arrow[ur, "i"] \arrow[r, "j"] & M_\infty \end{tikzcd}$$

$x \in p[i(T)] - p[T]$

$\mathbb{R}^{M_\infty} = \mathbb{R}^*$ and M_∞ sufficiently well-founded

Then using $T^* \longrightarrow j(T^*)$:

$A_G \subseteq M_\infty$. No divergence models in M_∞,

$A_G \subseteq j(\mathrm{Hom}_{<\lambda})$

<u>Remark</u> $M_\varphi \prec_{\Sigma_1} M_{\mathcal{P}(\mathbb{R})}$ in $L(A_G, \mathbb{R}^*)$

This is an easy by-product: In each case, it is how we showed $L(A_G, \mathbb{R}^*) \models AD^+$

$M_{\mathcal{P}(\mathbb{R})} \prec_{\Sigma_1} V$

But to prove: $M_\varphi \prec_{\Sigma_1} V$ in $L(A_G, \mathbb{R}^*)$ takes more work.

Wellordered unions

We are proving:

Theorem Γ is nonselfdual, pwo(Γ) and $\exists^{\mathbb{R}}\Gamma \subseteq \Gamma$.
Then Γ is closed under well-ordered unions.

We did Case 1 Γ closed under $\wedge^\omega, \vee^\omega$ but not $\forall^{\mathbb{R}}$

Case 2 Γ closed under $\forall^{\mathbb{R}}$. So Γ is closed under $\wedge^\omega, \vee^\omega$.
By pwo(Γ): $\check{\Gamma}$ not closed under well-ordered unions.
We have ρ_1 for Γ and ρ_2 for $\check{\Gamma}$.
We show $\rho_1 < \rho_2$. Notice $\bigcup_\rho \Gamma \supseteq \check{\Gamma}$. We have $A \in \bigcup_\rho \Gamma$
$A \in \check{\Gamma} - \Delta$. Write A as Σ^1_1-bounded union. Play game
I x U: universal Γ-set. I ~~plays a code~~ ~~or a sequence~~ plays $x \in A$ say α
II y, z II plays y coding $\beta > \alpha$ and
$z \in A_\alpha - \bigcup_{\xi<\alpha} A_\xi$ for some ~~..~~ $\alpha \geq \varphi(x)$.
Standard: I cannot win. Let τ be a WS for II.
Define a relation $<$ ~~pwo~~ on A:
$x < y$ iff $x, y \in A$ iff $(\tau(y))_1 \notin A_{(\tau(x))_0}$
Then $<$ is a well-founded relation of length ρ_1,
since ρ_1 is regular. By coding lemma: $\rho_1 > \rho_2$
Likewise show $\rho_2 > \rho_1$. Contradiction $\boxtimes$

Case 3 Γ not closed under ~~..~~ $\forall\omega^\omega$, $\wedge^\omega$. ~~..~~ $\forall^{\mathbb{R}}$
Actually we only use "not closed under ~~..~~"
Pf Here δ cf(λ) $= \omega$. (λ of type 1). Get ρ.
$\bigcup_\rho \Gamma \supseteq \exists^{\mathbb{R}}\check{\Gamma}$. As before let S be a universal Σ^1_1-set
Note: Γ closed under $\wedge, \vee$

Play the game and define $\prec$ as before. By coding lemma: $\bigcup_\rho \Gamma \subseteq \Sigma_1^\lambda$ and $\bigcup_\rho \Gamma = \Sigma_1^\lambda$.

Theorem (Chuang) Let Γ be nonselfdual closed under $\forall^{\mathbb{R}}$ and $\vee$. There is no strictly increasing or decreasing sequence of length $(\delta(\Gamma))^+$. Here
$\delta(\Gamma)$ = sup of all lengths of $\Delta = \Gamma \cap \check{\Gamma}$ - pwo's.

Pf Let $\{A_\alpha\}_{\alpha<\delta^+}$ be a strictly increasing sequence of Γ-sets.
Wma: $\bigcup_{\xi<\alpha} A_\xi \subsetneq A_\alpha$. Since $pwo(\Gamma) = pwo(\exists^{\mathbb{R}}\Gamma)$.
Let φ be the norm corresponding to $\{A_\alpha\}_\alpha$
The associated pwo is in $\exists^{\mathbb{R}}\Gamma$. Let U be universal for Γ.
Define:
$C(x,y,z) \iff \exists \alpha<\delta^+ \; U_x = A_\alpha \;\&\; y,z \in A \;\&\; \varphi(y)=\alpha \;\&\; \varphi(z)=\alpha$
Let $S \subseteq C$ be a choice set by Coding Lemma, $S \in \exists^{\mathbb{R}}\Gamma$
Now: any set in Δ is a δ-union of ~~&~~ Δ-sets, so
S is a δ-union of sets in $\exists^{\mathbb{R}}\Gamma \subseteq \check{\Gamma}$. Say $S = \bigcup_{\xi<\delta} S_\xi$
Define:

$$
\begin{aligned}
(x_1,y_1,z_1) \leq_\beta (x_2,y_2,z_2) &\iff \text{both in } S_\beta \;\&\; \varphi(y_1) \leq \varphi(y_2) \\
&\iff \quad \text{—"—} \quad \& \quad y_1 \in U_{x_2} \\
&\iff \quad \text{—"—} \quad \& \quad z_2 \notin U_{x_1}
\end{aligned}
$$

So $\leq_\beta \in \check{\Gamma}$. This gives an injective map $\delta^+ \to \delta \times \delta$ by $\alpha \mapsto (\beta,\gamma)$. For α let β be the least s.t. some $(x,y,z) \in S_\beta$ and $\varphi(y)=\alpha$. Then let $\gamma = |(x,y,z)|_{\leq_\beta}$

Case 2: Γ closed under $\exists^{\mathbb{R}}$
Pf As before, just $\prec \in \exists^{\mathbb{R}}(\Gamma \,\&\, \check{\Gamma})$

Q: Can one remove the assumption on the closure under $\forall$ in Chuang's theorem?

Rem There are 2 cases where $\forall$ ....

Will need: (black box)

Theorem (Steel, Woodin)
$AD \Rightarrow$ Suslin cardinals are closed ~~and~~ below their sup
$AD^+ \Rightarrow$ The Suslin cardinals are closed

Theorem (Martin, Steel) ~~ZF~~(AD) if $\kappa <$ sup of Suslins and T is a tree on $\omega \times \kappa$ then T is weakly homogeneous

Type 1 Case κ is a limit Suslin cardinal.

First consider $cf(\kappa) = \omega$. Let $\Lambda = S(\kappa)$. So Λ is closed under $\exists^{\mathbb{R}}, \forall^{\mathbb{R}}$ and $\neg$. Let $\lambda = o(\Lambda)$.

Claim $\lambda = \kappa$.
Pf If $\lambda > \kappa$ then is a Λ-pwo of length κ. By Code L $S(\kappa) \in \Lambda$, ↯. So $\lambda \leq \kappa$. Will show that there are Λ-pwo's of length $\kappa' < \kappa$.
Let $(B_\xi \mid \xi < \kappa')$ be ~~an~~ increasing, $B_\xi \in S(\kappa')$.
Find $\Gamma' \subseteq \Lambda$ with $\exists^{\mathbb{R}} \Gamma' \subseteq \Gamma'$, pwo$(\Gamma')$ ...
By closure properties of $S(\kappa)$: $S(\kappa) \supseteq \Sigma^{\kappa}_1$.
By Coding lemma: $S(\kappa) \subseteq \Sigma^{\kappa}_1$.

We show: Scale(Σ^{κ}_0). A25

Now given $A \in \Sigma^{\kappa}_0$: Write $A = \bigcup_{n \in \omega} A_n$ where $A_n \in S(\kappa_n)$. Define a scale in the standard way.

Then propagate by periodicity.

By Kunen-Martin: all the norms there are $\le \kappa^+$.

By Coding lemma, $\kappa^+ =$ sup of lengths of Σ^{λ}_1-wf relations.

So: κ^+ is regular.

→ Quick argument on the behavior for Σ^{λ}_n.

<u>Type 2,3 Case</u> $cf(\kappa) > \omega$.

Let $\Delta = S(<\kappa)$. Recall: Π^{λ}_{-1} not closed under $\exists^{\mathbb{R}}$

As before $\lambda = \kappa$.

Σ^{κ}_0 is closed under wellordered unions, so
$$S(\kappa) = \bigcup_{\kappa} S(<\kappa) \subseteq \Sigma^{\kappa}_0 .$$

So either $S(\kappa) = \Sigma^{\kappa}_0$ or $S(\kappa) = \Sigma^{\kappa}_{-1}$. We show the latter does not occur.

Let $\Gamma =$ the collection of all $A = p[T]$ where T is homogeneous on κ

Have: $S(\kappa) = \exists^{\mathbb{R}}\Gamma$ as every tree on $\omega \times \kappa$ is weakly homogeneous. So $\Gamma \notin \Delta$

<u>Claim</u> $\forall^{\mathbb{R}}\Gamma \subseteq S(\kappa)$

<u>Cor</u> $S(\kappa) = \Sigma^{\lambda}_0$

<u>Proof</u> Let $A(x) \leftrightarrow (\exists y) B(x,y)$ where ~~[illegible]~~ = $B \in \Gamma$, i.e. $B = p[T]$ for some homogeneous T on $\omega \times \omega \times \kappa$.

Play the game. Fix $x \in A$.

I II
y $\vec{\alpha}$

II has a w.s., as this is a closed game for II.

By the usual determinacy arguments for homogeneous

So $S_\kappa = \Sigma_0^\lambda$.

<u>Cor</u> $\Pi_{-1}^\lambda = \{p[T] \mid T \text{ is homogeneous}\}$

Now prove Scale$(S(\omega))$.

<u>Claim</u> Scale(Σ_0^λ), i.e. Scale$(S(\omega))$

<u>Pf</u> ~~Let~~ Basically same as before.

Now show: Scale(Π_{-1}^λ).
Let ρ be s.t.
$\Gamma = \Sigma_1^1$-bounded ρ-union of Δ-sets
Let $A \in \Gamma$ and U be universal Σ_1^1. Let $C = \{y \mid U_y \in A\}$
As before: $C \in \Gamma$, $C = p[T]$ some T on $\omega \times \kappa$, $U = p[S]$.
Define V on $(\omega)^3 \times \kappa$ by
$(s, t, u, \vec{\alpha}) \in S \Leftrightarrow (s, t, u) \in S$ & $(t, \vec{\alpha}) \in V$
Clearly: $A \in p[V]$. Modify as follows: get V'' on $\omega \times \omega \times \kappa$
s.t. for $(s, \vec{\alpha}) \in V''$: $\alpha_0 > \alpha_i$ $i > 0$.
The corresponding scale is in
For instance: Show $<_i^* = \bigcup_{\alpha < \rho} B_\alpha$ where

$(x, y) \in B_\alpha \leftrightarrow \exists \beta < f(\alpha)\ [(x, y) \in A$ & $\varphi_i(x) = \beta$ &
$\neg (y \in A$ & $\varphi_i(y) \leq \beta)]$

where $f: \rho \to \kappa$ is cofinal. Show this is a Σ_1^1-bounded union.

21. 7. 9:30 John Steel

Remarks

(1) AD^+ (+ $V = L(P(\mathbb{R}))$?) $\iff$

$M_\wp \models AD$ + $M_\wp \prec_{\Sigma_1} L(P(\mathbb{R}))$ (+ $V = L(P(\mathbb{R}))$?)

our proof of DMT involved showing
$L(A_G, \mathbb{R}^*) \models$ rhs

(2) $AD^+ \Rightarrow$ (a) Scale (Σ^2_1)

(b) $M_{\underset{\sim}{\Delta}^2_1} \prec_{\Sigma_1} M_\wp \prec_{\Sigma_1} M_{P(\mathbb{R})} \prec_{\Sigma_1} V$ — many

Ref: The Derived model Thm paper.

(3) $AD^+ \Rightarrow$ the class of Suslins is closed

All sets are ∞-Borel

Ordinal determinacy

(Ketchersid on website)

<u>Thm</u> Let λ be a limit of Woodins + $< \lambda$ strongs
then $A_G = \mathrm{Hom}^*$. So

$L(A_G, \mathbb{R}^*) \models$ All sets of reals are Suslin, $\wp = P(\mathbb{R})$

<u>Pf</u> Ow we have a largest Suslin in $L(A_G, \mathbb{R}^*)$

$\kappa = \delta_{\mathrm{Hom}^*}$. Let T on $\omega \times \kappa$, $p[T]$ is a w f relation
on $\mathbb{R}^*$ of rank $\geq \kappa$, $T \in L(A_G, \mathbb{R}^*)$,

Let $T \in V[h]$, $h \in HC^*$. Let h be a site μ-generic,
$\mu < \nu < \lambda$.

$V[h] \models \nu$ is $< \lambda$-strong

Let g be $V[h]$-generic for col $(\omega, 2^{2^\nu})$ and
$V[h][g] \models \mathrm{Hom}_\eta = \mathrm{Hom}_{<\lambda}$ $\eta < \lambda$.

Let $j : V[h] \to M[h]$ witnesses that ν is δ^+-strong
where $\delta =$ the 3^{rd} Woodin $> \eta$.

Then $V[h][g] \models j(T)$ is δ^+-absolutely complemented.
So

$$V[h][g] \models p[j(T)] \text{ is } \mathrm{Hom}_{<\lambda}$$
$$\models (S, U) \text{ is a } <\lambda \text{ a.c. pair s.t.}$$
$$p[U] = p[j(T)].$$

Now: (1) $p[U] \cap \mathbb{R}^*$ is a wf relation.
(2) $p[T] \cap p[S] = \emptyset$, so $p[T] \cap \mathbb{R}^* \subseteq p[U]$.
So $p[U] \cap \mathbb{R}^*$ is a Hom^* wf relation of rank $\geq \kappa$.
Contradiction. $\boxtimes$

<u>Exercise</u>: if λ is a limit of Woodins and $\exists \kappa < \lambda$ s.t. κ is $<\lambda$ strong then
$$L(A_G, \mathbb{R}^*) \models \Pi^2_1 \text{ sets are Suslin}$$
Equivalently: $\Theta_0 < \Theta$

<u>Notation</u> $D(M, \lambda)$ for $M \models ZFC + \lambda$ a limit of Woodins "is" $L(A_G, \mathbb{R}^*_G)$ for G $Col(\omega, <\lambda)$-generic.

<u>Exercise</u> (a) $D(M_\omega, \lambda_\omega) \models V = L(\mathbb{R})$
(b) $D(M_{\omega+\omega}, \lambda_{\omega+\omega}) \models V = L(\mathbb{R})$
(c) $D(M_{\omega+\omega}, \lambda_\omega) \models \mathbb{R}^\# \in \mathcal{S}$

— ∗ — ∗ — ∗ — ∗ —

<u>CONVERSE DIRECTION</u>

Given $M \models AD^+$, realize it as a derived model.

LARGEST SUSLIN CARDINAL CASE

Suppose in V we have: AD^+ + there is a largest Suslin.
Let T on $\omega \times \kappa$ witnesses: κ = the largest Suslin.
T^* as before, so $V = L(T^*, \mathbb{R})$ (Σ_1 fact)
(Every $A \subseteq \mathbb{R}$ is "countably captured" over T
$A \cap L[T,x] \in L[T,x]$ on a cone of x)

Claim For a cone of reals z:
$$\mathbb{R} \cap L[T,z] = \{x \in \mathbb{R} \mid x \text{ is } OD(T,z)\}$$
$$\overset{\shortparallel}{\mathbb{R} \cap L(T^*,z)}$$

Pf If not, let
$f(z)$ = the least $OD(T,z)$-real not in $L[T,z]$.
$A(z,n,m) \overset{\text{def}}{\iff} f(z)(n) = m$
Have $A \cap L[T,z] \in L[T,z]$ on a cone of z. Gives $f(z) \in L[T,z]$. $\boxtimes$

Let x_0 = base of a cone z from above. Claim holds.
If a is countable transitive with $x_0 \in a$:
$$\mathcal{P}(a) \cap L[T,a] = \{b \subseteq a \mid b \text{ is OD from } \{T,a\} \text{ and members of } a\}$$
Pf: Exercise; reduce this to the case of reals.

Let $\mathbb{P}$ be the Prikry tree forcing corresponding to the Martin measure on $\mathcal{D}$:
Conditions: (s,F) when s is a finite sequence of Turing degrees and
$$F : \mathcal{D}^{<\omega} \to \mathcal{P}(\mathcal{D})$$
$$F(s) \text{ is of measure } 1$$
$$T_F = \{u \in \mathcal{D}^{<\omega} \mid (\forall k \in dom(u))(u(k) \in F(u \restriction k))\}$$

then let

$(s, F) \leq (t, G)$ iff
$s \in T_G$ and $t \subseteq s$ and $F(v) \subseteq G(v)$ for $s \subseteq v$

If ~~…~~ g is $\mathbb{P}$-generic then
$\bigcup \{ s \mid \exists F \ (s, F) \in g \} \in \mathcal{D}^\omega$ s.t.
$\forall d \in \mathcal{D} \ \exists k \ d \leq_T s(k)$

Basic property Let $\{\varphi_i \mid i \in \omega\}$ be a family of sentences in the forcing language. Let $(s, F) \in \mathbb{P}$ then $\exists G$ s.t. $(s, G) \leq (s, F)$ and for all i
$(s, G) \Vert \varphi_i$ (Proof without DC - ~~…~~ Trang's paper)

Theorem (Woodin) (AD) If R, S are sets of ordinals then on a T cone of x:
$\mathrm{HOD}_R^{L[S, R, x]} \models \omega_2^{L[S, R, x]}$ is Woodin

Let a be ctbl transitive, admit a well-order ~~…~~ rudimentary in a. Let $x \in \mathbb{R}$ be s.t. a is coded by some $y \leq_T x$. Set

$$\delta^x = \omega_2^{L[T^*, x]}$$
$$Q_a^x = \mathrm{HOD}_{T^*, a}^{L[T, x]} \mid \delta^x + 1 \quad (V_{\delta^x} \text{ of this HOD equipped with its canonical w.o.})$$

Note: Q_a^x depends only on $[x]_T$
So δ^d, Q_a^d for $d \in \mathcal{D}$ makes sense.

Claim Let x_0 be as in the previous claim. Let a be countable transitive with $x_0 \in a$. Then for a cone of d: $\mathcal{P}(a) \cap \mathbb{Q}^d_a = \mathcal{P}(a) \cap L[T^*, a]$.

Now take g Prikry / V.

$\langle d_i \mid i \in \omega \rangle = \bigcup \{ s \mid (\exists F)(\langle s, F \rangle \in g) \}$

assume each $\langle d_i \mid i \in \omega \rangle$ meets certain measure one sets in martin $\times \cdots \times$ martin.

Let $\mathbb{Q}_0 = \mathbb{Q}^{d_0}_{\emptyset}$

$\mathbb{Q}_{i+1} = \mathbb{Q}^{d_{i+1}}_{\mathbb{Q}_i}$

$\delta_i = \delta^{d_i}_{\mathbb{Q}_{i-1}}$.

Then $\mathcal{P}(\delta_i)^{\mathbb{Q}_i} = \mathcal{P}(\delta_i)^{\mathbb{Q}_{i+1}}$ (by restricting to measure one sets at beginning)

Let $\mathbb{Q} = \bigcup_i \mathbb{Q}_i$

Consider $L[T^*, \mathbb{Q}]$.

Exercise For any i, $\mathcal{P}(\delta_i) \cap L[T^*, \mathbb{Q}] = \mathcal{P}(\delta_i)^{\mathbb{Q}_i}$

$\triangleright$ (Claim)

Claim $\mathbb{R}^V = \mathbb{R}^*_G$ for some G - $\mathrm{Col}(\omega, <\sup_i \delta_i)$ gen / $L[T^*, \mathbb{Q}]$

Pf x gen / $L[T^*, \mathbb{Q}]$: $x \leq d_i \Rightarrow x$ gen over $\mathbb{Q}^{d_i}_{\mathbb{Q}_i}$ by Vopěnka.

$\therefore$ " for poset of size $< \sup \delta_i \cdots$

$V = L[T^*, \mathbb{R}^V] \subseteq L(A_G, \mathbb{R}^V)$

However, we actually have = there, as otherwise we get a # for $L[T^*, \mathbb{R}^V]$ by forcing.)

☒

<u>Proof of:</u> $M_{\mathcal{P}(\mathbb{R})} \prec_{\Sigma_1} V$

<u>Cases:</u>

1. There is a largest Suslin ($\Rightarrow \Theta$ is regular)
   Exercise (Take hulls)
2. No largest Suslin cardinal + $cf(\Theta) = \omega$
3. No largest Suslin cardinal + $cf(\Theta) > \omega$.

<u>Case 2</u> Strategy. Prikry force to get a model N s.t. $D(N, \omega_1^V) = V$. Let $\langle \alpha_i : i < \omega \rangle$ be s.t. $\alpha_i \nearrow \Theta$.

<u>Recall:</u> ($ZF + DC_{\mathbb{R}} + AD$) Let X be a set. The Solovay sequence (relative to X) ~~is defined by:~~
$\langle \Theta_\alpha^X \mid \alpha \leq \gamma_X \rangle$ is defined by
$\Theta_0 = \sup\{\alpha \mid \exists \pi : \mathbb{R} \xrightarrow{\text{onto}} \alpha \text{ s.t. } \pi \text{ is } OD_X\}$
$\Theta_{\alpha+1}^X = \sup\{\alpha' \mid \exists \pi : \mathbb{R} \xrightarrow{\text{onto}} \alpha' \text{ s.t. } \pi \text{ is } OD_{X,A}$ for some/all A with $|A|_W = \alpha\}$
~~limits~~ sups for α limit.

<u>Remark</u> (AD^+). For $\Theta_\alpha^X < \Theta^X$:
$\Theta_{\alpha+1}^X = \sup\{\xi \mid \exists \pi : \mathcal{P}(\Theta_\alpha^X) \xrightarrow{\text{onto}} \xi \text{ s.t. } \pi \text{ is } OD_X\}$
~~Since cf(Θ) = ω~~

Now let α_i be as above.

<u>Notation</u> $\kappa_i = \sup \underset{\sim}{\delta}^1_1(A)$ where $|A|_W = \Theta_\alpha$.

W: (a) $\Theta_{\alpha_i+1} = \sup\{\xi \mid \exists \pi : \mathcal{P}(\kappa_i) \to \xi \;\; \pi \text{ is OD}\}$

(b) $\Theta_{\alpha_i+1} = \Theta^{\mathrm{HOD}_{\mathcal{P}(\kappa_i)}}$

Here

$\mathrm{HOD}_{\mathcal{P}(\kappa_i)} = \{A \mid \forall B \in \text{t.c.}(A \cup \{A\})\;\; B \text{ is } \mathrm{OD}_C,$ some $C \in \mathcal{P}(\kappa_i)\}$ look at $\Delta - B_C$

(c) $\mathrm{HOD}_{\mathcal{P}(\kappa_i)} = \mathrm{HOD}_{Y_i}$ where

$Y_i = \{A \subseteq \mathbb{R} \mid |A|_W < \Theta_{\alpha_i+1}\}$

In case 2 we have $AD_{\mathbb{R}}$ so can construct Solovay measure

For each i let μ_i be the supercompact measure on $\mathcal{P}_{\omega_1}(\mathcal{P}(\kappa_i))$

μ_i's are unique, hence OD. Let

$X_i = \{\sigma \in \mathcal{P}_{\omega_1}(\mathcal{P}(\kappa_i)) \mid$ ① $\mathrm{HOD}_{\sigma \cup \{\sigma\}} \models AD^+ + \neg AD_{\mathbb{R}}$ &

& σ collapses to $\mathcal{P}(\kappa^\sigma) \cap \mathrm{HOD}_{\sigma\cup\{\sigma\}}$ where

② $\Theta^{\mathrm{HOD}_{\sigma\cup\{\sigma\}}}$ is a successor on the Solovay sequence.

$\kappa^\sigma = \sup_{\kappa} \delta^1_\kappa(A)$

$|A|_W =$ the predecessor of $\Theta^{\mathrm{HOD}_{\sigma\cup\{\sigma\}}}\}$

$\leq_W$

Claim $\mu_i(X_i) = 1$ all i

Fix i: Let $M = \prod_\sigma \mathrm{HOD}_{\sigma\cup\{\sigma\}} / \mu_i$ where

the ultrapower is taken in $\mathrm{HOD}_{\mathcal{P}(\kappa_i)}$.

the ultraproduct is well-founded since $\mathrm{HOD}_{\mathcal{P}(\kappa_i)} \models DC$

Let $\pi : \mathbb{R} \xrightarrow{\text{onto}} \mathcal{P}(\kappa_i)$ in $\mathrm{HOD}_{\mathcal{P}(\kappa_i)}$ (this exists)

Let $\sigma^\infty = [id]_{\mu_i}$ then we have Łoś:

$M \models \varphi[\sigma^\infty]$ iff $\forall^*_{\mu_i} \sigma \;\; \mathrm{HOD}_{\sigma\cup\{\sigma\}} \models \varphi[\sigma]$

this follows from normality of μ_i.

Note:

- σ^{μ} collapses to $\mathcal{P}(\mathbb{R}_i)$
- $\mathbb{R} \cap M = \mathbb{R}$
- $\mathcal{P}(\mathbb{R}) \cap M = \underbrace{\{A \subseteq \mathbb{R} \mid |A|_w < \Theta_{\alpha_i+1}\}}_{Y_i} = \mathcal{P}(\mathbb{R}) \cap HOD_{\mathcal{P}(\mathbb{R}_i)}$

$\subseteq$ obvious; the ultrapower is in $HOD_{\mathcal{P}(\mathbb{R}_i)}$

$\supseteq$ If $B \in Y_i$ let $f(\sigma) = B \cap \sigma$. Then $[f]_{\mu_i} \in M$. ⊠

Rem Above: $HOD_{\sigma \cup \{\sigma\}}$ played a role of $L(T^*; \mathbb{R})$

Let T_0 ~~t~~ $= \{\langle \sigma_0, \ldots \sigma_n \rangle \mid \sigma_i \in X_i$ for $i \leq n\}$

$T = \{\langle \sigma_0, \ldots \sigma_n \rangle \in T_0 \mid$

(1) $\mathcal{P}(\mathbb{R})^{HOD} = \mathcal{P}(\mathbb{R})^{HOD_s}$

(2) $\sigma_i \subseteq \sigma_j$ whenever $i < j$

(3) $\sigma_k \in HOD_{\sigma_i \cup \{\sigma_i\}}$ all $k \leq i$

(4) σ_k is countable in $HOD_{\sigma_i \cup \{\sigma_i\}}$ all $k < i$

(5) Let $\Theta_i = \Theta^{HOD_{\sigma_i \cup \{\sigma_i\}}}$ then

$HOD_{s \restriction (i+1)} \models \Theta_i$ is Woodin &

& $\mathcal{P}(\Theta_i) \cap HOD_{s \restriction (i+1)} = \mathcal{P}(\Theta_i) \cap HOD_s$

Want: Let $s = \langle \sigma_0, \ldots, \sigma_n \rangle \in T$. Then

$\forall^*_{\mu_{i+1}} \sigma$ $s^\frown\langle \sigma \rangle \in T$. Let $H = HOD_s$

Then $H = HOD_H^{HOD_{\sigma_n \cup \{\sigma_n\}}}$.

Proof $H \subseteq HOD_{\sigma_n \cup \{\sigma_n\}}$ so RHS makes sense.

$\subseteq$ is obvious.

$\supseteq$: Follows from the fact that σ_n is OD from s.

Let $\sigma = \langle \sigma_0, \ldots \sigma_n \rangle \in T$. Then $\forall^*_{\mu_{n+1}} \sigma$ $s^\frown\langle \sigma \rangle \in T$.

<u>Proof</u> Enough to check this for (5).

Let $t = s^\frown\langle\sigma\rangle$ & $H = HOD_t$.

We know: $H = HOD_H^{HOD_{\sigma\cup\{\sigma\}}}$

<u>Theorem</u> (Koellner-Woodin) Assume ZF + DC + AD.
Let X, Y be sets. Then
$HOD_X \models (\Theta_0^{X,Y})^V$ is Woodin

We want to show:
$H \models \Theta^\sigma$ is Woodin where $\Theta^\sigma = \Theta^{HOD_{\sigma\cup\{\sigma\}}}$

<u>Proof</u> Work in $HOD_{\sigma\cup\{\sigma\}}$. So $H = HOD_H$.

Note: $\Theta = \Theta_0^{X,Y}$ some $Y \subseteq \mathbb{R}$

this is because $HOD_{\sigma\cup\{\sigma\}}$ has largest Suslin, Θ is successor in the Solovay seq. (hence regular) and DC

More detailed:

$\forall^*_{\mu_{\omega+1}}\sigma$ $HOD_{\sigma\cup\{\sigma\}} \models AD^+ + \neg AD_{\mathbb{R}}$ so $HOD_{\sigma\cup\{\sigma\}}$ has a largest Suslin, $\Theta^{HOD_{\sigma\cup\{\sigma\}}}$ is a successor on the Solovay sequence && ($\overset{?}{\Rightarrow}$) regular in $HOD_{\sigma\cup\{\sigma\}}$. Also DC holds in $HOD_{\sigma\cup\{\sigma\}}$

Now apply the above theorem to $\Theta^\sigma = \Theta_0^{X,Y}$ inside $HOD_{\sigma\cup\{\sigma\}}$: Get: $HOD_H = H$ and $H \models \Theta^\sigma$ is Woodin.

To see that $\forall^*_{\mu_{\omega+1}}\sigma$ $\mathcal{P}(\Theta_n)\cap HOD_s = \mathcal{P}(\Theta_n)\cap HOD_{s^\frown\langle\sigma\rangle}$
Just need
$\forall^*_{\mu_{\omega+1}}\sigma$ $\mathcal{P}(\Theta_n)\cap HOD_{s^\frown\langle\sigma\rangle} \subseteq \mathcal{P}(\Theta_n)\cap HOD_s$.

If not: Assume ~~for~~ $\forall^*_{\mu_{\omega+1}}\sigma$ A_σ witnesses this.
Take A_σ be the least such. By normality

of $\mu_{\omega+1}$ and the fact that Θ_n is fixed countable ordinal:

$\exists A\, \forall^*_{\mu_{\omega+1}} \sigma\ A = A_\sigma$ but $A \notin$ OD from S because $\mu_{\omega+1}$ is.

We look at $\mathrm{HOD}^{L[x,G]}$

M_1 = minimal proper class $L[E]$-model s.t. $L[E] \models \exists$ Woodin cardinal and that is fully iterable

Minimal = $M_1^\#$ without last extender.

<u>Def</u> M is M_1-like iff M is a proper class $L[E]$-model and $M \models \exists$ exactly one Woodin cardinal

δ^M = the unique Woodin of M

Deal mostly with $\delta^M < \omega_1^V$.

Note: $M = L[M|\delta^M] \sim L[a]$ when $a \in HC$

<u>Question</u> Suppose $M_1|\delta^{M_1} \in L[x]$, $\delta^{M_1} < \omega_1^{L[x]}$. How much does $L[x]$ know about the unique iteration strategy for M_1?

<u>Def</u> (Uniqueness) Let $\mathcal{T}$ be a normal IT on M

Recall $\delta(\mathcal{T}) = \sup\{lh\, E_\alpha^{\mathcal{T}} \mid \alpha < lh(\mathcal{T})\}$

$$M(\mathcal{T}) = \bigcup_{\alpha < lh(\mathcal{T})} M_\alpha^{\mathcal{T}} \| lh(E_\alpha^{\mathcal{T}})$$

<u>Def</u> $\mathcal{T}$ is <u>maximal</u> iff $L(M(\mathcal{T})) \models \delta(\mathcal{T})$ is Woodin

Otherwise $\mathcal{T}$ is short,

Then: If $\mathcal{T}$ is short ~~then~~ and $b = \Sigma(\mathcal{T})$ for Σ any strategy for M_1 then $M_b^{\mathcal{T}}$ is well-founded.

Let α be least s.t. $J_{\alpha+1}(M(\mathcal{T})) \models \delta(\mathcal{T})$ is not Woodin.
Then either

- $\rho^{\omega}_{J_\alpha(M(\mathcal{T}))} < \delta^{\mathcal{T}}$ or else
- there is some set A (or function f) definable over $J_\alpha(M(\mathcal{T}))$ that witnesses the failure of Woodinness

Define $Q(\mathcal{T}) = J_\alpha(M(\mathcal{T}))$. Then:

$b =$ the unique cofinal s.t. $Q(\mathcal{T}) \subseteq M_b^{\mathcal{T}}$

(Minimality of M_1). Then we can find b in $L[x]$ by absoluteness.

Now assume $\mathcal{T}$ is maximal. Then we may not be able to find b in $L[x]$. Why: (In $L[x]$) do genericity iteration which makes x generic. (Do it outside). Then $\mathcal{T}$ can be recovered in $L[x]$. We can recover $\mathcal{T}\restriction\lambda$. If $\mathcal{T}\restriction\lambda$ is maximal then

$L(M(\mathcal{T}\restriction\lambda)) \models \delta(\mathcal{T}\restriction\lambda)$ is Woodin

So for $b = \Sigma(\mathcal{T}\restriction\lambda)$ we have $i_b(\delta^{M_1}) = \delta(\mathcal{T}\restriction\lambda)$.
So x is $\mathbb{B}^N_{\delta^N}$ generic, then $N = L(M(\mathcal{T}\restriction\lambda))$
But δ^N is a cardinal in the generic extension.
But then $lh(\mathcal{T}_b{}^\frown b) = (\omega_1+1)^{L[x]}$. But the genericity iteration cannot take so long. (here $M_1 \xrightarrow{\mathcal{T}} N$)

Summary:

- So $lh(\mathcal{T}) = \omega_1^{L[x]}$, $\mathcal{T}$ has no branch in $L[x]$.
- x is generic for $\mathbb{B}_{\delta(\mathcal{T})}^{L(M(\mathcal{T}))}$ ⊠

What we have: We know the final model in $L[x]$: $M_b^{\mathcal{T}} = L(M(\mathcal{T}))$.

This works in general: if $\mathcal{T}$ is maximal in M and $b = \Sigma(\mathcal{T})$ then $L(M(\mathcal{T})) = M_b^{\mathcal{T}}$ (and b does not drop.)

<u>Theorem</u>: Let N be M_1-like, $\mathcal{T}$ a tree on N, $\mathcal{T}$ is maximal and b, c be cofinal wf branches of $\mathcal{T}$. Then $b = c$.

<u>Lemma</u> Let N be M_1-like and Γ be a proper class of ordinals. Then

$$\mathrm{Hull}^N(\Gamma) \cap \delta^N \text{ is cofinal in } \delta^N$$

<u>Prf</u> Let $\gamma = \sup(\mathrm{Hull}^N(\Gamma) \cap \delta^N) < \delta^N$.

Exercise: $\mathrm{Hull}^N(\gamma \cup \Gamma) \cap \delta^N = \gamma$, because δ^N is regular in N

Let M = collapse of $\mathrm{Hull}^N(\Gamma)$.

then $\gamma = \delta^M$. So $L[N|\delta^M] \subseteq M \models \delta^M$ not Woodin, ↯

⊠

Pf of the theorem:

$$\begin{tikzcd} N \arrow[r, bend left, "i_b"] \arrow[r, bend right, "i_c"'] & L(M(\mathcal{T})) \end{tikzcd} \quad \text{since } \varphi(\mathcal{T}) \text{ exists.}$$

with $i_b(\delta^N) = i_c(\delta^N) = \delta(\mathcal{T})$

Let $\Gamma = \{\alpha \mid i_b(\alpha) = i_c(\alpha) = \alpha\}$. Γ is a proper class.
But $\mathrm{Hull}^{L(M(\mathcal{T}))}(\Gamma) \subseteq \mathrm{rng}(i_b) \cap \mathrm{rng}(i_c)$.
This implies $b = c$.

Lemma Take any two cofinal non-dropping branches b, c of a normal iteration tree $\mathcal{U}$ on M.
Let $\xi = i_b(\gamma) = i_c(\eta)$ with $\xi < \delta(\mathcal{T})$
Then $\gamma = \eta$ and $i_b \restriction (\gamma+1) = i_c \restriction (\gamma+1)$.

Proof Exercise: notice that from the point where the branches start to diverge we have overlaps of the extenders, and the ordinals above the critical pts. are not in the intersection of ~~so~~ the ranges. ~~So~~ Use this to analyze the situation. This will give the equalities ~~and also~~.

Def For N that is M_1-like and $s \in \mathrm{On}^{<\omega}$ let
$\gamma^N_s = \sup(\mathrm{Hull}^{N|\max s}(s^-) \cap \delta^N)$, $s^- = s - \max(s)$.
Let $H^N_s = \mathrm{Hull}^{N|\max(s)}(\gamma^N_s \cup s^-)$
So $H^N_s \cap \delta^N = \gamma^N_s$.

Lemma If $\mathcal{T}$ is a maximal iteration on N and b, c are cofinal non-dropping, N is M-like and $i_b(s) = i_c(s) = s$ then $i_b \restriction H^N_s = i_c \restriction H^N_s$,

Since $i_b \restriction \gamma^N_s = i_c \restriction \gamma^N_s$ by the last lemma and $i_b(s) = i_c(s) = s$ by hypo:

(wlog $\max(s) > \delta^N$ and this means $s \in \mathrm{wfp}(M^{\mathcal{T}}_b) \cap \mathrm{wfp}(M^{\mathcal{T}}_c)$)

Theorem Assume $M_1^\#$ exists and is fully iterable. Let $M_1^\# \leq_T x$. Let G be $Col(\omega_1 <\kappa)$-generic over $L[x]$ where κ = least inaccessible of $L[x]$. Then

$$HOD^{L[x,G]} = L(M_\infty, \Lambda_\infty)$$

where

M_∞ is an iterate of M

$\Lambda_\infty = \Sigma_{M_1} \restriction \{\mathcal{T} \mid \mathcal{T}$ on M_∞ and $\mathcal{T} \in L_{\kappa_\infty}(M_\infty)\}$

~~and~~ where

κ_∞ = the 1st inaccessible of $M_\infty > \delta^{M_\infty}$

Proof M_∞ = direct limit of all Σ-iterates of M_1 via a finite stack of $\vec{\mathcal{T}}$ of normal trees with $\vec{\mathcal{T}} \in HC^{L[x,G]}$ under the comparison maps.

Working in $L[x,G]$

$I = \{(N,s) \mid N$ is M_1-like, δ^N is ctbl, $s \in On^{<\omega}$ and N is strongly s-iterable$\}$

Here: N is s-iterable iff for any finite stack $\langle \mathcal{T}_0, .., \mathcal{T}_n, \mathcal{U}\rangle \in HC$ s.t. $\mathcal{T}_0$ is on N, $\mathcal{T}_i$ is maximal for all i or has a last model $M_{\alpha_i}^{\mathcal{T}_i}$ ~~where~~ with no drops on the branch $[0,\alpha_i]_{\mathcal{T}_i}$ and $\mathcal{T}_{i+1}$ is on $L(M(\mathcal{T}_i))$ if $\mathcal{T}_i$ is maximal and last model of $\mathcal{T}_i$ otherwise.

Also: $\mathcal{U}$ is a tree on $L(M(\mathcal{T}_n))$ if $\mathcal{T}_n$ maximal and last model of $\mathcal{T}_n$ if $\mathcal{T}_n$ is short.

Then: (1) If $\mathcal{U}$ is short then $\mathcal{U}$ has a cwfb b s.t. $Q(M(\mathcal{U})) \trianglelefteq M_b^{\mathcal{U}}$.

(2) If $\mathcal{U}$ is maximal then in $L[x,G]^{Col(\omega, \max(s))}$:
there are branches $b_0, \ldots b_n, b_{n+1}$ that are (nondropping) cofinal in $\mathcal{T}_0, \ldots \mathcal{T}_n, \mathcal{U}$ respectively s.t. $i_{b_k}(s) = s$ for all $k \leq n+1$.

Rem For any $s \in On^{<\omega}$ there is a Σ_{M_1}-iterate of M_1, call it N, in $HC^{L[x,G]}$ s.t. $L[x,G] \models N$ is s-iterable

Say that ~~N~~ N is strongly iterable iff N is iterable and for a stack $\langle \mathcal{T}_0 \ldots \mathcal{T}_n \rangle$ $\langle \mathcal{U}_0 \ldots \mathcal{U}_k \rangle$ as above on N and $b_0 \ldots b_n$, $c_0 \ldots c_k$ in $L[x,G]^{Col(\omega, \max(s))}$ two sequences witnessing s-iterability, with $M(\mathcal{T}_n) = M(\mathcal{U}_k)$, we have

$$i_{b_n} \circ \ldots \circ i_{b_0} \restriction H_s^N = i_{c_k} \circ \ldots \circ i_{c_0} \restriction H_s^N$$

Σ_{M_1} is an s-iteration strategy on N as N is a Σ_{M_1}-iterate of M_1

Exercise. Hint: Σ_{M_1} has the Dodd-Jensen property.

Say $(N,s) \leq (M,t)$ for $(N,s), (M,t) \in I$ iff there is a good stack with the last model M and $s \subseteq t$.

(N,s) indexes H_s^N in the direct limit system

$\pi_{(N,s),(M,t)} : H_s^N \to H_t^M$ is the common value of all maps witnessing s-iterability of N

$$\pi_{(N,s)(M,t)} = \pi_{(R,u)(M,t)} \circ \pi_{(N,s)(R,u)}$$ by strong s-iterability

Let $\mathcal{F}$ be this direct limit system.

M_∞ = direct limit of $\mathcal{F}$

Claim $\leq^{\mathcal{F}}$ is directed.

<u>Prf</u>

$$\begin{tikzcd}
(N,s) \arrow[drr] & & \\
(M,t) \arrow[rr] & & (R, s\cup t) \\
M_1 \arrow[r, "\Sigma_{M_1}"] & (P, s\cup t) \in I \arrow[ur] &
\end{tikzcd}$$

Notice: we can have maximal trees in this comparison, so may not have cofinal branches. If the tree is maximal, we are done.

R is strongly $s\cup t$ iterable. Notice $R \in HC^{L[x,G]}$, as the outcome of the iteration may be uncountable in $L[x]$.

Let M_∞^+ = the direct limit of all Σ_{M_1} iterates of M_1 that are in $HC^{L[x,G]}$ via good stacks.

Then there is an embedding $\pi : M_\infty \longrightarrow M_\infty^+$

π is defined by a straightforward translation between the two direct limit systems.

$\forall z \in M_\infty : z = \pi_{(N,s)}(\bar{z})$ with $\bar{z} \in H^N_s$.

$$\begin{tikzcd}
(N,s) \arrow[dr] & & \\
& (R,s) \arrow[r, "\psi", "\Sigma_{M_1}"'] & M_\infty^+ \\
(P,s) \arrow[ur, "\text{compare}"'] & &
\end{tikzcd}$$

Here $M_1 \xrightarrow{\Sigma_{M_1}} P$.

We have $On \subseteq M_\infty$ since $\forall (N,s)\, \forall t\, \exists (P, s\cup t) \geq^{\mathcal{F}} (N,s)$.

Let $\delta_\infty = \delta_{M_\infty}$. Then one can show: $\delta_\infty = \delta^{M_\infty^+}$ and $\pi \restriction (\delta_\infty + 1) = id$. So $M_\infty = M_\infty^+$. ($\pi \neq id$ seems possible.)

Then $M_\infty \subseteq \mathrm{HOD}^{L[x,G]}$. Let
$\Lambda = \Sigma_{M_1,\mathcal{A}} \upharpoonright$ trees on M_1 that belong to
$M_\infty | \kappa_\infty$
where κ_∞ = the first inaccessible $> \delta_\infty$.

<u>Claim</u> $\Lambda \in \mathrm{HOD}^{L[x,G]}$

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Under $AD^+ + V = L(\mathcal{P}(\mathbb{R}))$

WTS $M_{\mathcal{P}(\mathbb{R})} \prec_{\Sigma_1} V$

Cases ① Θ regular. ($\exists$ largest Suslin card ~~regular~~ or not - in the latter case: not sure if V is a derived model)
② No largest Suslin + Θ $cf(\Theta) = \omega$
③ — " — + $cf(\Theta) > \omega$

We defined

$T = \{ \langle \sigma_0 \ldots \sigma_n \rangle \mid \sigma_i \in X_i$ and
1. $\mathcal{P}(\mathbb{R}) = \mathcal{P}(\mathbb{R})^{HOD_s}$
~~2.~~ $\forall i \leq n$: ~~$\sigma_k \subseteq \sigma_i$ and~~
2. $\sigma_k \subseteq \sigma_i$ and $\sigma_k \in HOD_{\sigma_i \cup \{\sigma_i\}}$ all $k \leq i$
3. σ_k countable in $HOD_{\sigma_i \cup \{\sigma_i\}}$ all $k < i$
4. Let $\Theta_i = \Theta^{HOD_{\sigma_i \cup \{\sigma_i\}}}$. Then $\mathcal{P}(\Theta_i) \cap HOD_{s \restriction (i+1)} = \mathcal{P}(\Theta_i) \cap HOD_s$ and $HOD_{s \restriction (i+1)} \models \Theta_i$ is Woodin

Lemma (Essentially Vopěnka) Let $s = \langle \sigma_0, \ldots, \sigma_n \rangle \in T$.
Then $\exists$ partial ordering $\mathbb{P}$ s.t.
(1) $HOD_s \models \mathbb{P}$ is a c.b.a. of size Θ_n
(2) Let $\kappa^{\sigma_n} = \sup_n \delta^1_{1n}(A)$ where $A \in HOD_{\sigma_n \cup \{\sigma_n\}}$ and $|A|_W$ = the predecessor of Θ_n on the Solovay sequence.