27.7.2010 9:30 Grigor Sargsyan -1-

<u>Conjecture</u> PFA $\Rightarrow \exists \Gamma \subseteq \mathcal{P}(\mathbb{R})$ s.t. $L(\Gamma,\mathbb{R}) \models AD^+ +$ $V_\Theta^{HOD^{L(\Gamma,\mathbb{R})}} \models \exists$ superstrong

<u>Def</u> Σ is an IS if it is a (κ,κ)-IS or κ-IS of M_Σ, the structure, is fine structural then Σ can be (κ,ω,κ)-IS (Here $M_\Sigma = \Sigma(\emptyset)$)

<u>Def</u> (Hull condensation for normal trees) Suppose Σ is an IS. then Σ has <u>hull condensation</u> iff whenever $\mathcal{T}$ is according to Σ and $\mathcal{U}$ is another tree, both on M then:
if $\sigma : lh(\mathcal{U}) \to lh(\mathcal{T})$ is $(\mathcal{U},\mathcal{T})$ order-preserving and $\pi_\alpha : M^{\mathcal{U}}_\alpha \to M^{\mathcal{T}}_{\sigma(\alpha)}$ are s.t.
$\pi_0 = id$ and all diagrams commute

$$\begin{tikzcd} M^{\mathcal{U}}_\beta \arrow[r, "\pi_\beta"] & M^{\mathcal{T}}_{\sigma(\beta)} \\ M^{\mathcal{U}}_\alpha \arrow[u, "\pi^{\mathcal{U}}_{\alpha,\beta}"] \arrow[r, "\pi_\alpha"] & M^{\mathcal{T}}_{\sigma(\alpha)} \arrow[u] \end{tikzcd}$$

Here $(\mathcal{U},\mathcal{T})$-order-preserving means the conjunction of
(i) $\alpha <_{\mathcal{U}} \beta \iff \sigma(\alpha) <_{\mathcal{T}} \sigma(\beta)$
(ii) $\xi = \mathcal{U}(\alpha+1) \iff \sigma(\xi) = \mathcal{T}(\sigma(\alpha+1))$ $\quad \sigma(\alpha)+1 = \sigma(\alpha+1)$
Under these condition: $\mathcal{U}$ is ~~made~~ according to Σ.

<u>Def</u> Σ has hull condensation iff whenever $\mathcal{T}$ is according to Σ and $\vec{\mathcal{U}}$ is a hull of $\vec{\mathcal{T}}$ then $\vec{\mathcal{U}}$ is according to Σ.

<u>Def</u> Σ has branch condensation if whenever $\vec{\mathcal{T}}$ is according to Σ, $\vec{\mathcal{U}}$ is a stack according to Σ without last model and c is a branch of $\vec{\mathcal{U}}$ s.t. if Q is the last model of $\vec{\mathcal{T}}$ and $R = M^{\vec{\mathcal{U}}}_c$ and there is $\sigma : M^{\vec{\mathcal{U}}}_c \to Q$ is s.t. the diagram commutes, then $c = \Sigma(\vec{\mathcal{U}})$.

$$\begin{tikzcd} & Q \\ M_\Sigma \arrow[ur, "\vec{\mathcal{T}}"] \arrow[dr, "\vec{\mathcal{U}}"'] & \\ & R \arrow[uu, "\sigma"'] \end{tikzcd}$$

<u>Open:</u> Does branch condensation imply hull cond? Is there some Σ without hull cond?

Def Σ is weakly commuting iff $\underset{\vec{T}}{\text{whenever}}$ $\vec{T}, \vec{U}$ are are according to Σ and $M_\Sigma \xrightarrow{\vec{U}} Q$ then the corresponding maps agree

Notation If $\vec{T}$ is according to Σ ~~then~~ with last model Q then $\Sigma_{Q,\vec{T}}$ is the induced strategy on Q

Def Σ is commuting iff for all $Q, \vec{T}$ as above $\Sigma_{Q,\vec{T}}$ is weakly commuting

Def Σ is positional iff $\Sigma_{Q,\vec{T}} = \Sigma_{Q,\vec{U}}$ for all $Q, \vec{T}, \vec{U}$.

Exercise (Via ~~ch.~~ Dodd-Jensen proof) Positional $\Rightarrow$ Commuting + Branch Condensation
(Also a version of hull cond would do it)

Exercise Suppose Σ is a strategy with hull condensation and $\pi : N \xrightarrow[\Sigma_1]{} M_\Sigma$. then Σ^π has hull condensation.
Open: Is something like this known for branch cond?

Background constructions

Suppose δ is a cardinal and V_δ is δ^+ (in fact, $\delta+1$) -iterable. Assume Σ is a δ^+-IS. ~~[illegible]~~ with hull condensation and $M_\Sigma \in V_\delta$. then we can do full background construction relative to Σ. We let $L_\delta[E, \Sigma] = N$ be this model. By FSIT N inherits a strategy from the background universe.

Exercise Suppose the strategy of V_δ has hull condensation. Show that the induced strategy of N also has hull cond.

__Stacking mice__ Suppose δ is a Woodin cardinal and V_δ is $(\delta+1)$-iterable for stacks of trees in $L_\omega(V_\delta)$. Let $\mathcal{N} = L[E]^{V_\delta}$. Let
$S(\mathcal{N}) = \bigcup\{M \mid M$ is a sound mouse over $\mathcal{N}$ projecting to $\delta\}$

__Lemma (Steel)__ $cf(o(S(\mathcal{N}))) \geq \delta$

__Proof__ If not: let $\gamma = cf(o(S(\mathcal{N}))) < \delta$. Let $f : \gamma \to o(S(\mathcal{N}))$ be cofinal. [Construct $\langle N_\xi \mid \xi < \delta\rangle$, $\langle \pi_\xi \mid \xi < \delta\rangle$ s.t.
1. $\pi_\xi : N_\xi \longrightarrow V_\lambda$ (λ large)
2. $cr(\pi_\xi) = \kappa_\xi$, $V_{\kappa_\xi} \subseteq N_\xi$]

Construct $\langle X_\xi \mid \xi < \delta\rangle$: letting λ large
1. $X_\xi \cap \delta \in \delta$
2. $X_\xi^\omega \subseteq X_\xi$ if $cf(\xi) > \omega$ or ξ successor

$\langle X_\xi\rangle$ elementary chain and $\gamma \in X_0$
Define $g : \delta \to \delta$ s.t. $g(\xi) = \sup\{h(\xi) \mid h \in X_\xi\}$
__Claim__ Let $N_\xi = \text{trcol}(X_\xi)$ if $X_\xi^\omega \subseteq X_\xi$, $\pi_\xi : N_\xi \to V_\lambda$ inverse to the collapse, $\kappa_\xi = cr(\pi_\xi)$. Then $\mathcal{P}(\kappa_\xi)^{\mathcal{N}} \subseteq \mathcal{P}(\kappa_\xi)^{N_\xi}$. If not

__Fact__ For any $A \subseteq \delta$ $\exists \kappa$ A-reflecting + the projection of the liftup of the initial segment of $\mathcal{N}$ that collapses $\mathcal{P}(\kappa_\xi)^{N_\xi}$ is δ. End of Fact

Then pick κ s.t. $\kappa = \kappa_\kappa$, κ reflects g and $\langle X_\xi \cap \delta \mid \xi < \delta\rangle$
Let E be an extender witnessing this. Then E witnesses that κ is superstrong in $\mathcal{N}$ or some initial segment of E
__Claim__ $j_E(g)(\kappa) \geq j_E(h)(\kappa)$ all $h \in \mathcal{N}$
(Exercise: get a superstrong out of this.)
__Proof__ Let $\pi : N_\kappa \longrightarrow V_\lambda$. Then $\pi(h)$ is defined, all $h \in \mathcal{N}$.
So $j_E(g)(\kappa) = \sup_{h \in X^*_\kappa} h(\kappa)$ (1) where $j_E(\langle X_\xi \mid \xi < \delta\rangle) = \langle X^*_\xi \mid \xi < j_E(\delta)\rangle$

$$\textcircled{1} \geq \sup_{h \in X_\kappa} h(\kappa) = \sup_{h \in N} j_E^{(h)}(\kappa). \qquad \boxtimes$$

Universality Suppose V_δ is $(\delta+1)$-iterable for trees in $L_\omega(V_\delta)$. Let $N = L[E]^{V_\delta}$. Then N wins the coiteration with any W of height $\leq \delta$.

Proof This is like Mitchell-Schindler, just that Woodinness replaces "no Woodins". See ATHM.

Universality (Cheapo) Assume: W is $(\delta+1)$-iterable, If $W \in V_\kappa$ where κ is the least measurable then in fact N does not move. (Can show it without the iterability hypo.) (Need that the strategy of W is moved correctly by the extenders in V_δ.)

Thick hulls Suppose $V_\delta, \delta, N, S(N)$ are as above. Let $C \subseteq o(S)$ be a club in $o(S)$. Let $H = \mathrm{Hull}_{\Sigma_1}^{S(N)}(C)$. Then H is universal.

Stack moves to stack Suppose $V_\delta, \delta, N, S(N)$ are as above. Suppose $\mathcal{T} \in V_\delta$ is a tree on $S(N)$ with last model W. Then $W = S(W|\delta)$

Lemma V_δ, δ, N as above. Let (W, Σ) be s.t. $W \in V_\kappa$ where κ = the least inaccessible. Let Σ be $(\delta+1)$-IS for W which has branch condensation. Let ξ be s.t. W is embeddable into some N_ξ, the ξ-th model of the construction. Assume $W \rightsquigarrow N_\xi$ via $\mathcal{T}$. Then

$\Sigma_{N_\xi, \mathcal{T}}$ = the induced strategy of N_ξ coming from the background universe.

Assume Σ is moved correctly.

Pf Let $\mathcal{N}$ be the induced strategy. $\mathcal{N} \overset{?}{=} \Sigma_{\mathcal{N},\sigma}$.

$$\begin{tikzcd}
V_\delta \arrow[rr, bend left, "u^*"] & & i(N_\xi) \\
W \arrow[urr, bend left, "i(T)", "i(j)"'] \arrow[r, "T", "j"'] & N_\xi \arrow[ur, "i"] \arrow[dr, "u{,}c"'] & \\
& & M^u_c \arrow[uu, "\sigma"']
\end{tikzcd}$$

<u>Claim</u> $i(j) = \sigma \circ i_c \circ j$ (because $i(j) = i \circ j$)

Now use branch condensation.

$\boxtimes$ Sketch of Proof

Application V_δ, $\mathcal{N}$, $S(\mathcal{N})$ as above.

Def P is <u>suitable</u> iff $P \models \eta$ is the only Woodin, $\forall \xi < \eta$: $L_p^P(P|\xi) \models \xi$ is not Woodin and $P \models L_p^P(P|\xi) = \bigcup \{M \trianglelefteq P \mid M$ is a pm over $P|\xi$ without overlaps$\}$

Def Suppose Σ is $(\delta+1)$-iterable for a suitable P cbbl. Σ is <u>fullness preserving</u> iff whenever $P \underset{\Sigma}{\rightsquigarrow} Q$ then $\forall \xi$ cutpoint of Q $L_p(Q|\xi) \trianglelefteq Q$.

<u>Lemma</u> V_δ, $\mathcal{N}$, P, Σ as above. Then the least strong of $\mathcal{N}$ ~~...~~ up to δ is a limit of Woodins.

<u>Pf</u> ETS: $S(\mathcal{N})$ has at least one overlapping extender. Suppose no overlaps in $S(\mathcal{N})$. Then $S(\mathcal{N}) = L_p(\mathcal{N})$ (2). Iterate P to make $\mathcal{N}$-generic. Get $P \rightsquigarrow Q$, $\mathcal{N}$ generic over Q. Then $L_p(\mathcal{N}) \in Q[\mathcal{N}]$ if AD. Also $o(L_p(\mathcal{N})) \cancel{=} \delta^Q$, So $\mathrm{cf}(o(S(\mathcal{N}))) = \omega$. Contradiction.

(2) Need to be checked – holds only in certain situations.

27.9.2010 14:00 Grigor Sargsyan.

Here is a proof that avoids (2): ETS $\exists \eta > \kappa$ strong to $\delta^{\mathcal{N}}$ s.t. $L_p(\mathcal{N}|\eta) \models \eta$ is Woodin (Notice: $L_p(\mathcal{N}|\eta) \triangleleft \mathcal{N}$ by universality)

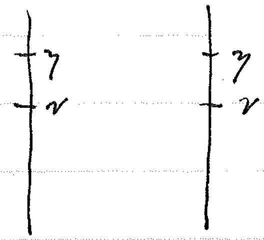
Let $\mathcal{N}^* = L[E]^{\mathcal{N}}$ built using extenders with crit pts $> \kappa$. Then $\mathcal{N}^*$ is universal. So there is some η s.t. $\mathcal{N}^*|\eta$ is a Σ-iterate of P (compare $\mathcal{N}^*$ with P; P is not moved). $\eta <$ the first

$<\delta$ strong. Let ν be the Woodin of $N^*|\gamma$. Then $N^* \models \nu$ is a Woodin (because ν is a cutpoint, & and $N^*|\gamma$ is full - strategy is fullness preserving.)

Claim $Lp(N|\nu) \models \nu$ is Woodin.

S-constructions : $N|\nu$ is generic for $\mathbb{B}_\nu^{N^*}$ with ν generators

Also via S-constructions



$N^*|\gamma[N|\nu] \cong Lp_{\Gamma_\omega}(N|\nu)$

S-constructions.

(Exercise)

N N^* $Lp(N|\nu) \models \nu$ Woodin. ⊠

[Exercise Construct a directed system in $M_1(x)$ where x codes $M_2^\#$.]

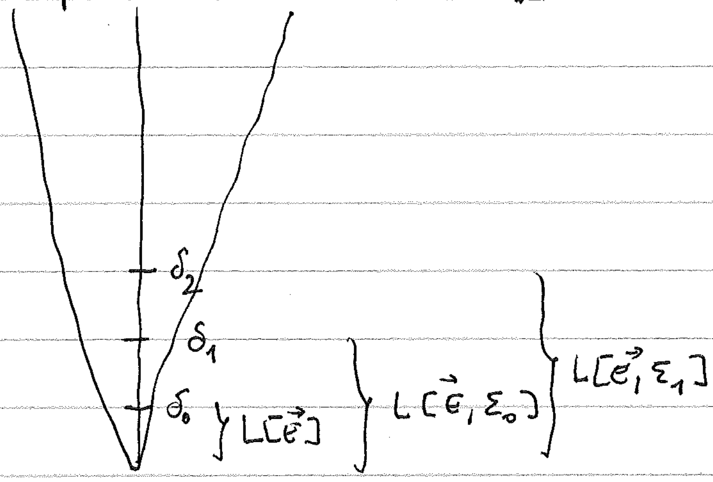
Exercise 1 Suppose $x \in \mathbb{R}$ codes $M_2^\#$. Let κ be the least inaccessible of $M_1(x)$. Let $G \subseteq col(\omega, <\kappa)$ be generic over $M_1^\#(x)$. Construct a DLS using $M_2^\#$ and show that the Woodin cardinal of M_∞ is κ^+. Q: ~~what is~~ ~~HOD^{M_1}~~? Find W s.t. $M_\infty = HOD^W$.

Exercise 2 Let δ_0 be the first Woodin of M_ω. Let $P = M_\omega | (\delta_0^{+\omega})^{M_\omega}$. Show that P has an (ω_1+1)-IS in $L(\mathbb{R})$.

Exercise 3 Back to Example 1. Let δ be the Woodin of $M_1(x)$ and let κ be the least $<\delta$-strong of $M(x)$. Show that if "$N = M_\infty$ at κ" then N is $<\delta$ iterable in $M_1(x)$. And use to construct a model P with 2 Woodins which computes many successors correctly & and is $<\delta$-iterable.

HOD premouse below $AD_{\mathbb{R}}$ + Θ measurable.

$P=$



<u>Def</u> P is a hod-pm iff there is a sequence of
$\langle \delta_\alpha \mid \alpha \leq \lambda \rangle$, $\langle \Sigma_\alpha \mid \alpha \leq \lambda - 1 \rangle$, $\langle \mu_\alpha \mid \alpha \leq \lambda \rangle$, $\langle P(\alpha) \mid \alpha \leq \lambda \rangle$ s.t.

1. $\delta_{\alpha+1}$'s are the only Woodin cardinals of P
2. Σ_α is the strategy of $P(\alpha) \trianglelefteq P$
3. $P(0) = L_{p_\omega}(P \mid \delta_0)$, $P(\alpha+1) = L_{p_\omega}^{\Sigma_\alpha}(P \mid \delta_{\alpha+1})$

<u>Exercise</u> Show that $\mu_{\alpha+1}$ is not a Woodin cardinal of P.
(Genericity iteration). Moreover $\delta_{\alpha+1}^{+P(\alpha+1)} < \delta_{\alpha+1}^{+}$

4. If α is limit with $cf(\alpha)$ not measurable then
$P(\alpha) = \cancel{L_{p}^{\oplus_{\beta<\alpha}\Sigma_\beta}} \; L_{p_\omega}^{\oplus_{\beta<\alpha} \Sigma_\beta}(P \mid \delta_\alpha)$
<u>Rem</u> $\bigoplus_{\beta<\alpha} \Sigma_\beta$ is the strategy of $P(\alpha)$.

$$\Sigma_\alpha = \bigoplus_{\beta<\alpha} \Sigma_\beta \qquad \text{In this case } \delta_\alpha^{+P(\alpha)} = \delta_\alpha^{+}$$

$$\delta_\omega^{+} = o(L_p^{\Sigma_\omega}(P \mid \delta_\omega))$$

5. If $cf(\alpha)$ is measurable then
$P(\alpha) = L_{p_\omega}^{\oplus_{\beta<\alpha} \Sigma_\beta}(P \mid \delta_\alpha)$

$P|\delta_{\alpha+1}$ is a Σ_α-mouse over $P(\alpha)$ Rem: $\Sigma_\alpha \neq \bigoplus_{\beta<\alpha} \Sigma_\beta$

$(\delta_\alpha^+)^{P(\alpha)} = \delta_\alpha^+$

(3') No δ_α is measurable

(6) For every α: if $\eta \in (\delta_\alpha, \delta_{\alpha+1})$: $P|\eta$ is $<\mathrm{On}$ iterable above δ_α.

(7) if λ is a limit then
$$P = P(\lambda) = L_{P_\omega}^{\bigoplus_{\beta<\alpha}\Sigma_\beta}(P|\delta_\lambda)$$
if λ is a successor then
$$P = P(\lambda) = L_{P_\omega}^{\Sigma_{\lambda-1}}(P|\delta_\lambda)$$

[Definition (P,Σ) is a hod pair]

Exercise Suppose P is a hod premouse s.t. $\lambda^P > 0$.
Then $P \models$ there is a unique strategy of $P(0)$
(If σ was such a strategy has 2 branches then $\mathrm{cf}(\delta(\sigma)) = \omega$.)
More generally:
$$P \models (\forall \alpha < \lambda)(\text{there is a unique strategy for } P(\alpha))$$

Definition (P,Σ) is a hod pair if P is a hod pm, Σ is an ω_1-IS for P with hull condensation s.t. if $P \xrightarrow[\sigma \text{ via } \Sigma]{} Q$ then $\Sigma^Q = \Sigma_{Q,\sigma} \restriction Q$

Definition $P \trianglelefteq_{hod} Q$ iff $\exists \alpha$ $P = Q(\alpha)$.

Comparison

Rem In general, the notion of comparison is meaningless.
If in the coiteration of M, N $\delta_0^{M'} \neq \delta_0^{N'}$ then M', N' came from different hierarchies and thus $M' \ntrianglelefteq N'$ and $N' \ntrianglelefteq M'$.

Def Suppose (P, Σ) is a hod pair. Then Σ is fullness preserving (FPR) iff for every $P \xrightarrow{\vec{\mathcal{T}}} Q$ via Σ s.t. $i^{\vec{\mathcal{T}}}$ exists, letting $\Lambda = \Sigma_{Q, \vec{\mathcal{T}}}$, for any $\alpha < \lambda^Q$ and all $\eta \in (\delta_\alpha, \delta_{\alpha+1}]^Q$ that is a strong cutpoint (i.e. η not overlapped by an extender): $Q | \eta^{+Q} = Lp^{\Lambda_{Q(\alpha)}}(Q|\eta)$ and $Q(\alpha) = Lp_\omega^{\oplus_{\beta<\alpha} \Lambda_{Q(\beta)}}(Q|\delta_\alpha)$ if $cf(\alpha)$ is measurable

REM If (P, Σ), (Q, Λ) are hod pairs s.t. Σ, Λ are FPR

Def We say comparison holds for (P, Σ) and (Q, Λ) if there is a stack $\vec{\mathcal{T}}$ on P according to Σ with last model R and a stack $\vec{\mathcal{U}}$ ~~on~~ according to Λ on Q with last model S s.t.

1. $i^{\vec{\mathcal{T}}}, i^{\vec{\mathcal{U}}}$ exist
2. either $R \trianglelefteq_{hod} S$ and $(\Lambda_{S, \vec{\mathcal{U}}})_R = \Sigma_{R, \vec{\mathcal{T}}}$
   or else $S \trianglelefteq_{hod} R$ and $(\Sigma_{R, \vec{\mathcal{T}}})_S = \Lambda_{S, \vec{\mathcal{U}}}$

Fact (AD^+) [Given a good ~~pc~~ (inductive-like) pointclass]
Given a set of reals A and a triple (N, δ, Σ) we say that A is Suslin captured by (N, δ, Σ) iff

1. δ is Woodin in N and Σ is an (ω_1+1)-IS, N is "ctble".
2. There are δ^+-a.c. trees T, U on N s.t. if $N \xrightarrow[\mathcal{U} \text{ via } \Sigma]{i} M$ then for any $g \in Col(\omega, i(\delta))$-generic$/M$
   $p[i(T)]^{M[g]} = A \cap M[g]$

<u>Theorem</u> (Woodin) Given a good (inductive-like) pointclass Γ there is a pair (R, Φ) and $F: \mathbb{R} \to V$ defined on a cone, if $x \in \mathrm{dom}(F)$ then

$$F(x) = \langle N_x^*, M_x, \delta_x, \Sigma_x \rangle \quad \text{s.t.}$$

1. M_x is $M_1^\Phi(x)$ (so x codes R)
2. δ_x is the Woodin in N_x^*, M_x, $N_x^* | \delta_x = M_x | \delta_x$
3. Σ_x is the IS of M_x
4. $N_x^* = L[\Sigma', M_x]$ where $\Sigma' = \Sigma \restriction (M_x | \kappa_x)$ $(= L[M_x | \delta_x])$ and $\kappa_x =$ the least inaccessible of M_x

(think of (R, Φ) is M_1 and Γ is Σ^1_2)

5. For every $A \in \underset{\sim}{\Gamma}$, $\overbrace{x \in \mathrm{dom}(F)}^{\text{for a cone of } x}$ ~~s.t.~~ $F(x) = (N_x^*, \delta_x, \Sigma_x)$ Suslin captures A

END OF FACT

<u>Theorem</u> (AD^+). Suppose (P, Σ), (Q, Λ) are hod pairs s.t. Σ and Λ have branch condensation and are FPR. Suppose there is a Suslin cardinal $\kappa > w(\mathrm{Code}(\Sigma), \mathrm{Code}(\Lambda))$. Then comparison holds for (P, Σ) and (Q, Λ) via normal trees.

<u>Proof</u> Let Γ be a good pointclass s.t. $\exists$ Suslin cardinals $> \Gamma$, $\mathrm{Code}(\Sigma \oplus \Lambda)$. Let F be as in the theorem for Γ. Let x be s.t. $(N_x^*, \delta_x, \Sigma_x)$ Suslin capture $\mathrm{Code}(\Sigma)$, $\mathrm{Code}(\Lambda)$.

<u>Aside</u>: $\langle N_\beta, P_\beta, \Sigma_\beta \mid \beta < \Omega \rangle$ is the output of the maximal hod pair construction of N_x^* off

1. $N_0 = L[\vec{E}]^{N_x^* | \delta_0}$ if N_0 has a Woodin cardinal

then let δ_0 be the least and let

$P_0 = (\mathcal{N}_0 | \delta_0^{+\omega})^{\mathcal{N}_0}$

2. Given $(\mathcal{N}_\alpha, P_\alpha, \Sigma_\alpha)$ let $\mathcal{N}_{\alpha+1} = L[\vec{E}, \Sigma_\alpha]^{\mathcal{N}_\alpha^* | \delta_\alpha}$

If $\mathcal{N}_{\alpha+1}$ has a Woodin cardinal above $o(P_\alpha)$ let $\delta_{\alpha+1}$ be the least

$P_\alpha \trianglelefteq P_{\alpha+1}$ - required.

3. Given $(\mathcal{N}_\alpha, P_\alpha, \Sigma_\alpha)$ for $\alpha < \lambda$, λ limit:

let $P_\lambda^* = \bigcup P_\alpha$ and $P_\lambda = L_{P_\omega}^{\oplus_\lambda}(P_\lambda^*)$.

If P_λ is a hod mouse let Σ_λ be its induced strategy and $\mathcal{N}_\lambda = L[\vec{E}, \Sigma_\lambda]$

<u>Claim</u> $\exists \beta$ s.t. (P_β, Σ_β) is a tail of (P, Σ)

((R, Ω) is a tail of (S, Φ) iff $\exists \vec{\mathcal{T}}$ stack on S s.t. the last model is R & $\Omega = \Phi_{R, \vec{\mathcal{T}}}$)

<u>Proof</u> (By induction)

<u>Case 0</u>: (P_0, Σ_0) exists. Compare P, Σ with $\mathcal{N}_0$.

By universality, P goes to some $\mathcal{N}_0 | \eta$ so P_0 exists.

27.7.2010 17:00 Richard Ketchersid – Discussion -1-

Theorem ($AD + DC_{\mathbb{R}}$) if κ is a limit of Suslin cardinals and κ-ordinal determinancy holds and $\kappa <$ sup of all Suslin cards then κ is Suslin

κ-ordinal determinancy: if $f: \kappa^\omega \to \omega^\omega$ is continuous then $G^\kappa_{f^{-1}[A]}$ is determined where $A \in S_{<\kappa}$.

Note: $\Lambda = S_{<\kappa}$ is a projectively closed algebra

$\langle A_\alpha \mid \alpha < \kappa \rangle$ with $A_\alpha \in \Lambda$ are mutually disjoint and can pick T_α s.t. $A_\alpha = p[T_\alpha]$ we are done – we have a new κ-Suslin set – But this is not in general possible, so we need some tool to make the idea work.

Theorem There is a sequence $\langle S_\alpha \mid \alpha < \kappa \rangle$ of ∞-BC with each $S_\alpha \in \Lambda$ ($S_\alpha \subseteq \gamma < \kappa$) s.t. $A_{S_\alpha} \neq \emptyset$ and $A_{S_\alpha} \cap A_{S_\beta} = \emptyset$ for $\alpha \neq \beta$.

Proof Fix an ∞-BC with $A_S \notin \Lambda$. ~~Look at~~ Define $T \mathrel{E} T'$ iff $A_T = A_{T'}$. Look inside $L[E, S]$. Inside $L[E,S]$ find $\langle S_\alpha \mid \alpha < \kappa \rangle$ an antichain (i.e. S_α are E-incomparable) in $BC^{L[E,S]}_{<\kappa}$.

Pf ① If κ is regular, then $BC^{L[E,S]}_{<\kappa}$ is κ-complete and κ-c.c., so is complete. So there is $S' \in BC^{L[E,S]}_\kappa$ s.t. $S \mathrel{E} S'$. $= BC^{L[E,S]}/E$

② If κ is singular: Choose $\langle S_\alpha \mid \alpha < \gamma \rangle$ s.t. $\bigvee_{\alpha<\gamma} S_\alpha / E \notin BC^{L[E,S]}_{<\kappa}/E$, mutually incomparable modulo E.

Exercise: Get $\langle T_{\alpha,\beta} \mid \beta < \kappa_\alpha \rangle$ antichain in

in $BC^{L[E,S]}_{<\kappa_\alpha}$ in ~~[...]~~ $S_\alpha | E$. Where $\kappa_\alpha \nearrow \kappa$ regular.
Put these together. $\boxtimes$

Def S is a strong ∞-BC, $S \subseteq \gamma$, iff for a club of $\sigma \in \mathcal{P}_{\omega_1}(\gamma)$: $S \cap \sigma \cong S_\sigma$ then $A_{S_\sigma} \subseteq A_S$.
Borel code

S "$\subseteq$" γ : $G(S)$ $\begin{array}{llllll} \mathrm{I} & \alpha_0 & & \alpha_2 & & \\ \mathrm{II} & & \alpha_1 & & \alpha_3 & \dots f \end{array}$
II wins iff $S_{f[\omega]}$ is a BC and $A_{S_{f[\omega]}} \subseteq A_S$.

There is a continuous $\pi : \gamma^\omega \to \omega^\omega$ so that $\pi(f)$ s.t. $\pi(f)$ ~~[...]~~ $\cong S_{f[\omega]}$. (Look at the payoff set $B = \{x \in \mathbb{R} \mid A_x \subseteq A_S\}$)

S is a strong ∞-BC iff II ~~[...]~~ has a w.s. in $G(S)$

Fact If S is strong ∞-BC ~~[...]~~, $S \subseteq \gamma$ then A_S is γ-Suslin.
$x \in A_S \iff (\exists f)(f$ is closed under strategy for II in $G(S)$ & $\underbrace{x \in A_{\pi(f)}}_{\Sigma^1_1})$

if $t \in {}^{<\omega}\mathrm{rng}(f)$ then the strategy responds with an element of $\mathrm{rng}(f)$.

Now produce strong ∞-BC from ∞-BC:
Fix an ∞-BC S.
$\mathbb{Q}^x_S = BC^{HOD_S^{L[S,x]}} / E^x_S$, so
$T \; E^x_S \; T' \iff (A_T = A_{T'})^{L[S,x]}$
Then:

We blackbox these two — 3 —

- $L[S,x] \models CH$ on a cone
- $\mathrm{HOD}_S^{L[S,x]} \models \omega_2^{L[S,x]}$ is inaccessible $(\omega_2^{L[S,x]} = \Theta^{L[S,x]} \overset{\text{def}}{=} \Theta_S^x)$

<u>Claim</u> $\mathbb{Q}_S^x = BC_{<\Theta_S^x}^{\mathrm{HOD}_S^{L[S,x]}} / E_S^x$

<u>Proof</u> $\mathbb{Q}_S^x$ is Θ_S^x-c.c. Otherwise we contradict CH in $L[S,x]$.

Now $\mathbb{Q}_S^x \subseteq BC_{<\Theta_S^x}^{\mathrm{HOD}_S^{L[S,x]}}$; let D_S^x be enumeration in length Θ_S^x of all MACs.

S^x = that code E_S^x is equivalent to. Then

$\langle \mathbb{Q}_S^x, D_S^x, S^x \rangle \in \langle \mathrm{HOD}_S^{L[S,x]} | \Theta_S^x, S^x \rangle = N_S^x \quad \text{''} \subseteq \text{''} \ \omega_2^{L[S,x]}$

Let

$$S_*^x = \bigwedge_{A \in D_S^x} \bigwedge_{T, T' \in A} \neg(T \wedge T') \wedge S^x \wedge \bigwedge_{A \in D_S^x} \bigvee A$$

<u>Exercise</u>: For z's (anywhere)

$z \in A_{S_*^x} \iff z$ is generic for $\mathbb{Q}_S^x$ over $\mathrm{HOD}_S^{L[S,x]}$ &

& $\mathrm{HOD}_S^{L[S,x]}[z] \models z \in A_S$

In V: $A_{S_*^x} \subseteq A_S$

$S_* = \prod_S S_*^x / \mu \qquad (\mu = \text{Martin measure})$

$N_S^\infty = \prod_S N_S^x / \mu = H_S^\infty | \Theta_S^\infty = \prod_S \mathrm{HOD}^{L[S,x]} / \mu$

$\Theta_S^\infty = \prod_S \Theta_S^x / \mu$

If Θ_S^∞-Det holds $\Rightarrow S_*$ is a strong ∞-BC

Will show:

$\Theta^\infty_S < \kappa$ for $S \in \mathcal{S} = S(<\kappa)$.

We can then effectively turn:

$\langle S_\alpha \mid \alpha < \kappa \rangle \mapsto \langle S^*_\alpha \mid \alpha < \kappa \rangle$ where $S^*_\alpha \subseteq \kappa$

then

$$\bigvee_{\alpha \leq \beta < \kappa} S^*_\alpha \times S^*_\beta \text{ is strong } \infty\text{-BC}$$

This is an exercise. Here use ordinal determinacy by showing that I cannot win.

Next

(A) $\prod_S \Theta^x_S / \mu < \kappa$ $S \in \mathcal{S}$

(B) S_* is strong if Θ^∞_S-det

We defined "model operator". A way of thinking about F:

$F(M)$ = a mouse $N \trianglerighteq M$ over M least with $N \models \varphi$

where φ is a Σ_1-formula.

Eg: $\varphi \equiv \exists$ extender above n Woodin cardinals ; or

$\varphi \equiv \exists$ extender + V closed under $M_n^\#$

Now applications (Steel)

Theorem 1 (Steel) CH + Homogeneous presaturated ideal on ω_1 $\Rightarrow$ HC is closed under $M_n^\#$ for all n

Theorem 2 (Woodin) ω_1-dense ideal on ω_1 $\Rightarrow$ HC is closed under all $M_n^\#$

Theorem 3 (Schindler) Precipitous ideal on ω_1 + slight strengthening of BPFA + V closed under all $M_n^\#$'s, $n<\omega$.

Fact Suppose V closed under $M_n^\#$ but $M_{n+1}^\#$ does not exist. Then by the $K^{M_n^\#}$-existence dichotomy, K^c is $<$On iterable + K exists)

If $\exists$ precipitous ideal on κ then $\kappa^{+K} = \kappa^{+V}$.

Suppose moreover $j(\omega_1^V) = \omega_2^V$ where j is the generic UP map.

Get easy contradiction.

Proof of Theorem 1 Aim is to prove inductively:

(A_n) H_{ω_2} closed under $M_n^\#$

(B_n) H_{ω_2} closed under $(M_n^\#)^\#$ $\rightarrow$ Sharp for an inner model that is closed under $M_n^\#$.

The structure of the argument then is:

$(A_n) \Rightarrow (B_n) \Rightarrow (A_{n+1})$

In the following write $\mathcal{J}$ for mouse operators.

Def. • Let $\mathcal{J}$ be a mouse operator which is total on H_{ω_2} we say that $\mathcal{J}$ has the <u>extension property</u> iff $j(\mathcal{J})$ extends $\mathcal{J}$ and $j(\mathcal{J}) \restriction HC^{V[G]}$ is definable in $V[G]$. Here $j: V \xrightarrow[G]{} M$ is a fixed generic embedding from above.

• <u>ω_1 is inaccessible to the reals</u> iff $L^{\mathcal{J}}_{\omega_2}(x) \models$ "ω_1^V is inaccessible" for all $x \in \mathbb{R}^V$.

<u>Exercise</u> If there is a precipitous ideal then $(A)_0$

<u>Now do $(B_n) \Rightarrow (A_{n+1})$.</u>

We don't have global K, ~~so~~ now, so we cannot do the easy argument ~~for~~ from above. Look at $(M_n^\#)^\#(\mathbb{R}^V)$. By CH, $\mathbb{R}$ has size ω_1. Get $M_{n+1}^\#$. Notice: B_n is used here. !

<u>Case 1</u> $M_{n+1}^\#$ exists in $(M_n^\#)^\#(\mathbb{R}^V)$

<u>Case 2</u> Otherwise. Let $K = K^{(M_n^\#)^\#(\mathbb{R})}$. Then $j(K)$ is definable inside $j((M_n^\#)^\#(\mathbb{R})) = j((M_n^\#)^\#(\mathbb{R}^{V[G]}))$. Show $j((M_n^\#)^\#)(\mathbb{R}^{V[G]})$ is countably iterable in $V[G]$, hence definable. But the generic extension is definable. This gives $j(K) \in V$.

<u>Remark</u> $(M_n^\#)^\#(\mathbb{R}^V)$ is not a model of choice. Technically, K is actually $K^{HOD^{(M_n^\#)^\#(\mathbb{R}^V)}}$. Alternatively, we can add well-ordering of reals to $(M_n^\#)^\#(\mathbb{R}^V)$ by a homogeneous forcing that preserves $(M_n^\#)^\#$. END OF REMARK

Let E_j be the $j(K)$ extender derived from j. We show: $E_j \restriction \alpha \in j(K)$. For this we show:

~~$(j(\kappa), \mathrm{Ult}(j(\mathbb{K})), \alpha)$ is~~
$(j(\kappa), \mathrm{Ult}(j(\mathbb{K}), E_j|\alpha), \alpha)$ is iterable. (Work inside $j((M_n^\#)^\#(\mathbb{R}^{V[G]})))$ This gives us: $E_j|\alpha$ is on the $j(\mathbb{K})$-sequence. Then ω_1^V would be Shelah in $j(\mathbb{K})$. The iterability of the above phalanx is proved in a standard way, using the Łoś' theorem for j, combined with an absoluteness argument.

Proof of Thm 2. Since we don't have CH, we cannot construct one local universe as in the proof of Thm 1. Now describe local universes: Pick $A_0 \subseteq \omega_1$ s.t. ~~$L[A_0]$~~, $\omega_1^V = \omega_1^{L[A_0]}$

Let $A \subseteq \omega_1$. Let

$$\bar{N}_A = L_{\omega_2}[A_0, A_1, f] \text{ where } f \text{ is a Skolem function for } (J_{\omega_1}(\mathbb{R}), \in, I)$$

collection of countable mice

By adding f to $\bar{N}_A$ we guarantee that $\bar{N}_A$ is closed under $M_n^\#$

Let

$$N = (\bar{N}_A)^\#$$

Ralf crossed the board through. There are some issues in the notes he will correct. He only sketched the argument in words, but I could not make sense of that. He will correct the notes and let us know.

Proof of Thm 3. Here we inductively prove:

$(A)_n$: HC is closed under $M_n^\#$

$(B)_n$: H_{ω_2} —"—

$(C)_n$: V —"—

Structure: $(A)_n \Rightarrow (B)_n \Rightarrow (C)_n \Rightarrow (A)_{n+1}$

Mentioned above — New — Similar to that done before - the K-argument

Here is the new argument $(B)_n \Rightarrow (C)_n$:
Assume $M_n^\#(A)$ does not exist. Construct the stack $S(A) = Lp(A)$ and show $cf(S(A) \cap On) > \omega$. Covering argument take ~~countable~~ $\aleph_1$-sized substructure; the collapse is of the form $Lp(\bar{A})$ for $|\bar{A}| = \omega_1$. Then $M_n^\#(\bar{A})$ exists by $(B)_n$. Then lift to get $M_n^\#(A)$.
We in fact get $cf(S(A) \cap On) > \omega_1$.
In $V[G]$ when G generic for $Col(\omega_1, Lp(A) \cap On)$ we have $cf(On \cap Lp(A)) = \omega_1$. Look at the tree of attempts to construct an extension $M \triangleright Lp(A)$. Specialize T. Look at
"$\exists A \subseteq \omega_1$ $\exists n$-small A-prm M s.t. $cf(M \cap On) = \omega_1$,
T_M is special and M is a mouse"
Here T_M is constructed over M the same way as T over $Lp(A)$.

27.7.2010 14:00 Ralf Schindler

Open questions: Do the following statements imply PD?
(1) Projective measure + category + all Π^1_{2n+1} relations have Π^1_{2n+1} uniformizing functions
(2) Any two projective sets are Wadge comparable
(3) There is a homogeneous presaturated ideal on ω_1.

Next topic: The core model induction $L(\mathbb{R})$

Definition (Suslin capturing) Let $A \subseteq \mathbb{R}$. Let N be a countable transitive model and let T, U be trees in N s.t. (a) $N \models ZFC + \delta_0 < \delta_1 < \cdots < \delta_k$ are Woodin cardinals
(b) In $N^{Col(\omega, \delta_k)}$ T, U project to complements
(c) There is an ω_1-IS Σ for N s.t. if $i: N \to N^*$ given

by Σ then $p[i(T)]^{N^* \operatorname{col}(\omega, i(\delta_k))} \subseteq A$
$p[i(U)]^{N^* \operatorname{col}(\omega, i(\delta_k))} \subseteq \mathbb{R} - A$.

Then we say N is a coarse (A,k)-Woodin mouse.

Using the extender algebra we can make any real generic over an iterate of N, so

$$A = \bigcup_{\substack{i: N \to N^* \\ g \text{ generic}}} p[i(T)]^{N^*[g]}$$

__Example__ Let $T \in M_n^\#$ be the tree of all attempts to find to find
- a countable M together with $\sigma : M \to M_n^\# | \alpha$
- $x \in \mathbb{R}$ s.t. x is generic$/M$ at $\sigma^{-1}(\delta_0)$
- $M[x] \models \varphi(x)$ where φ is Σ^1_n given in advance and fix.

Let U be defined similarly with the only difference that $M[x] \models \neg\varphi(x)$.

By extender algebra: $M \models \varphi(x) \Leftrightarrow V \models \varphi(x)$.
Elaborate on this idea to produce an (A,k)-mouse.

__Definition__ (Witness & W_α^*) Let $A \subseteq \mathbb{R}$ and suppose A, $\mathbb{R} - A$ have scales s.t. the associated pwo's are in $J_\alpha(\mathbb{R})$. Then for all $x \in \mathbb{R}$ and all k there is a coarse (A,k)-Woodin $\underbrace{\text{mouse}}_{N \text{ with iteration strategy in } J_\alpha(\mathbb{R})}$. We denote this statement by W_α^*

The core model induction shows W_α^* by induction on α.
W_2^* is achieved by the method discussed in the morning.

<u>Fact</u> $W_\alpha^* \Rightarrow AD^{J_\alpha(\mathbb{R})}$

<u>Definition</u> β is called <u>critical</u> iff there is $A \subseteq \mathbb{R}$ s.t. both A and $\mathbb{R}-A$ have scales s.t. the associated prewellorderings are in $J_{\beta+1}(\mathbb{R})$ but A and $\mathbb{R}-A$ do not have a scale in $J_\beta(\mathbb{R})$.

<u>Lemma</u> (Steel, Scales in $L(\mathbb{R})$) (W_β^*) Let β be critical. Then one of the following is true.

(1) $\beta = \eta+1$ for some critical η

(2) $cf(\beta)=\omega$ or ($cf(\beta)>\omega$ and β inadmissible);
in both cases β is a limit of critical points

If (1) or (2) holds, we refer to the situation as "inadmissible case".

(3) β is not a limit of critical points and, letting
$\alpha = \sup$ of all critical points $< \beta$ then
<u>either</u> $[\alpha,\beta]$ is a weak gaps
<u>or else</u> $\beta = \eta+1$ for some η and $[\alpha,\eta]$ is a strong gap.
Case (3) is referred to as "end of gap case".

<u>Fact</u> $(W_\alpha^*) \Rightarrow AD^{J_\alpha(\mathbb{R})}$

<u>Sketch of proof</u>: If not: Then there is a non-determined set of reals in $J_\alpha(\mathbb{R})$, i.e. for some $\gamma<\alpha$ there is such a set.
The least such γ, ends a gap, so $\gamma+1$ begins a gap.
If γ is not critical then by Kechris-Woodin: the gap ending by γ is strong, so $AD^{J_{\gamma+1}(\mathbb{R})}$, a contradiction.
Hence γ must be critical.

Now pick $A \subseteq \mathbb{R}$, $A \in J_{\gamma+1}(\mathbb{R})$. WTS A is determined.
Pick N a coarse $(A, 10)$-Woodin mouse. ~~Pick~~ Let $T, U \in N$ witness this. Then

$N \models p[T]$ is determined

Say I has a w.s. in the game for $p[T]$, denote it by σ.
Let $x \in \mathbb{R}^V$. Now assume $\sigma * x \notin A$. Now use genericity iteration $N \xrightarrow{i} N^*$ to absorb x. Then use the fact that $i(T), i(U)$ capture A + elementarity to get a contradiction.

$\boxed{}$ (A is determined)

<u>Rem</u> the above is just a sketch. One has to argue carefully that there is enough iterability to do genericity iterations.

Now let φ be a Σ_1-formula. There are formulae φ^n, $n \in \omega$ s.t. for all $z \in \mathbb{R}$ and all $\alpha \in On$

$$J_{\alpha+1}(\mathbb{R}) \models \varphi(z) \iff (\exists n \in \omega)\ J_\alpha(\mathbb{R}) \models \varphi^n(z).$$

<u>Def</u> Let φ be Σ_1 and $z \in \mathbb{R}$. A (φ, z)-witness is a z-premouse N s.t. for some $T, U \in N$:

- $N \models ZFC$
- $N \models \delta_0 < \delta_1 < \dots < \delta_{10}$ are Woodin
- U, T are complementing in $N^{Col(\omega, \delta_{10})}$
- For some k: In $N^{Col(\omega, \delta_0)}$

  $p[T] =$ the Σ_k-theory of $J_\gamma(\mathbb{R})$ where $\gamma =$ the least s.t. $J_\gamma(\mathbb{R}) \models \varphi^k(z)$.
- N is $(\omega_1 + 1)$-iterable.

Lemma If there is a (φ, z)-witness then $L(\mathbb{R}) \models \varphi(z)$.

Proof Sketch

$Th = \bigcup \{ p[i(T)] \cap N^*[g] \mid i : N \to N^*$ is a countable iteration map and g is $col(\omega_1, i(\delta_0))$-generic / $N^* \}$

By the Dodd-Jensen property, Th is a consistent complete theory of a model of "I am $J_\gamma(\mathbb{R})$". But then $J_\gamma(\mathbb{R}) \models \varphi^k(z)$, etc... $\boxtimes$

(W_α) is the statement: If φ is Σ_1, $z \in \mathbb{R}$ and $J_\alpha(\mathbb{R}) \models \varphi(z)$ then there is a (φ, z)-witness with (ω_1+1)-IS Σ s.t. $\Sigma \restriction HC \in J_\alpha(\mathbb{R})$.

Theorem If α is a limit ordinal then $W_\alpha^* \Rightarrow W_\alpha$.

The proof of this theorem expands the proof of the mouse set theorem in $L(\mathbb{R})$.

Let Γ be a good pointclass. Eg: $\Gamma = \Sigma_1(J_\alpha(\mathbb{R}))$ where α begins a proper weak gap.

$C_\Gamma(z) = \{ x \in \mathbb{R} \mid x$ is $\Delta^{(z)}$ in a countable ordinal$\}$ i.e.

$(\exists \alpha < \omega_1)(\exists A \in \Delta)(x = x' \iff (x', z, y) \in A$ where $\|y\| = \alpha)$

Mouse set theorem: $C_\Gamma(z) = \mathbb{R} \cap M$ where M is a z mouse, with strategy in Δ.

Warmup for the proof of the MST. Fix Γ.

For every y there is $z \geq_T y$ s.t. $\exists z$ mouse R which has a real outside of $C_\Gamma(z)$. [There comes an idea drawn as a picture on the board which I was unable to reproduce here.]

28.7.2010 16:30 Discussion: Paul Larson

A version of tree production lemma:

<u>TPL II</u> Let δ be Woodin, φ, ψ be formulas and a, b sets. Suppose:
- For all $G \subseteq \mathbb{Q}_{<\delta}$ and all $<\delta$-generic $h \in V[G]$ and all $x \in \mathbb{R}^{V[h]}$

$$V[h] \models \varphi(x, a) \iff M \models \psi(x, j(b))$$

Then there are $<\delta$ absolutely complementing T, U on $\omega \times \mathrm{On}$ s.t. in all $<\delta$-generic extensions

$$p[T] = \{x \in \mathbb{R} \mid \varphi(x, a)\}$$

<u>Proof</u> Essentially identical with that of the similar theorem in Larson's book. The formulation of the theorem in the book is slightly different than here.

29.7.2010, 9:30 Grigor Sargsyan

<u>Theorem</u> (AD^+) Suppose (P,Σ) and (Q,Λ) are two hod pairs s.t. Λ and Σ have BC and are FPR. Suppose $\exists$ Suslin cardinal $> w(\Sigma\oplus\Lambda)$. Then comparison holds for (P,Σ) and (Q,Λ).

<u>Proof</u> Let Γ be a good pointclass s.t. $\mathrm{Code}(\Sigma\oplus\Lambda)\in\Gamma$.
Let x be s.t. $F(x)=(N_x^*, \delta_x, \Sigma_x, M_x)$ Suslin captures $\mathrm{Code}(\Sigma\oplus\Lambda)$.

<u>Claim</u> $\exists\beta$ s.t. the β-th model of the hod pair construction of N_x^* is an iterate of Q. That is:

Let $\langle N_\xi, P_\xi, \Sigma_\xi \mid \xi<\Omega\rangle$ be the hod pair construction of N_x^*. $\exists\beta$ s.t. (P_β,Σ_β) is a tail of (Q,Λ).

<u>Proof</u> Why capturing? Need $N_x^*\models P, Q$'s δ_x+1 iterable & Σ_x moves both Σ,Λ correctly.

By induction construct $\langle \mathcal{U}_\alpha, Q_\alpha \mid \alpha\le\beta\rangle$ s.t.

1. $\mathcal{U}_0$ is normal tree on $Q(0)$ according to Λ, Q_1 is the last model, $Q_1(0)=P_0$, $\Sigma_0=\Lambda_{Q_1(0)}$
2. $Q_\alpha(\alpha-1)=P_{\alpha-1}$, $\Sigma_{\alpha-1}=\Lambda_{Q_\alpha(\alpha-1)}$
3. $\mathcal{U}_\alpha$ is a tree on $Q_\alpha(\alpha)$ with last model $Q_{\alpha+1}$ s.t. $Q_{\alpha+1}(\alpha)=P_\alpha$ and $\Sigma_\alpha=\Lambda_{Q_{\alpha+1}(\alpha)}$.

<u>Facts</u> Why $Q_1(0)=P_0$. By universality $Q(0)$ iterates to an initial segment of N_0, say $N_0|\eta$. By FPR of Λ: $N_0|\eta$ is full. So the Woodin in $N_0|\eta$ is Woodin in N_0.

[Since the Woodinness of ν is preserved ~~by~~ in $Lp(N_0|\nu)$, if Woodinness is killed in N then there must be an extender overlapping ν, ↯]

$\mathcal{U}_0$ is the tree from $Q(0)$ to P_0. Moreover $\Lambda_{Q_1(0)}$ is Σ_0, by the Branch condensation lemma.

Continue this process with changed min condition:

None of δ^1_α's is an inaccessible limit of inaccessibles. (To make the situation simpler.)

Point: the strategies after the first round agree and this agreement will be maintained. Hence the only difference is in the extender sequences

The comparison terminates by the min condition.

Here is some detail of the above:

<u>Claim</u> $\Lambda_{Q_\alpha(\alpha)} = \Sigma_\alpha$ in V. (In V: by absoluteness argument.)

<u>Proof</u> Let $\mathcal{T}$ be on P_α in N_x^*

$$\begin{tikzcd}
 & & & N^* \arrow[r, "\mathcal{T}^*, c"] & P_\alpha \\
Q \arrow[r, squiggly, "\vec{u}"'] \arrow[urrrr, "\pi(\vec{u})"] & Q_\alpha \trianglerighteq Q_\alpha(\alpha) = P_\alpha \arrow[rrr, "\mathcal{T}, c"'] \arrow[urrr] & & & M_c^{\mathcal{T}} \arrow[u, "\sigma"']
\end{tikzcd}$$

Let $c = \Sigma_\alpha(\mathcal{T})$, $\pi = \sigma \circ i^{\mathcal{T}}_c$. By elementarity

$i^{\pi(\vec{u})} = \sigma \circ i^{\mathcal{T}}_c \circ i^{\vec{u}}$

So $\mathcal{T}^\frown M_c^{\mathcal{T}}$ is by Λ, from branch condensation. ⊠

<u>$\diamond$- comparison argument</u>, or <u>Comparison in ZFC context</u>

~~theorem~~ Intended applications: Divergent models of AD^+ implies $(\exists M)$ $\mathbb{R} \in M$, $On \subseteq M$ & $M \models AD_{\mathbb{R}} + \Theta$ regular.

Given (P, Σ) and (Q, Λ) s.t. λ^P, λ^Q are limit we want to compare them.

<u>Minimal disagreements</u> Suppose (P, Σ) and (P, Λ) are hod pairs with $\Sigma \neq \Lambda$

<u>Definition</u> (Essential components) Suppose P is a hod pm with λ^P limit and $\vec{\mathcal{T}}$ a stack on P. $\vec{\mathcal{T}} = \langle \vec{\mathcal{T}}_\alpha, M_\alpha, M_\alpha^*, i_{\alpha\beta} \mid \alpha < \beta < \eta \rangle$ are the essential components of

1. Letting E be the first extender in $\vec{\mathcal{T}}$, $M_0 = P$, $M_0^* \trianglelefteq P$ minimal s.t. $E \in M_0^*$ and $\vec{\mathcal{T}}_0$ is the largest IS of $\vec{\mathcal{T}}$ based on M_0^*. M_1 = the last model of $\vec{\mathcal{T}}_0$.
2. M_α = the last model of $\bigoplus_{\beta<\alpha} \vec{\mathcal{T}}_\beta$ of α limit; M_α is the last model of $\vec{\mathcal{T}}_{\alpha-1}$.
3. Let E be the first extender used on $\vec{\mathcal{T}}$ after $\bigoplus_{\beta<\alpha} \vec{\mathcal{T}}_\alpha$; $M_\alpha^* \trianglelefteq_{hod} M_\alpha$ the least s.t. $E \in M_\alpha^*$ and $\vec{\mathcal{T}}_{\alpha+1}$ is the longest IS based on M_α^*.
4. $i_{\alpha\beta} : M_\alpha \to M_\beta$

For hod pairs (P, Σ), (P, Λ) as above: $\vec{\mathcal{T}}$ is a <u>minimal disagreement</u> iff letting $\langle \vec{\mathcal{T}}_\alpha, M_\alpha^*, M_\alpha, i_{\alpha,\beta} \mid \alpha \leq \beta \leq \eta \rangle$ be the essential components then

1. $\lambda^{M_\alpha^*}$ is a successor
2. if $\alpha < \eta$ then $\Sigma_{M_\alpha^*} = \Lambda_{M_\alpha^*}$
3. $\Sigma_{M_\eta^*} \neq \Lambda_{M_\eta^*}$ and $\vec{\mathcal{T}}_\eta$ witnesses it, i.e.

$$\Sigma_{M_\eta^*}(\vec{\mathcal{T}}_\eta) \neq \Lambda_{M_\eta^*}(\vec{\mathcal{T}}_\eta)$$

Lemma There is a minimal disagreement.

Proof (Sketch) If it is at limit stage, look at the essential components. Some has such a disagreement at a successor stage, as otherwise ~~...~~ the iteration would be ill-founded.

Given a hod pair (P, Σ) let Σ has branch condensation

$B(P,\Sigma) = \{Q \mid \exists\ \Sigma\text{-iterate } R \text{ of } P \text{ s.t. } Q \trianglelefteq_{hod} R\}$

$I(P,\Sigma) = \{Q \mid Q \text{ is a } \Sigma\text{-iterate of } P \text{ and } i: P \to Q \text{ exists}\}$

$\Gamma(P,\Sigma) = \{A \mid \exists Q \in B(P,\Sigma) \text{ s.t. } A \leq_W Code(\Sigma_Q)\}$

Definition (ZFC) Suppose $\Gamma \subseteq \mathcal{P}(\mathbb{R})$ is s.t. $L(\Gamma, \mathbb{R}) \models AD^+$ and (P,Σ) is a hod pair with λ^P limit. Then Σ is Γ-fullness preserving (Γ-FPR) iff for all $Q \in I(P,\Sigma)$ and all $\alpha < \lambda^Q$ and $\eta \in (\delta_\alpha, \mu_{\alpha+1})^Q$, η strong cutpoint of Q the following holds:

$$Q \mid \eta^{+Q} = Lp^{\Gamma, \Sigma_{Q(\alpha)}}(Q \| \eta) =$$

$$= \bigcup \{M \text{ is sound}, \overbrace{\text{projecting}}^{\Sigma_{Q(\alpha)}\text{-mouse}} \text{ to } Q|\eta \text{ with strategy in } \Gamma\}$$

Theorem Suppose (P,Σ), (Q,Λ) are two hod pairs with λ^P, λ^Q limit. Σ, Λ ~~...~~ have BC. Assume that letting $\Gamma = \Gamma(P,\Sigma) \cup \Gamma(Q,\Lambda)$ then $L(\Gamma,\mathbb{R}) \models AD^+$, $\Gamma = \mathcal{P}(\mathbb{R})^{L(\Gamma,\mathbb{R})}$ and Σ, Γ are Γ-FPR.

Then comparison holds for (P,Σ) and (Q,Λ).

<u>Proof</u> First: We can compare $\begin{matrix} P \xrightarrow{\sigma} R \\ Q \to R \end{matrix}$ s.t. $\forall \alpha < \lambda^R$ $\Sigma_{R(\alpha)} = \Lambda_{R(\alpha)}$.

How to do it: Use our comparison in AD^+-models to do the following: In the AD^+-models, because we have comparison there. We get (R, Σ_R) and (R, Λ_R) and $\forall \alpha < \lambda^R$ $\Sigma_{R(\alpha)} = \Lambda_{R(\alpha)}$. Let $(R, \Sigma), (R, \Lambda)$ be two hod pairs s.t. $\forall \alpha < \lambda^R$ $\Sigma_{R(\alpha)} = \Lambda_{R(\alpha)}$. Want to compare. Construct a "diamond" sequence:

$$\begin{tikzcd}[row sep=small] & R_1 \arrow[dr] & & & R_4 \arrow[dr] & \\ R \arrow[ur, "\vec{\sigma}_0"] \arrow[dr] & & R_3 \arrow[urr, "\sigma_1"] \arrow[drr] & & & \cdots \\ & R_2 \arrow[ur] & & & R_5 \arrow[ur] & \end{tikzcd}$$

Basic step: Gives $\Sigma_{R_3(\alpha)} = \Lambda_{R_3(\alpha)}$ Basic

$\vec{\sigma}_0$ is a minimal disagreement:

$$\begin{tikzcd} & & R_2 \arrow[dr, "\sigma_2"] & \\ R = R_0 \arrow[r, "\vec{\sigma}_0"] & R_1 \arrow[ur, "\vec{\sigma}_{1,c}"] \arrow[dr, "\vec{\sigma}_{1,b}"'] & & R_4 \\ & & R_3 \arrow[ur, "\sigma_3"'] & \end{tikzcd}$$

Call this a <u>bad block</u>

$\forall \alpha < \lambda^{R_4}$: $\Sigma_{R_4(\alpha)} = \Lambda_{R_4(\alpha)}$

<u>Claim</u> There is a Bad sequence of length ω_1, i.e. $\langle B_\alpha \mid \alpha < \omega_1 \rangle$ with

$$B_\alpha = \langle (R^i_\alpha \mid i \leq 4), (\vec{\sigma}_i \mid i \leq 3), c_\alpha, b_\alpha, i^c_\alpha = i^{\sigma_2} \circ i^{\sigma_1}_c \circ i^{\sigma_0}, i^b_\alpha = i^{\sigma_3} \circ i^{\sigma_1}_b \circ i^{\sigma_0} \rangle \text{ s.t.}$$

1. B_α is a bad block
2. $R^\alpha_4 = R^{\alpha+1}_0$
3. The direct limit using i^b's is equal to that using i^c's.

4. $R^0_\beta = \text{dir lim} \langle R^0_\alpha, i_{\alpha\gamma} \rangle_{\alpha\leq\beta}$ of β limit

Let $M_0 \prec M_1 \prec H_{\omega_2}$ elementary with
$$\begin{tikzcd} M_0 \arrow[r, "\pi_0"] \arrow[dr, "\pi"'] & H_{\omega_2} \\ & M_1 \arrow[u, "\pi_1"'] \end{tikzcd}$$

and $\vec{B} \in \text{rng}(\pi_0)$. $\kappa_0 = \text{cr}(\pi_0)$, $\kappa_1 = \text{cr}(\pi_1)$.

<u>Exercise</u> : $\pi = i^c = i^b$ where

$$\begin{tikzcd} R_{\kappa_0} \arrow[r, bend left, "i^c"] \arrow[r, bend right, "i^b"'] & R_{\kappa_1} \end{tikzcd}$$

<u>Claim</u> $\quad j^c = j^b$ where

$$\begin{tikzcd} R^1_{\kappa_0} \arrow[r, bend left, "j^c"] \arrow[r, bend right, "j^b"'] & R_{\kappa_1} \end{tikzcd}$$

<u>Proof</u> : Standard.

Now use branch condensation for $\Sigma_{R^1_{\kappa_0}}$ for stacks
$\mathcal{T}^1_{\kappa_0} {}^\frown c$, $k : R^2_{\kappa_0} \to R^0_{\kappa_1}$
and the bottom iteration $\Rightarrow \mathcal{T}^1_{\kappa_0} {}^\frown c$ is by $\Sigma_{R^1_{\kappa_0}}$ ↯ ⊠

29. 7. 2010 RALF SCHINDLER 14:00.

Fix a weak gap $[\alpha, \beta]$ where $\alpha < \beta$. Let n be least such that $\rho^{n}_{J_\alpha(\mathbb{R})} = \mathbb{R}$. Then the pointclasses $\Sigma^{J_\beta(\mathbb{R})}_{n+2i}$ have the scale property. Assume W^*_β. We want to prove W^*_β.

<u>Fact</u> Every set of reals in $J_\beta(\mathbb{R})$ has a scale whose prewellorderings are in $J_\beta(\mathbb{R})$. Also, Every $\Sigma^{J_\beta(\mathbb{R})}_{n}$ is countable union of sets in $J_\beta(\mathbb{R})$.

<u>Definition</u> N is α-suitable iff $ht(N) = ((\delta^N)^{+\omega})^N$ where δ^N is the unique Woodin cardinal in N. Setting $\Gamma = \Sigma_1^{J_\alpha(\mathbb{R})}$:
For all $n \in \omega$, $C_\Gamma((N|(\delta^N)^{+n})^N) = N|((\delta^N)^{+n+1})^N$

An iteration strategy Σ is α-<u>fullness preserving</u> iff for all iterations $N \rightsquigarrow N^*$ according to Σ with iteration map i, the premouse N^* is also α-suitable. If there is a drop on the main branch of $N \rightsquigarrow N^*$ we require that $J_\alpha(\mathbb{R}) \models$ "N^* is iterable".

<u>Plan</u> Construct witnesses to $(W^*_{\beta+1})$ as Σ-mice N^* with finitely many Woodin cardinals where Σ is a nice IS for an α-suitable mouse. A Σ-mouse is an F-mouse for an appropriate F.

<u>Lemma</u> Let N be an α-suitable pm. Let $A \in J_\beta(\mathbb{R})$.
Say $A \in (OD^{<\beta})^{J_\beta(\mathbb{R})}$. There is a term $\tau \in Col(\omega, \delta)$ s.t.
for comeager many g that are $Col(\omega,\delta)$-generic$/N$:

$$\tau^g = A \cap N[g]$$

(So τ "captures" A.)

<u>Proof</u> (Sketch)

$(p, \sigma) \in \tau$ iff $\cdot$ $p \in Col(\omega, \delta)$, σ is a nice name for a real ($\in N$)

$\cdot$ for comeager many g: $p \in g \to \sigma^g \in A$

Claim 1 $\tau \in N$

The proof of this uses that N is Γ-full and also that $\Sigma_1^{J_\beta(\mathbb{R})} = \Sigma_1^{J_\delta(\mathbb{R})}$ ⊠

Claim 2 τ captures A ⊠

~~Lemma~~

Definition A sjs (self-justifying system) is a countable collection of sets $A \subseteq \mathbb{R}$ s.t. $\forall$: $\mathcal{A} \subseteq J_\beta(\mathbb{R})$
- the universal $\Sigma_n^{J_\beta(\mathbb{R})}$ set is the union of a countable collection of sets in $\mathcal{A}$
- $\mathcal{A}$ is closed under complements
- $\forall A \in \mathcal{A}$ there is a scale on A s.t. the individual pwo's are in $\mathcal{A}$

$DC_{\mathbb{R}} \Rightarrow$ There is a sjs.

Lemma Let N be α-suitable and let $\mathcal{A}$ be a sjs s.t. every $A \in \mathcal{A}$ is captured over N. If $\pi : \bar{N} \longrightarrow N$ is s.t. all $\tau^N_{A,(\delta^+)^N}$ are in $\operatorname{rng}(\pi)$ then $\bar{N}$ is α-suitable. Moreover $\pi^{-1}(\tau^N_{A,(\delta^+)^N}) = \tau^{\bar{N}}_{A,(\bar{\delta}^+)^{\bar{N}}}$ and $\operatorname{rng}(\pi)$ is cofinal in δ, and if $\pi \restriction \delta = id$ then $\pi = id$.

Proof (Sketch) Fix $A \in \mathcal{A}$, $\leq_m \in \mathcal{A}$. Have τ_m capture $\leq_n$ over N. For g $Col(\omega,\delta)$-generic$/N$: (from a given comeager set.)

$U_m^g = \{ (x \restriction m, \varphi_0(x), .., \varphi_{m-1}(x)) \mid x \in A \cap N[g] = \tau_A^g \}$

All $U_m \in N$ $\forall m$. Let $U = \bigcup_m U_m$. Easy: $A \cap N[g] \subseteq p[U] \subseteq A$.

Now $\Vdash^N_{Col(\omega,\delta)} (\forall x)(x \in \tau_A \longrightarrow \underbrace{(x \restriction m, (\varphi_0(x), .. \varphi_{m-1}(x)))}_{\vec{\mathcal{L}}} \in \check{U}_m)$

$\Vdash^{\bar{N}}_{Col(\omega,\bar{\delta})}$ — " — $\pi^{-1}(\tau_A)$ — " — $\underbrace{\pi^{-1}(\varphi_0(x)) \dots\dots}_{\vec{\mathcal{L}}}$ $\in \pi^{-1}(\check{U}_m)$

So if $\bar{U} = \bigcup_n \bar{U}_n$ then if $x \in \pi^{-1}(\tau_A)^g \Rightarrow (x, \vec{\alpha}) \in \bar{U}$

So $x \in p[U] \subseteq A$.

15:30 29.7.2010 RALF SCHINDLER

Definition Let $A \subseteq \mathbb{R}$, $A \in J_\beta(\mathbb{R})$. We say that N is A-iterable iff there is an iteration strategy Σ for N s.t. whenever $N \rightsquigarrow N^*$ is an iteration ~~strategy~~ according to Σ then

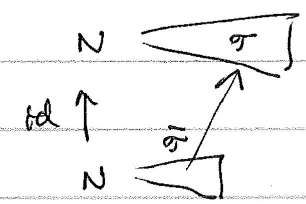
$$i(\tau^N_{A,(\delta+n)^N}) = \tau^{N^*}_{A,(i(\delta)+n)^{N^*}}$$

Theorem Given $A \in J_\beta(\mathbb{R})$ there is an ~~A~~ α-suitable A-iterable premouse.

Let $\mathcal{A} \subseteq J_\beta(\mathbb{R})$ be an sjs. Then there is actually one ~~sjs~~ N α-suitable A-iterable for all $A \in \mathcal{A}$. This iteration strategy for N will be guided be the terms τ_A for $A \in \mathcal{A}$ in the following sense:

$$b = \Sigma(\mathcal{T}) \text{ iff } i_b(\tau^N_A) = \tau_A^{M_b^{\mathcal{T}}}$$

Therefore Σ satisfies condensation: (Hull condensation)



$\bar{\mathcal{T}} \to \mathcal{T}$ $\mathcal{T}$ according to Σ.
Then $\bar{\mathcal{T}}$ is according to Σ.

Now fix a universal $\Sigma_1^{J_\beta(\mathbb{R})}$ set that is a countable union of sets ~~of the form~~ from $\mathcal{A} \subseteq J_\beta(\mathbb{R})$ when $\mathcal{A}$ is sjs. Let N be α-suitable s.t. there is one strategy Σ for N witnessing N is A-iterable for all $A \in \mathcal{A}$ + Σ has condensation.

Now for every $i<\omega$ we want to produce "Σ-mice" with i Woodin cardinals.

1st Step: Produce $L^\Sigma(N)$. (Inside the right universe.)
We want to construct this as an F-mouse. Need to find what F is. We let
$L^\Sigma(N) = L[N,S]$ $\quad S \subseteq On$.
$J_\gamma^\Sigma(N) = J_\gamma[N, S\restriction\gamma]$ $\to$ this is amenable, as $\mathcal{T}$ was chosen
$S\restriction[\gamma, \gamma+lh(\mathcal{T})) = \{\gamma+\xi \mid \xi \in b\}$ to be minimal with $\mathcal{T} \in J_\gamma^\Sigma(N)$ s.t. $b \notin J_\gamma^\Sigma[N]$.
Then $L^\Sigma(N)$ is a fine structural model by construction for Σ.

In general: Produce $K^{c,\Sigma}(N)$.
The difference between this and the $L^\Sigma(N)$ case:
In $L^\Sigma(N)$ case we don't need to core down, whereas in the $K^{c,\Sigma}(N)$ we may need to core down even at the stages where we are adding the predicate for the new branch.

Definition (Σ-premouse) A structure of the form
$$J_\gamma[N, \vec{E}, S]$$

For the right γ, γ' $S\restriction(\gamma,\gamma']$ codes the branch (or an IS thereof) b through the $<_{J_\gamma[N,\vec{E},S]}$-least $\mathcal{T}$ (not yet dealt with) where $b = \Sigma(\mathcal{T})$.

Inductively we want to ~~produce~~ prove that $M_n^{\#,\Sigma}$ exists for all x (x need not be a real – depends on the application). We ~~now~~ now claim that the $M_n^{\#,\Sigma}(x)$ witness $(W^*_{\beta+1})$.

Now let B be a universal $\Sigma_n^{J_\beta(\mathbb{R})}$-set, $B = \bigcup_m A_m$, all $A_m \in \mathcal{A}$ hence captured over N

$N^* = M_n^{\#,\Sigma}(N, x)$ can see the sequence of terms $\tau_{A_m}^N$.

Claim N^* has a term capturing B.

Ralf sketched the idea but I was not able to record it.

29.7.2010 4:50 DISCUSSION: JOHN STEEL

Sketch of a proof of N_x^* exists

Assuming AD^+. Let Γ be lightface inductive like with scale(Γ).
Let $A \in \Delta$. Assume every set in $\check{\Gamma}$ is Suslin.
Let Γ_1 be an inductive like pointclass with scale(Γ_1) and $\Gamma \subsetneq \Delta_1$.
Let T, U be trees on a universal Γ_1-set and its complement.
Look at $HOD_{(T,U)}^{L[T,U,x]}$ where x is of sufficiently large Turing degree

$$N = HOD_{(T,U)}^{L[T,U,x]} \models \omega_2^{L[T,U,x]} \text{ is Woodin}.$$

Let ~~$\models$~~ δ_0 = the <u>least</u> s.t. $C_\Gamma(V_{\delta_0}^N) \models \delta_0$ is Woodin
i.e. δ_0 is Woodin w.r.t. all sets in $C_\Gamma(V_{\delta_0}^N, <)$
Let T_Γ ~~the~~ a tree of scale on the universal Γ set

$$L(T_\Gamma, V_{\delta_0}^N, \underset{\substack{\uparrow \\ \text{w.o. on } V_{\delta_0}^N}}{<}) \models \delta_0 \text{ is Woodin}$$

Let $\Gamma \subsetneq \Omega \subseteq \Gamma_1$ where Ω another pointclass,
(R, S) trees for universal Ω-set + its complement,
one easily defined from (T, U)

$$L_\alpha(R, S, V_{\delta_0}^N) \models \cancel{\delta_0 \text{ not Woodin}}$$
$$\supseteq C_\Omega(V_{\delta_0}^N)$$

So $L_\alpha(R, S, V_{\delta_0}^N) \models \delta_0$ is not Woodin.

Why: Otherwise take a hull $L_{\bar{\alpha}}(\bar{R}, \bar{S}, V_\eta^N)$ where $\eta < \delta_0$.
Then $C_\Gamma(V_\eta^N) \subseteq L_\alpha(\bar{R}, \bar{S}, V_\eta^N)$!

Now working in N: let P be countable, $\pi : P \to V_\gamma^N$
where γ large with (T, U) in rng(π), $\delta_0 \in$ rng(π).

(*) $N \models P$ is ω_1-iterable by choosing the unique cofinal branch b of $\mathcal{T}$ s.t. $C_\Omega(M(\mathcal{T})) \subseteq M_b^{\mathcal{T}}$.

since such branch is the unique cofinal realizable branch.

If
$$\begin{tikzcd} P \arrow[r, "\pi"] \arrow[dr, "i_b"', "\mathcal{T}"] & V_\gamma \\ & M_b^{\mathcal{T}} \arrow[u, "\sigma"'] \end{tikzcd}$$
then $C_\Omega(M(\mathcal{T})) \subseteq M_b^{\mathcal{T}}$ since: $\sigma(i_b(\bar{T},\bar{U})) = (T,U)$

$P \models \underbrace{\text{"I am } C_\Omega \text{ full at } \delta_0\text{"}}_{\varphi}$ is true of any coding $x = x(V^P_{\delta_0+1}, g)$ where g is $Col(\omega, \delta_0^P)$-gen

as certified by $\bar{T}$. So $\Vdash^P_{Col}$ "$(\varphi, x_g) \in p[\bar{T}]$". But this corresponding statement is true in V_γ, so also in P. Hence M_b satisfies the same statement of $i_b(\bar{T})$. Since $\sigma: i_b(\bar{T}) \mapsto T$, φ is really true of any $x = x_g(i_b(V^P_{\delta_0+1}), g)$.
Also $M_b^{\mathcal{T}} \models$ There is $f: \delta(\mathcal{T}) \to \delta(\mathcal{T})$ witnessing non-Woodinness of $\delta(\mathcal{T})$ in $C_\Omega(M(\mathcal{T}))$

Remark: $\mathcal{T}$ based on $V^P_{\delta_0+1}$. But we cannot have two distinct branches with this property, i.e. that $\delta(\mathcal{T})$ is not $C_\Omega(M(\mathcal{T}))$-Woodin in $M_b^{\mathcal{T}}$. So (*) defines an iteration strategy. Let Σ_P^N = this strategy as defined in N. Then Σ^P extends to V via the same definition because N has (T,U) and thus is sufficiently correct. Now

$N \models \Sigma_P$ has condensation (hull or branch)

Why:
$$\begin{tikzcd} & Q \arrow[r, "f"] & V_\gamma^N \\ P^{(\bar{T},\bar{U})} \arrow[ur, "\mathcal{T}"] \arrow[r, "\mathcal{U}"'] & R \arrow[u, "\sigma"'] & \end{tikzcd}$$
Here proof shows $C_\Omega(M(\mathcal{U})) \subseteq R$ (The argument with term relations is a local version of this.)

But then Σ_P has condensation in V.
To get A-iterability just put trees for $A, \neg A$ in the range of
$\pi : P \longrightarrow V^N_\gamma$.

To get full properties of $N^*_x, \delta_x, \Sigma_x, M_x$. $A \subseteq \mathbb{R}$.
We have

1. $N^*_x \models \delta_x$ is Woodin
2. ~~Σ_x~~ (N^*_x, Σ_x) Suslin captures A at δ_x.

$(M_x = N^*_x | \delta_x)$ $M_x \restriction \delta_x = N^*_x \restriction \delta_x$

$\Sigma_x \restriction$ trees on ~~[illegible]~~ $M_x \restriction \delta_x$ which $\in M_x | \kappa_x$.

$N^*_x = L(M_x, \Sigma_x)$

$M_x = M_1^{(P,\Sigma_P)}(x)$ "$x \geq_T P$"
$\delta_x =$ The Woodin of $M^{(P,\Sigma_P)}(x)$
$\kappa_x = 1^{st}$ inaccessible of $M_x > \delta_x$
~~[illegible]~~

Correction: $\Sigma_x =$ the canonical strategy for $M_x \restriction$ trees on $M_x | \delta_x$ which are $\in M_x | \kappa_x$.

Proof that the above works follows the proof that it works when $(P, \Sigma_P) = \emptyset$ (For M_1 and its strategy). However there is a problem: Σ_P determines itself on generic extensions?
Fix: In the paper "Derived models associated to mice" (Section 11?)

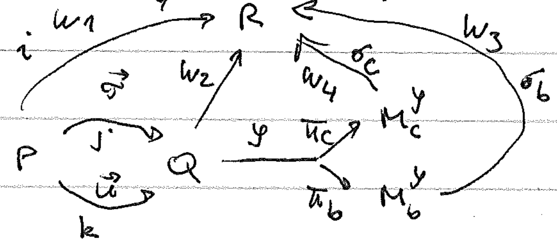
<u>Theorem</u> (Revising ZFC comparison) (P,Σ), (Q,Λ) two pairs s.t. letting $\Gamma = \Gamma(P,\Sigma) \cup \Gamma(Q,\Lambda)$ we have $L(\Gamma,\mathbb{R}) \models AD^+$.
Σ, Λ are Γ-FPR. Also assume: For any $S \in B(P,\overset{\Sigma}{\cancel{\Lambda}})$, $R \in B(Q,\overset{\Lambda}{\cancel{\Sigma}})$: Σ_S and Λ_R have branch condensation and Σ, Λ have hull condensation.
Then comparison holds for (P,Σ) and (Q,Λ).

<u>Remark</u> One may vary the argument; at initial steps one may avoid the use of hull condensation.

<u>Theorem</u> Suppose (P,Σ) is a hod pair s.t. Σ has BC and is FPR. Then Σ is positional. (i.e. $\Sigma_{Q,\vec{\tau}}$ is independent of $\vec{\tau}$. So $P \to Q \xrightarrow[\vec{\tau}, j]{\vec{u}, i} R$ $i = j$ commuting.)

<u>Proof</u> WLOG $\lambda^P = 0$. ~~[illegible]~~ $P \overset{\vec{\sigma}}{\underset{\vec{u}}{\rightrightarrows}} Q$.

Want $\Sigma_{Q,\vec{\sigma}} = \Sigma_{Q,\vec{u}}$. Let ~~$\mathcal{S}$~~ $\mathcal{Y}$ be a normal tree on Q.
Want: $\Sigma_{Q,\vec{\sigma}}(\mathcal{Y}) = \Sigma_{Q,\vec{u}}(\mathcal{Y})$. Can arrange the following



~~[illegible]~~

$\Sigma_{R,W_1} = \Sigma_{R,\vec{\sigma}^\frown W_2} = \ldots$

How to arrange: <u>Exercise</u> - do the AD^+-comparison argument
<u>Claim</u> $i = \sigma_c \circ \pi_c \circ j = \sigma_b \circ \pi_b \circ k$ By Dodd-Jensen proof.
Idea: D-J proof works as long as $P \overset{\vec{x}}{\underset{\vec{y}}{\rightrightarrows}} Q$ $\Sigma_{Q,\vec{x}} = \Sigma_{Q,\vec{y}} \Rightarrow i^{\vec{x}} = i^{\vec{y}}$

Moreover: $M_c^{\mathcal{Y}} = M_b^{\mathcal{Y}}$ $(= L_{p_\omega}(M(\mathcal{Y})))$.
$W_3 = W_4$ up to the last branch. ~~[illegible]~~ (Because the two models agree below $\delta(\mathcal{Y})$. Let W be the part of W_3, W_4 without the last branches, d the branch in W_3 and e in W_4.
Then $\mathrm{rng}(\sigma_e) \cap \mathrm{rng}(\sigma_b) \cap \delta^R (= \delta(W))$ is cofinal in $\delta(W)$.
So $d = e \Rightarrow \sigma_c = \sigma_e$. Now cancel in the above claim:

Get $\pi_b \circ k = \pi_c \circ j$ so $\mathrm{rng}(\pi_c) \cap \mathrm{rng}(\pi_b)$ is cofinal in $\delta(\mathcal{P})$.
So $c = b$.

The general case is treated similarly. ☒

The mouse set conjecture

Strong mouse capturing (SMC) is the statement.
Suppose (P, Σ) is a hod pair s.t. Σ has a branch condensation and is FPR. Then for all y coding P:

$x \in OD(y, \Sigma) \iff x$ is a Σ-mouse over y

Let $\Gamma = \Sigma^2_1(\mathrm{Code}(\Sigma))$

$C_\Gamma = \{x \subseteq \mathbb{R} \mid x \in OD(\Sigma, y)\}$

We need C_Γ for transitive a s.t. $P \in a$.
Mouse capturing for reals $\Rightarrow$ mouse capturing for transitive sets, i.e.

$b \in C_\Gamma(a) \iff \exists$ a-mouse M s.t. $b \in M$

SMSC (Strong mouse set conjecture) Assume AD^+ + no mouse with superstrong. Then SMC holds.

SMC for reals $\Rightarrow$ SMC for transitive sets can be proved, but is difficult for hod mice. This is the S-construction. The difficulty lies in the fact that we generically collapse the transitive set. If we have only ordinary mice, i.e. only extenders, this is not going to change the hierarchy (we can "fatten" the extenders). But in the case of hod mice we are introducing new iteration trees by collapsing, so we may introduce new branches.

Solution: Change the hierarchy. Then one can do S-constructions.

Theorem Assume AD^+ + no Γ s.t. $L(\Gamma, \mathbb{R}) \models AD_{\mathbb{R}} + \Theta$ regular.
Then SMC holds

Rem Con(Woodin limit of Woodins) $\Rightarrow$ Con($AD_{\mathbb{R}} + \Theta$ regular)

The computation of HOD Assume $AD^+ + SMC$.

Definition (Suitable pair) Notation: If P is a hod pm and λ^P is a successor ~~ordinal~~ then $P^- = P(\lambda^P - 1)$

(P, Σ) is a suitable pair if (P^-, Σ) is a hod pair s.t. Σ has branch condensation and is FPR and P is a hod pm

1. P is a Σ-premouse over P^-
2. P is $\Sigma^2_1(\mathrm{Code}(\Sigma))$-full i.e. $\forall \eta > \delta^P_{\lambda^P - 1}$ $C_{\Sigma^2_1}(P|\eta, \Sigma) \subseteq P$

Suppose (P, Σ) is a hod pair s.t. Σ has BC and is FPR.

$$[P,\Sigma] = \left\{ (Q,\Lambda) \;\middle|\; \begin{array}{l} \bullet\ (Q,\Lambda) \text{ is a hod pair;} \\ \bullet\ \Lambda \text{ has BC and is FPR} \\ \bullet\ (P,\Sigma) \text{ and } (Q,\Lambda) \text{ coiterate to the same thing} \end{array} \right\}$$

Aside If (P, Σ) is as above let

$M_\infty(P,\Sigma)$ = dir lim of all iterates of P via Σ
(More precisely the dir lim of $I(P,\Sigma)$.

Note $M_\infty(P,\Sigma) = M_\infty(Q,\Lambda)$ if (P,Σ) and (Q,Λ) coiterate to the same thing. Let $\pi : P \to M_\infty(P,\Sigma)$

Exercise Show ~~$M_\infty(P,\Sigma)$~~ $\mathrm{Code}(\Sigma)$ is ~~M_∞~~ $\delta_\lambda^{M_\infty(P,\Sigma)}$ Suslin

$\lambda^{M_\infty(P,\Sigma)}$ Code

$\mathbb{B}(P,\Sigma)$
$\shortparallel$
$\mathbb{B} = \{ B \mid B \subseteq [P,\Sigma] \times \mathbb{R} \times \mathbb{R}$; B is OD; $\forall (Q,\Lambda) \in [P,\Sigma]$ $p[B_{(Q,\Lambda)}]$ on the first coordinate is the set of codes of $Q \}$

<u>Definition</u> Suppose (P,Σ) is suitable and $B \in \mathbb{B}(P^-,\Sigma)$
*Then for P-cardinals $\kappa > o(P^-)$ there is a term τ capturing B, i.e. for any $g \subseteq Col(\omega,\kappa)$:

$$\forall R \in I(P^-,\Sigma) \cap P[g] \quad (\tau_g)_{(R,\Sigma^P_R)} = \mathbb{B}_{(R,\Sigma_R)} \cap (\mathbb{R}^{P[g]})^2$$

Then ~~the~~ standard term ~~...~~ doing the capturing. ~~...~~ is denoted by $\tau_{B,\kappa}$. (One can show that the standard such term is unique.)

* this "then" is nontrivial - it has to be proved, and it was proved for ordinary premice in one of John Steel's previous lectures.

14:00 GRIGOR SARGSYAN

~~Defini~~ Given suitable (P,Σ), $B \in \mathbb{B}(P,\Sigma)$ let $\tau^P_{B,i}$ be the term capturing B at $(\delta^{+i})^P$. Here $\delta^Q \overset{\text{def}}{=} \delta^Q_{\lambda_Q}$).
We let ~~$\gamma^P_{B,i}$~~

$$\gamma^P_{B,i} = \sup \mathrm{Hull}^P_{\Sigma_1}\{P^- \cup \{\tau_{B,k} \mid k \le i\}\} \cap \delta^P$$

<u>Theorem</u> $\gamma^P_{B,i} = \sup \mathrm{Hull}^P_{\Sigma_1}\{P^- \cup \gamma^P_{B,i} \cup \{\tau_{B,k} \mid k \le i\}\} \cap \delta^P$
and
$$H^P_{B,i} = \mathrm{Hull}^P_{\Sigma_1}(P^- \cup \gamma^P_{B,i} \cup \{\tau^P_{B,k} \mid k \le i\})$$

Definition (P,Σ) suitable, $B \in \mathbb{B}(P^-,\Sigma)$. The pair (P,Σ) is B-iterable iff player II has a w.s. in the following game on P:

1. $\vec{T}_0$ a stack on P^- according to Σ $\quad P_0 \overset{\vec{T}}{\rightsquigarrow} P_1$
2. I starts playing subrounds. If this lasts ω steps, I loses. I plays normal "correctly" guided maximal trees, $\mathcal{U}_i$; II plays cwfb's. Want $(M^{\mathcal{U}_i}_{b_i}, \Sigma_{M^{\mathcal{U}_i}_{b_i}})$ is suitable + the term relations are moved correctly: $i^{\mathcal{U}_{i+1}}_{b_{i+1}}(\tau^{M^{\mathcal{U}_i}_{b_i}}_{B,k}) = \tau^{M^{\mathcal{U}_{i+1}}_{b_{i+1}}}_{B,k}$ all i.
3. I exists after finitely many steps and starts the main round on P_2^-
4. If I uses ω subrounds II wins
5. II wins of the game has ω_1 steps.
6. Σ moves B correctly, if not, II loses

Definition Given a suitable pair (P,Σ) and $B \in \mathbb{B}(P,\Sigma)$: (P,Σ) is strongly B-iterable if the embeddings in the subrounds do not depend on the play, i.e: In B-iteration game, embeddings of the subrounds $i_b^{\mathcal{U}} \gamma^P_{B,k}$ are independent of of b. Def above says that it is independent of the play also.

Hypo Suppose (P,Σ) is a hod pair s.t. for some α
$$M_\infty(P,\Sigma)|\Theta_\alpha = V^{HOD}_{\Theta_\alpha}$$
Suppose for every $B \in \mathbb{B}(P,\Sigma)$ there is a suitable pair (Q,Λ) s.t. (Q,Λ) is strongly B-iterable, $(Q,\Lambda) \in [P,\Sigma]$.
Moreover $\forall B\ \forall (Q,\Lambda)\ \forall c$ s.t. (Q,Λ) is suitable, $(Q^-,\Lambda) \in [P,\Sigma]$: (Q,Λ) is B-iterable, $B, C \in \mathbb{B}(P,\Sigma)$, there is a B-tail

of (Q,Λ), (R,Ψ) which is C-iterable [B-tail means some iterate according to some strong B-iteration strategy.]

$I = \{(Q,\Lambda,B) \mid (Q,\Lambda)$ is suitable, $(Q,\Lambda) \in [P,\Sigma]$, (Q,Λ) is strongly B-iterable$\}$

$\mathcal{F}$ = the set of all $H^{Q,\Lambda}_{B,k}$ s.t. $(Q,\Lambda,B) \in I$

Define $\leq^*$ on I:

$(Q,\Lambda,\vec{B}) \leq^* (R,\Psi,\vec{C})$ iff

(R,Ψ) is a $\vec{B}$-tail of (Q,Λ) and $\vec{B} \subseteq \vec{C}$

We let

$$i_{(Q,\Lambda,\vec{B}),(R,\Psi,\vec{C})} : \bigcup_{k\in\omega} H^{Q,\Lambda}_{\vec{B},k} \longrightarrow \bigcup_{k\in\omega} H^{R,\Psi}_{\vec{C},k}$$

Let M_∞ = dir lim $(\mathcal{F}, \leq^*)$ under $i_{(Q,\Lambda,\vec{B}),(R,\Psi,\vec{C})}$

Hypo 2 ~~Suppose~~ M_∞ is wf and $\delta_\infty = \delta^{M_\infty} = \Theta_{\alpha+1}$

We prove:

Hypo 1 + Hypo 2 $\Rightarrow$ $M_\infty | \Theta_{\alpha+1} = V^{HOD}_{\Theta_{\alpha+1}}$

Proof Directly from the above: $M_\infty \subseteq$ HOD.

Let $A \subseteq \gamma < \Theta_{\alpha+1}$, A is OD. WTS $A \in M_\infty$.

Fix $(Q,\Lambda,B) \in I$. Show $A \in M_\infty$. Fix $(Q,\Lambda,B) \in I$ s.t. $i_{(Q,\Lambda,B),\infty} > \gamma$. Let

$C = \{((R,\Psi), x, y) \mid (R,\Psi,B) \in I$ and x codes R y codes some $\alpha_y < \gamma^{R,\Psi}_B$ & $i_{(R,\Psi,B),\infty}(\alpha_y) \in A)\}$

$C \in \mathbb{B}(P,\Sigma)$

Let $\tau_{C,\infty}$ = image of τ_C's from the direct limit (i.e. $(S,\Phi,C) \in I$: $\tau_{C,\infty} = i_{(S,\Phi,C),\infty}(\tau_C^{S,\Phi})$)

Then A is defined over M_∞ by (letting $\delta = \Theta_{\alpha+1}$)
$\beta \in A \Leftrightarrow \Vdash^{M_\infty}_{Col(\omega,\delta^+)} ((M_\infty|\delta^+, \Sigma_{M_\infty^-}), \dot{x}, \dot{y}) \in \tau_C$ where $\dot{x}$ is a code for M_∞^- & $\dot{y}$ is a code for β)

<u>Proof</u> Suppose $\beta \in A$, fix $(S,\Phi,B) \in A$ and $\bar{\beta} < \gamma_B^{S,\Phi}$ s.t. $i_{(S,\Phi,B),\infty}(\bar{\beta}) = \beta$ &

$\Vdash^{S}_{Col(\omega,\delta^{+S})} (S|(\delta^+)^S, \dot{\Sigma}^S, \dot{x}, \dot{y}) \in \tau_C$ where $\dot{x}$ is a code for S^-
$\dot{y}$ is a code for $\bar{\beta}$

then use elementarity of the direct limit maps. This proves "$\Rightarrow$" in the above equivalence, which is the heart of the argument $\boxtimes$

<u>Proving Hypo 1 + Hypo 2.</u>

Assume $\Theta_{\alpha+2}$ exists.

For Hypo 1 ETS: There is a hod pair (P,Σ) s.t.
$M_\infty(P^-, \Sigma_{P^-}) = V^{HOD}_{\Theta_\alpha}$,
(P,Σ_{P^-}) is suitable, Σ is FPR and has BC.

We say that Σ is <u>$\langle B_i \mid i<\omega\rangle$ - guided</u> iff $\{B_i\}_{i<\omega} \subseteq \mathbb{B}(P^-, \Sigma_{P^-})$ and $\forall C \in \mathbb{B}(P^-, \Sigma_{P^-})$ $\exists$ tail (Q,Λ) of (P,Σ) that Λ <u>respects</u> C.
"Λ-respects C" means: moves the term relations for C correctly.

Def (Σ, P) a hod pair, Σ has BC and is FPR.
$\vec{B} = \{B_i \mid i \in \omega\} \subseteq \mathbb{B}(P^-, \Sigma_{P^-})$ is $\vec{B}$-guided if
(a) $\sup \gamma_{B_i}^{P, \Sigma_{P^-}} = \delta^P$

(b) $(\forall \vec{\mathcal{T}})$ ($\Sigma(\vec{\mathcal{T}}) = b \iff b$ is the unique branch that moves the terms for all B_i's correctly)

(Take this with a grain of salt.)

Let Γ be a good pointclass s.t. $w(\Gamma) > \Theta_{\alpha+1}$. Let F be an N_x^* function for Γ. We can assume (by induction hypo) that we have (P, Σ) s.t. Σ has BC + is FPR and
$$M_\infty(P,\Sigma) \mid \Theta_\alpha = V_{\Theta_\alpha}^{HOD} \Rightarrow \mathrm{Code}(\Gamma) \in \underset{\sim}{\Delta}_\Gamma$$

Fix x s.t. $F(x) = \langle N_x^*, M_x, \delta_x, \Sigma_x \rangle$ captures Σ.

Backtrack Def (P, Σ) strongly respects B iff whenever

$$\begin{tikzcd} P \arrow[r, "\vec{\mathcal{T}}"] \arrow[dr, "i"'] & Q \\ & R \arrow[u, "\sigma"'] \end{tikzcd}$$

$\vec{\mathcal{T}}$ via σ, $i^{\vec{\mathcal{T}}} = \sigma \circ i$ then
$$i^{\vec{\mathcal{T}}}(\tau_B^{P, \Sigma_{P^-}}) = \tau_B^{Q, \Sigma_{Q^-}}$$
$$\sigma^{-1}(\tau_B^{Q, \Sigma_{Q^-}}) = \tau_B^{R, (\Sigma_{Q^-})^\sigma}$$

Exercise Find a problem here and solve it.

First We show that for every B there is a hod pair (P, Σ) s.t. Σ is FPR, has BC and strongly respects B.

Abbreviate: $\mathbb{B} = \mathbb{B}(P, \Sigma)$ for (P, Σ) s.t. $M_\infty(P, \Sigma) = V_{\Theta_\alpha}^{HOD}$

<u>Lemma</u> Let Γ be a good pointclass $> \Theta_{\alpha+1}$. Let (P, Σ) be s.t. Σ has BC + ~~Σ~~ is FPR, $Code(\Sigma) \in \Delta$.
Let F be an N_x^* function. Let $B \in \mathbb{B}$. Let x be s.t. $Code(\Sigma)$ and B are Suslin captured by $(N_x^*, \delta_x, \Sigma_x)$. For some β letting (P_β, Σ_β) be the β-th model of hod mouse construction of N_x^* some tail of Σ_β strongly respects B (In part P_β^- is Σ-iterate of P.)

<u>Proof</u> By comparison argument $\exists \gamma$ s.t. P iterates to P_γ via Σ and $\Sigma_\gamma = \Sigma_{P_\gamma}$. We need to show $P_{\gamma+1}$ exists and $\Sigma_{\gamma+1}$ is FPR + has BC.

1. $P_{\gamma+1}$ exists: What if $N_{\gamma+1}$ $= L[E, \Sigma_\gamma]^{N_x^* | \delta_x}$ projects across δ_γ. Let $M \trianglelefteq N_{\gamma+1}$ s.t. $\rho_M^\omega < \delta_\gamma$. Consider $(M, P_\gamma, \overbrace{\Sigma_{\gamma}}^{\Sigma_{\gamma}})$. We have (1)

   a) M is a Σ_γ-mouse over P_γ, $\rho_M^\omega < \delta_\gamma$ + M least such

   b) Σ_γ is FPR + has BC, $M_\omega(P_\gamma, \Sigma_\gamma) | \Theta_\alpha = V_{\Theta_\alpha}^{HOD}$

Let

$$\mathcal{F} = \{(N, Q, \Lambda) \mid (N, Q, \Lambda) \text{ satisfy a) + b) in place of } (M, P_\gamma, \Sigma_\gamma)\}$$

Let $\leq^*$ on $\mathcal{F}$ be given by

$(N, Q, \Lambda) \leq^* (N^*, Q^*, \Lambda^*)$ iff Q^* is a Λ-iterate of Q and $\Lambda^* = \Lambda_{Q^*}$

Here projectum stays below δ^Q as doing fine structural iterations one level down — FSIT Ch 6.

(1) Σ_γ comes from a strategy of M.

Let

$M =$ dir lim $(\mathcal{F}, \leq^*)$ under the iteration maps

By the same result, $\rho^\omega_M < \Theta_\alpha$ and $M \subseteq$ HOD. Contradiction, because $M_\infty(P, \Sigma) | \Theta_\alpha = V^{HOD}_{\Theta_\alpha}$. So $M \in M_\alpha(P, \Sigma)$ hence $M \in M$. This shows that $\rho^\omega_{M_{\gamma+1}} \not< \delta_\gamma$. $\boxtimes$

Next need to see: if $cf(\gamma)$ is measurable in P_γ then $(\delta_\gamma^+)^{P_\gamma} = (\delta_\gamma^+)^{N_{\gamma+1}}$. Suppose not. Let $M \trianglelefteq N_{\gamma+1}$ be s.t. $\rho^\omega_M = \delta_\gamma$. This is an easy exercise on FPR.

Now let $w(A) = \Theta_{\alpha+1}$ and

$A = \{(Q, \Lambda, x, C_{\Sigma^2_1(Code(\Lambda))}(x)) \mid (Q, \Lambda)$ sits at Θ_α and x is a real coding $x, Q \ldots\}$

$B = \{\sigma \mid \sigma^{-1}(A) = (Q, \Lambda, \vec{C})$ s.t. $\vec{C}$ is a sjs at $\delta^2_1(Code(\Lambda))\}$ for Σ^2_1

Assume: B is captured by N^*_x. Work in $N^*_x[g]$ where g is generic for $Col(\omega, \gamma)$. Let $\sigma \in N^*_x[g]$ be s.t.

$\sigma^{-1}(A) = ((P_\gamma, \Sigma_\gamma), \vec{C})$ $\vec{C}$ is sjs

Let $\pi : N \to N^* | \lambda$ $\lambda \gg \delta_x$

$\kappa = cr(\pi) \to \delta_x$

$V_\kappa \subseteq N$, $\vec{C} \in rng(\pi)$. Now $N \models \kappa$ is Woodin $\Rightarrow$ $Lp^{\Sigma_\gamma}_\omega(V^N_\kappa) \models \kappa$ is Woodin. So

$N_{\gamma+1} \models$ the least strong to δ_x is a limit of Woodins

Conclusion: $P_{\gamma+1}$ exists

2.8.2010 9:30 GRIGOR SARGSYAN

$\Sigma_{\gamma+1}$ has BC and is FPR

For FPR use universality + absoluteness: If not then there is $\vec{\mathcal{T}}$ on $P_{\gamma+1}$ according to $\Sigma_{\gamma+1}$ with last model Q and Q is not full. $\exists \alpha < \lambda^Q$, $\eta \in (\delta_\alpha, \mu_{\alpha+1})^Q$ and a ~~mouse~~ $(\Sigma_{\gamma+1})_{Q(\alpha)}$-mouse M over $Q|\eta$ s.t. $M \notin Q$. The same is true in $N_x^*[g]$, $g \subseteq col(\omega, \delta_{\gamma+1})$. Let $(\vec{\mathcal{T}}, M) \in N_x^*[g]$ witnessing that $\Sigma_{\gamma+1}$ is not FPR.

$$\begin{tikzcd} N_{\gamma+1} \arrow[r, "\vec{\mathcal{T}}"] & Q^* \\ P_{\gamma+1} \arrow[u] \arrow[r, "\vec{\mathcal{T}}"'] & Q \arrow[u] \end{tikzcd}$$

Let $S = (L[E, (\Sigma_{\gamma+1})_{Q(\alpha)}][Q|\eta])^{Q^*}$

Now S is universal, so

$M \trianglelefteq S \Rightarrow M \trianglelefteq Q$ $\boxtimes$ (FPR)

For BC use the stack arguments: "If BC fails, we can internalize it." We get

$$\begin{tikzcd} & R \\ P_{\gamma+1} \arrow[ur, "\vec{\mathcal{T}}"] \arrow[dr, "\vec{\mathcal{U}}"'] & \\ & Q \arrow[uu, "\sigma"'] \end{tikzcd}$$

Some branch c through $\vec{\mathcal{U}}$ s.t. $\sigma \circ i_c^{\vec{\mathcal{U}}} = i^{\vec{\mathcal{T}}}$

By absoluteness there is $\vec{\mathcal{T}}, \vec{\mathcal{U}}, c, R, Q, \sigma$ inside $N_x^*[g]$, $g \subseteq col(\omega, P_{\gamma+1})$.

In N_x^*:

$S = S(N_{\gamma+1}) = \bigcup\{M \mid \rho(M) = \delta_x \ \& \ M \text{ is } \Sigma_\gamma\text{-universal} \ldots\}$

S

$N_{\gamma+1}$

$R^* \longrightarrow$ Result of applying $\vec{\mathcal{T}}$ to S

$Q^* \nearrow$ — " — $\vec{\mathcal{U}}^\frown c$ to S.

$$\begin{tikzcd} P_{\gamma+1} \arrow[r, "\vec{\mathcal{T}}"] \arrow[dr, "{\vec{\mathcal{U}}, c}"'] & R \\ & Q \arrow[u] \end{tikzcd}$$

Claim Q^* is w.f.

Proof Extend σ to act on Q^*: get $\sigma^*: Q^* \longrightarrow R^*$ in the usual way.

Also let $b = \Sigma_{\gamma+1}(\vec{\mathcal{U}})$ and $S \xrightarrow{\vec{\mathcal{U}},\, b} W^*$. Note:
$Q^* \upharpoonright \pi_c(\delta_{\gamma+1}) = W^* \upharpoonright \pi_b(\delta_{\gamma+1})$ then Q^*, W^* compare above $\pi_c(\delta_{\gamma+1})$ to some $S^* = S(S^* \upharpoonright \delta_x)$
Let $\lambda = o(S)$. $cf(\lambda) \geq \delta_x$. So there is an ω-club $C \subseteq \lambda$ on which π_b, π_c agree. So there is an ω-club $D \subseteq \sigma_b[C] \cap \sigma_c[C]$. Then

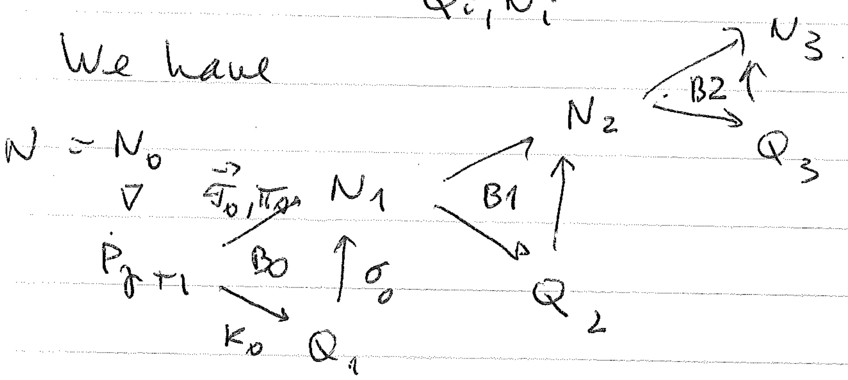
$$\mathrm{Hull}^{S^*}(D \cap [\pi_b(\delta_\gamma), \pi_b(\delta_{\gamma+1})]) \subseteq \mathrm{rng}(\pi_c) \cap \mathrm{rng}(\pi_b)$$

the hull on the LHS must be cofinal in $\delta_{\gamma+}$ by FPR. Otherwise we would get a Woodin cardinal in the interval $[\pi_b(\delta_\gamma), \pi_b(\delta_{\gamma+1})]$, a contradiction.

☒ (BC)

Now we want to show that some tail of $\Sigma_{\gamma+1}$ (strongly) respects B.
If not: In $N_x^*[g]$ we have the same. Let $N = N_x^* | (\nu^{+\omega})^{N_{\gamma+1}}$ where ν = the sup of the first ω Woodin cardinals of $N_{\gamma+1}$.
Let $\langle \mathcal{T}_i, k_i, \underbrace{\sigma_i, \pi_i}_{Q_i,\, N_i} \mid i \in \omega \rangle \in N_x^*[g]$ be as follows

We have



$$\begin{tikzcd}
N_0 \arrow[r] \arrow[dr] & N_1 \arrow[r] \arrow[dr] & N_2 \arrow[r] \arrow[dr] & N_3 & \cdots & N_i \arrow[r] \arrow[dr] & N_{i+1} \arrow[r] & \cdots & N_\omega \\
 & Q_0 \arrow[u] & Q_1 \arrow[u] & Q_2 \arrow[u] & & & Q_i \arrow[u] & &
\end{tikzcd}$$

Let $h \subseteq \mathrm{col}(\omega, <\nu)$ generic over N_x^*. Let $\mathbb{R}^* = \mathbb{R}^*_h$ ~~be the reals~~