Talks Tehran Oct 2015

<u>Remarkable Cardinals</u>

Let $\mathcal{A} = (A; \ldots)$, $\mathcal{B} = (B; \ldots)$ be two models in the same language, $\mathcal{L}$.

Let us consider the following game, $G(\mathcal{A}, \mathcal{B})$.

I	a_0	a_1	...
II	b_0	b_1	...

The game lasts ω steps. Rules:
$\{a_0, \ldots\} \subset A$, $\{b_0, \ldots\} \subset B$, and for all $\mathcal{L}$-formulae φ,

$$\mathcal{A} \models \varphi(a_0, \ldots, a_{k-1}) \iff \mathcal{B} \models \varphi(b_0, \ldots, b_{k-1}).$$

$G(\mathcal{A}, B)$ is closed: If II loses, she loses after finitely many steps.

Hence $G(\mathcal{A}, \mathcal{B})$ is determined by Gale–Steward.

<u>Lemma 1.</u> Let A be ctble. Then II wins $G(\mathcal{A}, \mathcal{B})$ iff there is an elementary embedding $j: \mathcal{A} \longrightarrow \mathcal{B}$.

Proof: "$\Longrightarrow$" Let I play $a_0, a_1, \dots$, where $A = \{a_0, a_1, \dots\}$. The winning strategy for II will produce an elementary embedding $\mathcal{A} \longrightarrow \mathcal{B}$.
"$\Longleftarrow$" Let $j: \mathcal{A} \longrightarrow \mathcal{B}$ be elementary. In responce to $a \in A$, played by I, let II play $b = j(a)$. ⊣

Generalization:

<u>Lemma 2.</u> II wins $G(\mathcal{A}, \mathcal{B})$ iff in $V^{\mathrm{Col}(\omega, A)}$ there is an elementary embedding $j: \mathcal{A} \longrightarrow \mathcal{B}$.

Proof: "$\Longrightarrow$" Let τ be a w.s. for II. Then τ is still a w.s. for II in $V^{\mathrm{Col}(\omega, A)}$. Then use Lemma 1.

"$\Leftarrow$" Let $p \Vdash \tilde{j} : \check{\mathcal{A}} \to \check{\mathcal{B}}$. Look at $\tilde{G}(\mathcal{A}, \mathcal{B})$, played in V:

I	a_0		a_1		...	
II		p_0, b_0		p_1, b_1		...

Rules: $\{a_0, \dots\} \subset A$, $\{b_0, \dots\} \subset B$, $p_0 \leq p$, $p_{n+1} \leq p_n$, $p_n \Vdash \tilde{j}(\check{a}_n) = \check{b}_n$.

Clearly, II has a w.s. in $\tilde{G}(\mathcal{A}, \mathcal{B})$. But this results in a w.s. for II in $G(\mathcal{A}, \mathcal{B})$ by hiding the side moves $p_0, p_1, \dots$ $\dashv$

<u>Corollary</u> 1. If there is an elementary embedding $j : \mathcal{A} \to \mathcal{B}$ (in V), then II wins $G(\mathcal{A}, \mathcal{B})$.

We shall now be interested in rank initial segments of V as our models.

Recall : κ is supercompact iff for every $\lambda > \kappa$ there is an elementary embedding $j : V \to M$ with critical point κ s.t. M is transitive and $j(\kappa) > \lambda$, ${}^{\lambda}M \subset M$.

<u>Lemma</u> 3 (Magidor ?) Let κ be supercompact. For all $\alpha > \kappa$ there is some $\beta < \kappa$ and some $j^* : V_\beta \to V_\alpha$ s.t. $j^*(\mathrm{crit}(j^*)) = \kappa$.

Proof : Fix α. Pick $j : V \to M$ with $\mathrm{crit}(j) = \kappa$ s.t. ${}^{\overline{\overline{V_\alpha}}}M \subset M$, $\alpha < j(\kappa)$. In particular, $\{ V_\alpha, j \upharpoonright V_\alpha \} \subset M$ and also $V_\alpha^M = V_\alpha$. Hence

$$M \models \exists \beta < j(\kappa)\ \exists j^* : V_\beta \to j(V_\alpha),\ j^*(\mathrm{crit}(j^*)) = j(\kappa).$$

(True as being witnessed by α, $j \upharpoonright V_\alpha$.) So

$$V \models \exists \beta < \kappa\ \exists j^*\ V_\beta \to V_\alpha,\ j^*(\mathrm{crit}(j^*)) = \kappa. \quad \dashv$$

The converse of Lemma 3 is true also.

<u>Corollary</u> 2. Let κ be supercompact. For all $\alpha > \kappa$ there is $\beta < \kappa$ s.t. $\underline{\text{II}}$ wins $G(V_\beta, V_\alpha)$.

We may obviously reformulate the conclusion of Cor. 2 as follows. Consider the following game, G_0^κ:

I	α	x_0	x_1	...
$\underline{\text{II}}$	β	y_0	y_1	...

Rules: $\alpha > \kappa$, $\beta < \kappa$, $\{x_0, x_1, \ldots\} \subset V_\beta$, $\{y_0, y_1, \ldots\} \subset V_\alpha$, and for all formulae φ,

$$V_\beta \models \varphi(x_0, \ldots, x_{k-1}) \iff V_\alpha \models \varphi(y_0, \ldots, y_{k-1}).$$

<u>Cor.</u> 3 If κ is supercompact, then $\underline{\text{II}}$ wins G_0^κ.

Let us also consider a variant of G_0^κ, call it G_*^κ:

I	α	x_0	x_1	$\ldots$
II	$\beta, \bar\kappa$	y_0	y_1	$\ldots$

Rules: As in G_1^κ, plus: $\bar\kappa < \beta$, if $x_n \in V_{\bar\kappa}$, then $y_n = x_n$, and if $x_n = \bar\kappa$, then $y_n = \kappa$.

<u>Definition</u> 1. κ is called remarkable iff II wins G_*^κ.

By Lemma 3 and the proof of Lemma 2:

<u>Corollary</u> 4. If κ is supercompact, then κ is remarkable.

In contrast to supercompact cardinals, remarkable cardinals exist in L.

<u>Lemma</u> 4. Let M be a transitive model of ZFC, and let $\mathcal{A}, \mathcal{B} \in M$. Then

$V \models$ II wins $G(\mathcal{A}, \mathcal{B})$ iff $M \models$ II wins $G(\mathcal{A}, \mathcal{B})$.

Proof: "$\Leftarrow$" ✓

"$\Rightarrow$" If $M \models$ I wins $G(\mathcal{A}, \mathcal{B})$ via τ.

Let $T =$ the tree of all finite initial segments of plays of $G(\mathcal{A}, \mathcal{B})$ in which I follows τ and II didn't yet lose.

If $V \models \tau$ is not a w.s. for I in $G(\mathcal{A}, \mathcal{B})$, then T is ill-founded in V, hence in M, hence M has an infinite play of $G(\mathcal{A}, \mathcal{B})$ in which I follows τ but does not win. ↯

$\dashv$

The proof of Lemma 4 plus Cor. 4 give:

<u>Lemma</u> 5. If κ is supercompact, then κ is remarkable in L.

Let's prove a stronger result, building upon Lemma 4.

<u>Lemma</u> 6. Suppose that M is a transitive model of ZFC, and let $\mathcal{A}, \mathcal{B} \in M$ be such that in V, there is an el. embedding $j: \mathcal{A} \to \mathcal{B}$. Then $\underline{\mathrm{II}}$ wins $G(\mathcal{A}, \mathcal{B})$ in M.

Proof: Lemma 1 and Lemma 4. $\dashv$

<u>Theorem</u> 1. Assume $0^\#$ exists. Every Silver indiscernible is remarkable in L.

Proof: Let κ be a Silver indiscernible, and let $\alpha > \kappa$.

Let $j : L \longrightarrow L$ be such that $\mathrm{crit}(j) = \kappa$ and $j(\kappa) > \alpha$. As $j \restriction V_\alpha^L : V_\alpha^L \longrightarrow V_{j(\alpha)}^L$ is elementary, $\underline{II}$ wins $G_*^{j(\kappa)}$ in L by (the proof of) Lemma 6, where $\underline{I}$ starts out by playing $j(\alpha)$.

By the elementarity of j, $\underline{II}$ wins G_*^κ in L, where $\underline{I}$ starts out by playing α.

As α was arbitrary, $\underline{II}$ wins G_*^κ in L. $\dashv$

<u>Theorem</u> 2. Let (κ, γ) be lexicographically least s.t. $\kappa < \gamma \leq \infty$ and

$L_\gamma \models$ "ZFC + $\underline{II}$ wins G_0^κ." then

κ is remarkable in L_γ.

Proof. Suppose that σ is a w.s. for $\underline{II}$ in G_0^κ:

$\underline{I}$	α	x_0	x_1	...
$\underline{II}$	β	y_0	y_1	...

By the proof of Lemma 2, for all $\alpha < \gamma$, $\alpha > \kappa$, there is then some $\beta < \kappa$ s.t. in $L_\gamma^{Col(\omega, \overline{\overline{V_\beta^L}})}$ there is an el. embedding $j_{\beta\alpha} : V_\beta^L \to V_\alpha^L$.

It suffices to see that $j_{\beta\alpha}(\mathrm{crit}(j_{\beta\alpha})) = \kappa$.

Suppose this were wrong for some α, β. Write $j = j_{\beta\alpha}$. Write $\lambda = \mathrm{crit}(j) < \bar{\kappa} = j^{-1}(\kappa)$, which we may assume to exist.

Let $\alpha < \min\{\bar{\kappa}, j(\lambda)\}$. As $j \upharpoonright V_\alpha^L : V_\alpha^L \to V_{j(\alpha)}^L$ is elementary, II wins ~~every~~ $G_0^{j(\lambda)}$ in L_κ, when I starts out by playing $j(\alpha)$.

Using j, II wins G_0^λ in $L_{\bar{\kappa}}$, where I starts out by playing α.

In other words, $L_{\bar{\kappa}} \models$ "II wins G_0^λ."

But $L_{\bar{\kappa}} \models ZFC$, so we have a contradiction to the minimality of (κ, γ). $\dashv$

On the other hand, if κ is remarkable in L_γ, then clearly II wins G_0^κ.