# Recent insights concerning the continuum problem

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# Set Theory and Cantor's Continuum Hypothesis

- Set theory started with the following theorem of Georg Cantor.
- Cantor (Nov 11, 1873, in a letter to R. Dedekind): $\mathbb{R}$ is uncountable. I.e., there are uncountably many real numbers.
- Cantor's first proof of this used nested intervals.
- But how many real numbers are there?
- Continuum Hypothesis (CH): For every uncountable $A \subset \mathbb{R}$ there is a bijection $f : \mathbb{R} \to A$.
- Cantor's Program: Show CH by "induction on the complexity" of $A \subset \mathbb{R}$.

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- Young (1906): Every uncountable $G_\delta$– oder $F_\sigma$–set $A \subset \mathbb{R}$ contains a perfect subset.
- Aleksandrov/Hausdorff (1916): Every uncountable *Borel* set $A \subset \mathbb{R}$ contains a perfect subset.
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# Cantor's Generalized Continuum Hypothesis

- In addition to sets of natural numbers, of reals, of sets of reals, etc., Cantor started considering sets *in general*.
- "By a 'set' we understand any gathering-together M of determined well-distinguished objects m of our intuition or of our thought, into a whole." (Cantor, 1995)
- This idea leads to the cumulative hierarchy of sets.
- For every set $x$ whatsoever, the *power set* $\mathcal{P}(x)$ exists.

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- For every set $x$ whatsoever, the *power set* $\mathcal{P}(x)$ exists.

- Cantor's Theorem (1892): Let $x$ be any set. There is no surjection $f : x \to \mathcal{P}(x)$.
- This time, Cantor's proof uses a diagonal argument.
- *How big* is $\mathcal{P}(x)$ in comparison to $x$?
- Generalized Continuum Hypothesis (GCH): For every infinite set $x$ and every $A \subset \mathcal{P}(x)$, there is either a surjection $f : x \to A$ or else a bijection $f : \mathcal{P}(x) \to A$.
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- We need to talk about axiomatizations of set theory in order to discuss CH and GCH.

# The axiom system ZFC (Zermelo–Fraenkel with choice)

- Any two sets with the same elements are equal.
- For all $x$ and $y$, $\{x, y\}$, $\bigcup x$, and $\mathcal{P}(x)$ exist.
- There is an infinite set.
- Separation. For all $x$ and for all formulae $\varphi(y)$, $\{y \in x \colon \varphi(y)\}$ exists.
- Replacement. For all $x$ and for all formulae $\varphi(y, z)$ such that for all $y \in x$ there is a unique $z$ with $\varphi(y, z)$, $\{z \colon \exists y \in x \varphi(y, z)\}$ exists.
- Every $x$ with $\emptyset \notin x$ admits a choice function.
- Every nonempty set has an $\in$–least element.

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- ZFC formalizes the idea (albeit somewhat indirectly) that the universe of set theory arises from nothing ($\emptyset$) through the operations $x \mapsto \mathcal{P}(x)$ and $x \mapsto \bigcup x$ in a cumulative fashion:
- If we define $V_\alpha = \bigcup\{\mathcal{P}(V_\beta)\colon \beta < \alpha\}$ for ordinals $\alpha$, then ZFC proves that every $x$ is an element of some $V_\alpha$. The $V_\alpha$'s are called *ranks*.
- Provably, there is no set of all sets. (By Cantor's Theorem: if $v$ were such a set, then there would be a surjection from $v$ onto $\mathcal{P}(v)$.)
- However, we may introduce a new category of objects, *classes* ("inconsistent multiplicities" in the language of Cantor), and there will be a class of all sets.

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# Back to Cantor's Program.

- Gödel (1938)/Cohen (1963): It is consistent with ZFC that all coanalytic sets of reals (in fact, all sets of reals whatsoever) satisfy CH, and it is also consistent that there is a conanalytic counterexample to CH. This is shown using Gödel's constructible universe $L$ and Cohen's method of forcing.
- So what is true?
- Gödel's Program: Decide statements which are independent from ZFC with the help of well–justified large cardinal axioms!

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# Large cardinals

- Replacement may be construed as a "large cardinal axiom." It says that for every formula $\varphi$ there is a rank $V_\alpha$ which refects $\varphi$, i.e.,

$$\varphi(x_1, ...x_k) \longleftrightarrow V_\alpha \models \varphi(x_1, ..., x_k)$$

  for all $x_1$, ..., $x_k \in V_\alpha$.
- The exploitation of this idea leads to stronger and stronger *reflection principles*: "If $V$ has a certain property, then there is a rank $V_\alpha$ which also has this property."

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- Here is a list of some of the large cardinal concepts which are on the market nowadays.
- Inaccessible < Mahlo < weakly compact < measurable < strong < Woodin < subcompact < supercompact < $I_0$ < ...
- Shelah/Woodin (1990): If there are infinitely many Woodin cardinals, then CH holds for all projective sets.
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- In order to decide CH, we need to search for new and plausible axioms.
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- In order to decide CH, we need to search for new and plausible axioms.
- There are two types of candidates out there: (1) $V$ is "constructible," and (2) $V$ is "saturated with respect to forcings."

# $V$ is constructible

- 1st candidate: $V = L$.
- Problem: "$V = L$" is refuted by large cardinals.
- Refined candidate: $V = L[E]$, where $E$ is an extender sequence.
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- (Foreman, Magidor, Shelah) $MA_{\omega_1}$, PFA, MM, $MM^{++}$.
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