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Addendum to "NS_{ω_1} is not Π_1 definable"

Modulo well-known results, the arguments of that paper may be simplified; Theorem 1 may also be slightly strengthened, as follows.

Theorem 1* (BMM + $\exists$ a Woodin cardinal) There is no $A \subset \omega_1$ and no Σ_1 formula φ in the language $\mathcal{L}_{\dot{\in}}$ of set theory s.t. for all $S \in \mathcal{P}(\omega_1)$, S is stationary iff $\varphi(S, A)$.

Proof: Let $S_0 = \{X \prec H_{\omega_2} : \overline{\overline{X}} = \aleph_1, X \text{ is transitive}\}$, a stationary set. Let g be $\mathbb{P}_{<\delta}$-generic over V, δ being a Woodin cardinal, $\mathbb{P}_{<\delta}$ being the associated full stationary tower; let $S \in g$. Then if

$$j : V \longrightarrow M \subset V[g]$$

is the generic elementary embedding given by g, M transitive, $j\text{''}H_{\omega_2} \in j(S_0)$, so that

$\mathrm{crit}(j) = \omega_2^V$.

We have $\mathcal{P}(\omega_1) \cap V \in M$: By BMM, $2^{\aleph_1} = \aleph_2$,*) so let $f : \omega_2 \longrightarrow \mathcal{P}(\omega_1) \cap V$ be bijective, $f \in V$. Then $j(f) \upharpoonright \omega_2^V = f$, and $j(f) \in M$, so $\mathcal{P}(\omega_1) \cap V = \mathrm{ran}(f) \in M$.

By MA_{ω_1}, NS_{ω_1} is not ω_1-dense. This was shown by A.D. Taylor, see [Tay 79]; see also e.g. Lemma 6.51 of [Woo 99]. As $\mathcal{P}(\omega_1) \cap V$ has size $\overline{\overline{\omega_2^V}} = \aleph_1$ in M, there is then some stationary set $S \subset \omega_1$ in M s.t. $T \setminus S$ is stationary for all stationary $T \in \mathcal{P}(\omega_1) \cap V$.

By ${}^{<\delta}M \cap V[g] \subset M$, it is still true in $V[g]$ that S is stationary and $T \setminus S$ is stationary for all stationary $T \in \mathcal{P}(\omega_1) \cap V$.

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*) This is a theorem of Todorcevic.

We may then continue as in the proof of theorem 1 of "NS_{ω_1} is not Π_1 definable," see pp. 4 bottom – 6. $\dashv$

The same simplification may be applied in the proof of theorem 2 of "NS_{ω_1} is not Π_1 definable."

If $p, q \in \mathbb{P}_{max}$, $q <_{\mathbb{P}_{max}} p$ as being witnessed by $i : p \to p^*$, then $\overline{\overline{p^*}} = \aleph_1$ in q, hence by $q \models MA_{\omega_1}$, there is a ~~[illegible]~~ positive set S in q such that $T \setminus S$ is positive for all positive $T \in p^*$ (here, "positive" refers to the distinguished ideal I^q of q; [Tay79] shows there is no ω_1-dense normal ideal under MA_{ω_1}). We may then let $L = \{ X \in \mathcal{P}(\omega_1) \cap q : \exists Y \in I^q \; X \setminus S \subset Y \}$, and proceed from there as on pp. 9–10 of "NS_{ω_1} is not Π_1 definable."

[Tay79] A.D. Taylor, "Regularity properties of ideals and ultrafilters," Ann. Math. Logic 16 (1979), pp. 33–55.

[Woo99] W.H. Woodin, "The axiom of determinacy, forcing axioms, and the nonstat. ideal," de Gruyter 1999.