SQUARE WITH BUILT-IN DIAMOND-PLUS

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Abstract. We formulate combinatorial principles that combine the square principle with various strong forms of the diamond principle, and prove that the strongest amongst them holds in L for every infinite cardinal.

As an application, we prove that the following two hold in L:

(1) For every infinite regular cardinal λ, there exists a special λ^+-Aronszajn tree whose projection is almost Souslin;
(2) For every infinite cardinal λ, there exists a *respecting* λ^+-Kurepa tree; Roughly speaking, this means that this λ^+-Kurepa tree looks very much like the λ^+-Souslin trees that Jensen constructed in L.

1. Introduction

In his seminal paper [13], Jensen initiated the study of the fine structure of Gödel's constructible universe, L, and proved that in this model, for every uncountable cardinal κ which is not weakly compact, there exists a κ-Souslin tree. These fine-structural-constructions of Souslin trees were then factored through the combinatorial principles $\diamondsuit$ and $\square$ (also due to Jensen), making the construction accessible to a wider audience of mathematicians.

In [9], Gray introduced a combinatorial principle that forms a strong combination of $\diamondsuit$ and $\square$, which he denoted by $\boxed{\diamondsuit}$. This principle turned out to be very fruitful, and, for instance, has recently been used to answer an old question of Hajnal in infinite graph theory [17].

In this paper, we introduce principles that combine $\square$ with stronger forms of $\diamondsuit$, such as $\diamondsuit^*$ and $\diamondsuit^+$. We study the implication between these principles, and prove that the strongest amongst them, $\boxed{\diamondsuit}_\lambda^\dagger$, holds in L for every infinite cardinal λ.

As $\diamondsuit$ and $\square$ have countless applications in infinite combinatorics, we expect the principles of this paper to prove fruitful, and allow deeper applications of the nature of L, outside of L.

In this paper, we demonstrate the utility of the new principles by presenting applications to the theory of trees. It is proved:

(1) $\boxed{\diamondsuit}_\lambda^* + \lambda^{<\lambda} = \lambda$ entails the existence of a $\square_\lambda$-respecting special λ^+-Aronszajn tree whose projection is almost Souslin (to be defined below);
(2) $\boxed{\diamondsuit}_\lambda^+$ entails the existence of a $\square_\lambda$-respecting λ^+-Kurepa tree;
(3) $\boxed{\diamondsuit}_\lambda^\dagger$ entails the existence of a $\square_\lambda$-respecting λ^+-Kurepa tree with the additional feature of having no λ^+-Aronszajn subtrees.

Before giving the definition of a *respecting* tree, we first motivate it by briefly recalling how Jensen constructed a λ^+-Souslin tree from the hypothesis that $\square_\lambda(E) + \diamondsuit(E)$ holds for some stationary subset $E \subseteq \lambda^+$.[1]

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[1]The full details may be found in, e.g., [6, Theorem IV.2.4] or [18, Lemma 11.68].

Let $\langle C_\alpha \mid \alpha < \lambda^+\rangle$ and $\langle S_\alpha \mid \alpha \in E\rangle$ witness the hypothesis. The construction of the tree T is by recursion, where at stage $\alpha < \lambda^+$, the α^{th}-level, T_α, is constructed. We start with T_0 a singleton, and for $T_{\alpha+1}$, we simply make sure that any node of T_α admits two incompatible extensions in $T_{\alpha+1}$. The heart of the matter is the definition of T_α for α limit nonzero, once $T \restriction \alpha = \bigcup_{\beta<\alpha} T_\beta$ has already been constructed. Here, for every node $x \in T \restriction \alpha$, one identifies a *canonical branch* $\mathbf{b}^\alpha_x$ which is cofinal in $T \restriction \alpha$ and goes through x. Of course, to be able to construct such a branch, we need to make sure that the process of climbing up through the levels of $T \restriction \alpha$ is always successful, i.e., that we never get stuck when trying to take a limit. For this, we advise with the $\Box$-sequence, ensuring in advance that if $\bar\alpha \in \operatorname{acc}(C_\alpha)$, then $\mathbf{b}^{\bar\alpha}_x$ would make an initial segment of $\mathbf{b}^\alpha_x$.[2]

But we also need to seal antichains! For this, we advise with the $\diamondsuit$-sequence, to decide whether T_α should be equal to $\{\mathbf{b}^\alpha_x \mid x \in T \restriction \alpha\}$, or only to some carefully-chosen subset of it.

Here is a possible abstraction of the above process.

Definition 1.1. Suppose that T is a downward-closed family of functions from ordinals to some fixed set Ω, so that $(T, \subset)$ forms a λ^+-tree. Denote $T \restriction X = \{t \in T \mid \operatorname{dom}(t) \in X\}$.

We say that T is $\Box_\lambda$*-respecting* if there exists a stationary subset $E \subseteq \lambda^+$, and a sequence of mappings

$$\langle \mathbf{b}^\alpha : T \restriction C_\alpha \to {}^\alpha\Omega \cup \{\emptyset\} \mid \alpha < \lambda^+\rangle$$

such that:

(1) $\langle C_\alpha \mid \alpha < \lambda^+\rangle$ is a $\Box_\lambda(E)$-sequence;
(2) $T_\alpha \subseteq \operatorname{Im}(\mathbf{b}^\alpha)$ for every $\alpha \in E$;
(3) if $\bar\alpha \in \operatorname{acc}(C_\alpha)$ and $x \in T \restriction C_{\bar\alpha}$, then $\mathbf{b}^{\bar\alpha}(x) = \mathbf{b}^\alpha(x) \restriction \bar\alpha$.

In particular, for every $\alpha \in E$, any node $y \in T_\alpha$ is essentially the limit of some canonical branch $\mathbf{b}^\alpha(x)$ for some $x \in T \restriction C_\alpha$.

While working on their paper, the authors of [2] were considering the problem of constructing, say, an $\aleph_3$-Souslin tree whose reduced $\aleph_0$-power is $\aleph_3$-Aronszajn, and whose reduced $\aleph_1$-power is $\aleph_3$-Kurepa. They realized that such a tree may be constructed, provided that there exists a respecting $\aleph_3$-Kurepa tree.

Question. *Can a λ^+-Kurepa tree be $\Box_\lambda$-respecting?*

At a first glance, this sounds unlikely, as λ^+-Kurepa trees are usually obtained in a top-down fashion (one outright identifies λ^{++} many whole functions from λ^+ to 2, and then verifies that the number of traces on any $\alpha < \lambda^+$ is rather small), while respecting trees are described in a bottom-up langauge. However, in this paper, we shall demonstrate that $\boxdiamond^+_\lambda$ allows the construction of such a λ^+-Kurepa tree. In fact, we shall construct a λ^+-Kurepa tree satisfying a considerably stronger form of $\Box_\lambda$-respecting, that is, $\boxtimes_\lambda(\lambda^+)$-respecting (see Definition 4.4 below).

In another front, we shall prove that $\boxdiamond^*_\lambda$ entails the existence of a $\Box_\lambda$-sequence $\vec C$ for which the λ^+-Aronszajn trees derived from the process of *walks on ordinals* [23] along $\vec C$ are respecting. This is also somewhat surprising, as the standard description of these trees is also top-down. However, the fact that these trees could be $\Box_\lambda$-respecting is already hinted in [21, Equation $(*)$ on p. 267], based on a concept affine to Definition 1.1 and implicit in the proof of [20, Theorem 4.1]. Of course, the derived trees obtained here will be moreover $\boxtimes_\lambda(\lambda^+)$-respecting.

For λ regular uncountable, we shall also ensure that the derived tree $\mathcal{T}(\rho_1)$ (which is a projection of the *special* tree $\mathcal{T}(\rho_0)$) is *almost Souslin*.[3] As for $\lambda = \aleph_0$, in [22], it was proved that Cohen

[2]Here, $\operatorname{acc}(A) = \{\alpha < \sup(A) \mid \sup(A \cap \alpha) = \alpha > 0\}$.

[3]The relevant definitions may be found on Section 4 below.

modification of a C-sequence on ω_1 makes its $\mathcal{T}(\rho_1)$ almost Souslin. In [10], a similar result was obtained from $\diamondsuit^*(\omega_1)$. Altogether, we conclude that in L, for every infinite regular cardinal λ, there exists a special λ^+-Aronszajn tree whose projection is almost Souslin.

1.1. **Organization of this paper.** In Section 2, we recall the definition of the principle $\boxtimes_\lambda$, introduce the principles $\boxtimes^*_\lambda$, $\boxtimes^+_\lambda$, $\boxtimes^\dagger_\lambda$, and discuss the interrelations between them.

In Section 3, we prove that if $V = L$, then $\boxtimes^\dagger_\lambda$ holds for every infinite cardinal λ.

In Section 4, we use the new principles to derive new types of λ^+-trees.

2. Hybrid squares and diamonds

In [9], Gray introduced the principle $\boxtimes_\lambda$ for λ a regular uncountable cardinal. In [1, §2], the definition was generalized to cover the case of λ singular. Then, in [3], the principle was generalized to cover the case $\lambda = \aleph_0$, as well. The outcome is as follows:

Definition 2.1 ([9],[1],[3])**.** $\boxtimes_\lambda$ asserts the existence of $\langle (C_\alpha, S_\alpha) \mid \alpha < \lambda^+\rangle$ such that:

1. C_α is a club in α of order-type $\leq (\omega \cdot \lambda)$;
2. $S_\alpha \subseteq \alpha$;
3. if $\bar\alpha \in \operatorname{acc}(C_\alpha)$, then
   (a) $C_{\bar\alpha} = C_\alpha \cap \bar\alpha$;
   (b) $S_{\bar\alpha} = S_\alpha \cap \bar\alpha$;
4. for every $X \subseteq \lambda^+$ and every club $D \subseteq \lambda^+$, there exists a limit $\alpha < \lambda^+$ with $\operatorname{otp}(\operatorname{acc}(C_\alpha)) = \lambda$, such that $S_\alpha = X \cap \alpha$ and $\operatorname{acc}(C_\alpha) \subseteq D$.

Note that $\langle S_\alpha \mid \alpha < \lambda^+\rangle$ forms a $\diamondsuit(E^{\lambda^+}_{\operatorname{cf}(\lambda)})$-sequence, and if λ is uncountable, then $\langle C_\alpha \mid \alpha < \lambda^+\rangle$ forms a $\square_\lambda$-sequence. By [3], $\boxtimes_\omega$ is equivalent to $\diamondsuit(\omega_1)$. By [19], $\square_\lambda + \diamondsuit(\lambda^+)$ does not imply $\boxtimes_\lambda$ for λ regular uncountable. By [16], $\square_\lambda + \diamondsuit(\lambda^+)$ is equivalent to $\boxtimes_\lambda$ for every singular cardinal λ.

Definition 2.2. $\boxtimes^*_\lambda$ asserts the existence of $\langle (C_\alpha, X_\alpha, f_\alpha) \mid \alpha < \lambda^+\rangle$ such that:

1. C_α is a club in α of order-type $\leq \lambda$;
2. X_α is a subset of $\mathcal{P}(\alpha)$, of size $\leq \lambda$;
3. $f_\alpha : C_\alpha \to X_\alpha$ is a function, and a surjection whenever $\operatorname{otp}(C_\alpha) = \lambda$;
4. if $\bar\alpha \in \operatorname{acc}(C_\alpha)$, then
   (a) $C_{\bar\alpha} = C_\alpha \cap \bar\alpha$;
   (b) $f_{\bar\alpha}(\beta) = f_\alpha(\beta) \cap \bar\alpha$ for all $\beta \in C_{\bar\alpha}$;
5. for every subset $X \subseteq \lambda^+$ and a club $C \subseteq \lambda^+$, the set
$$\left\{\alpha < \lambda^+ \mid C_\alpha \subseteq^* C \;\&\; X \cap \alpha \in X_\alpha\right\}$$
   contains a club;[4]
6. $\{\alpha < \lambda^+ \mid \operatorname{otp}(C_\alpha) = \lambda\}$ is stationary in λ^+.

Of course, $\langle X_\alpha \mid \alpha < \lambda^+\rangle$ forms a $\diamondsuit^*(\lambda^+)$-sequence, and $\langle C_\alpha \mid \alpha < \lambda^+\rangle$ forms a strong club-guessing sequence in the sense of [8]. In particular, by [12], $\boxtimes^*_\omega$ is actually stronger than $\diamondsuit^*(\omega_1)$.

Definition 2.3. $\boxtimes^+_\lambda$ asserts the existence of $\langle (C_\alpha, N_\alpha, f_\alpha) \mid \alpha < \lambda^+\rangle$ such that:

1. C_α is a club in α of order-type $\leq \lambda$;
2. N_α is a rud-closed transitive set, $\lambda = |N_\alpha| \subseteq N_\alpha$, with $\{f_\beta \mid \beta < \alpha\} \cup \{\alpha\} \subseteq N_\alpha$;
3. $f_\alpha : C_\alpha \to \mathcal{P}(\alpha) \cap N_\alpha$ is a function;
4. if $\bar\alpha \in \operatorname{acc}(C_\alpha)$, then

[4]Here, $A \subseteq^* B$ stands for the assertion that $\sup(A \setminus B) < \sup(A)$.

(a) $C_{\bar\alpha} = C_\alpha \cap \bar\alpha$;
(b) $f_{\bar\alpha}(\beta) = f_\alpha(\beta) \cap \bar\alpha$ for all $\beta \in C_{\bar\alpha}$;

(5) for every subset $X \subseteq \lambda^+$ and a club $C \subseteq \lambda^+$, there exists a club $D \subseteq \lambda^+$ such that for all $\alpha \in D$:
- $C_\alpha \subseteq^* C$;
- $X \cap \alpha, D \cap \alpha \in N_\alpha$;
- if $\mathrm{otp}(C_\alpha) = \lambda$, then $X \cap \alpha \in \mathrm{Im}(f_\alpha)$;

(6) $\{\alpha < \lambda^+ \mid f_\alpha \text{ is surjective}\}$ is stationary in λ^+;
(7) $\{N_\alpha \mid \alpha < \lambda^+\}$ is an increasing $\subseteq$-chain converging to H_{λ^+}.

Note that $\langle P(\alpha) \cap N_\alpha \mid \alpha < \lambda^+\rangle$ forms a $\diamondsuit^+(\lambda^+)$-sequence. As before, the result of [12] entails that $\boxed{\diamondsuit}^+_\omega$ is stronger than $\diamondsuit^+(\omega_1)$.

Definition 2.4. $\boxed{\diamondsuit}^\dagger_\lambda$ asserts the existence of $\langle (C_\alpha, N_\alpha, f_\alpha) \mid \alpha < \lambda^+\rangle$ such that Clauses (1)–(7) of Definition 2.3 hold, with Clause (6) strengthened to

(6) for every $n, m < \omega$, end every Π^n_m-sentence η valid in a structure
$$(\lambda^+, \in, \langle A_i \mid i < \omega\rangle), ^5$$
there are stationarily many $\alpha < \lambda^+$ for which all of the following hold:
- f_α is surjective;
- $\langle A_i \restriction \alpha \mid i < \omega\rangle \in N_\alpha$;
- $N_\alpha \models$ "η is valid in $(\alpha, \in, \langle A_i \restriction \alpha \mid i < \omega\rangle)$".

Note that if $\langle (C_\alpha, N_\alpha, f_\alpha) \mid \alpha < \lambda^+\rangle$ witnesses $\boxed{\diamondsuit}^\dagger_\lambda$, then $\langle N_\alpha \mid \alpha < \lambda^+\rangle$ is not far from being a witness to Devlin's notion of a $\diamondsuit^\sharp(\lambda^+)$-sequence (see [5]). Specifically, there are two differences:

- In $\diamondsuit^\sharp(\lambda^+)$, the models N_α are required to be p.r.-closed, while here they are only rud-closed;
- In $\diamondsuit^\sharp(\lambda^+)$, the reflection property is restricted to Π^1_2-sentences, while here there is no restriction on the complexity of the sentences.

Lemma 2.5. *$\boxed{\diamondsuit}^*_\lambda$ entails the existence of a $\boxed{\diamondsuit}^*_\lambda$-sequence $\langle (C_\alpha, X_\alpha, f_\alpha) \mid \alpha < \lambda^+\rangle$ with the additional property that for every club $D \subseteq \lambda^+$, there exists some limit $\alpha < \lambda^+$ with $\mathrm{otp}(C_\alpha) = \lambda$ such that $C_\alpha \subseteq D$.*

Proof. Let $\langle (C_\alpha, X_\alpha, f_\alpha) \mid \alpha < \lambda^+\rangle$ be a witness to $\boxed{\diamondsuit}^*_\lambda$. Fix $\varsigma : \lambda \to \lambda$ such that for all $i < \lambda$, $\varsigma(i) \leq i$ and $\varsigma^{-1}\{i\}$ is cofinal in λ. Let $\alpha < \lambda^+$ be arbitrary. For all $j < \lambda$, denote
$$C^j_\alpha = \{\gamma \in C_\alpha \mid \mathrm{otp}(C_\alpha \cap \gamma) \geq j\}.$$

Let $\pi_\alpha : \mathrm{otp}(C_\alpha) \to C_\alpha$ denote the monotone enumeration of C_α. Define $f'_\alpha : C_\alpha \to X_\alpha$ by stipulating
$$f'_\alpha(\beta) = f_\alpha(\pi_\alpha(\varsigma(\pi_\alpha^{-1}(\beta)))).$$

It is easy to see that if $\bar\alpha \in \mathrm{acc}(C_\alpha)$, then $\pi_{\bar\alpha} = \pi_\alpha \restriction \mathrm{otp}(C_{\bar\alpha})$, and hence $f'_{\bar\alpha}(\beta) = f'_\alpha(\beta) \cap \bar\alpha$ for all $\beta \in C_{\bar\alpha}$. In addition, if $\mathrm{otp}(C_\alpha) = \lambda$, then for all $j < \lambda$:
$$\mathrm{Im}(f'_\alpha \restriction C^j_\alpha) = \mathrm{Im}(f_\alpha) = X_\alpha.$$

Claim 2.5.1. *There exists some $j < \lambda$ such that for every club $D \subseteq \lambda^+$, there exists some limit $\alpha < \lambda^+$ with $\mathrm{otp}(C_\alpha) = \lambda$ and $C^j_\alpha \subseteq D$.*

[5] "Validity in a structure" here has the obvious meaning, cf. [5, p. 891].

Proof. Suppose not. Then for all $j < \lambda$, we may pick a club counterexample $D_j \subseteq \lambda^+$. Let $C = \bigcap_{j<\lambda} D_j$. By Clauses (5) and (6) of Definition 2.3, then, there must exist some $\alpha < \lambda^+$ with $\mathrm{otp}(C_\alpha) = \lambda$ and $C_\alpha \subseteq^* C$. Pick $j < \lambda$ such that $C^j_\alpha \subseteq C$. In particular, $C^j_\alpha \subseteq D_j$ contradicting the choice of D_j. □

Let $j < \lambda$ be given by the preceding claim. For all $\alpha < \lambda^+$, let

$$C^\bullet_\alpha = \begin{cases} C^j_\alpha, & \text{if } \mathrm{otp}(C_\alpha) > j; \\ C_\alpha, & \text{otherwise.} \end{cases}$$

Put $f^\bullet_\alpha = f'_\alpha \restriction C^\bullet_\alpha$, and $X^\bullet_\alpha = X_\alpha$. Then $\langle (C^\bullet_\alpha, X^\bullet_\alpha, f^\bullet_\alpha) \mid \alpha < \lambda^+ \rangle$ forms a $\boxdiamond^*_\lambda$-sequence with the additional desired property. □

Lemma 2.6. *For every infinite cardinal λ, $\boxdiamond^\dagger_\lambda \implies \boxdiamond^+_\lambda \implies \boxdiamond^*_\lambda \implies \boxdiamond_\lambda$.*

Proof. Let λ be an arbitrary infinite cardinal. The implication $\boxdiamond^\dagger_\lambda \implies \boxdiamond^+_\lambda$ is trivial. To see that $\boxdiamond^+_\lambda \implies \boxdiamond^*_\lambda$, let $\langle (C_\alpha, N_\alpha, f_\alpha) \mid \alpha < \lambda^+ \rangle$ be a $\boxdiamond^+_\lambda$-sequence. For all $\alpha < \lambda^+$, let

$$X_\alpha = \begin{cases} \mathcal{P}(\alpha) \cap N_\alpha, & \text{if } \mathrm{otp}(C_\alpha) < \lambda; \\ \mathrm{Im}(f_\alpha), & \text{otherwise.} \end{cases}$$

Then $\langle (C_\alpha, X_\alpha, f_\alpha) \mid \alpha < \lambda^+ \rangle$ forms a $\boxdiamond^*_\lambda$-sequence.

Out next task is showing that $\boxdiamond^*_\lambda \implies \boxdiamond_\lambda$. By [3], $\diamondsuit(\omega_1)$ implies $\boxdiamond_\omega$, hence we hereafter assume that λ is uncountable. In particular, $\omega \cdot \lambda = \lambda$.

Let $\langle (C_\alpha, X_\alpha, f_\alpha) \mid \alpha < \lambda^+ \rangle$ be given by Lemma 2.5. Fix a bijection $\psi : \lambda \times \lambda^+ \to \lambda^+$. Let $E = \{\alpha < \lambda^+ \mid \psi[\lambda \times \alpha] = \alpha\}$. For every $\alpha < \lambda^+$, let $\pi_\alpha : \mathrm{otp}(C_\alpha) \to C_\alpha$ denote the monotone enumeration of C_α, and then for $i < \lambda$, write

$$S^i_\alpha = \{\tau < \alpha \mid \psi(i, \tau) \in f_\alpha(\pi_\alpha(i))\}.$$

Claim 2.6.1. *There exists an $i < \lambda$ such that for every club $D \subseteq \lambda^+$ and every $X \subseteq \lambda^+$, there exists some limit $\alpha < \lambda^+$ with $\mathrm{otp}(\mathrm{acc}(C_\alpha)) = \lambda$, such that $S^i_\alpha = X \cap \alpha$ and $C_\alpha \subseteq D$.*

Proof. Suppose not, and so for all $i < \lambda$, pick a counterexample (D_i, X_i). Let $X = \{\psi(i, \tau) \mid i < \lambda, \tau \in X_i\}$. By Clause (5) of Definition 2.2, we may find a club $D \subseteq E \cap \bigcap_{i<\lambda} D_i$ such that $X \cap \alpha \in X_\alpha$ for all $\alpha \in D$. Now, fix a limit ordinal $\alpha < \lambda^+$ such that $\mathrm{otp}(C_\alpha) = \lambda$ and $C_\alpha \subseteq D$. In particular, $\alpha \in E$. Note that since λ is uncountable, moreover $\mathrm{otp}(\mathrm{acc}(C_\alpha)) = \lambda$. Next, as $\mathrm{otp}(C_\alpha) = \lambda$, we infer from Clause (3) of Definition 2.2 the existence of some $i < \lambda$ such that $X \cap \alpha = f_\alpha(\pi_\alpha(i))$. By definition of X and since $\pi[\lambda \times \alpha] = \alpha$, we conclude that

$$S^i_\alpha = \{\tau < \alpha \mid \psi(i, \tau) \in X \cap \alpha\} = X_i \cap \alpha.$$

By the choice of (D_i, X_i), it must then be the case that $C_\alpha \nsubseteq D_i$. However, $C_\alpha \subseteq D \subseteq D_i$. This is a contradiction. □

Let $i < \lambda$ be given by the previous claim. For all $\alpha \in E^{\lambda^+}_\omega$, let c_α be a cofinal subset of α of order-type ω. Finally, for all $\alpha < \lambda^+$, let

$$C^\bullet_\alpha = \begin{cases} C_\alpha \cap E, & \text{if } \sup(E \cap C_\alpha) = \alpha; \\ C_\alpha \setminus \sup(E \cap \alpha), & \text{if } \sup(E \cap \alpha) < \alpha; \\ c_\alpha, & \text{otherwise,} \end{cases}$$

and

$$S^\bullet_\alpha = \begin{cases} S^i_\alpha, & \text{if } \sup(E \cap C_\alpha) = \alpha; \\ \emptyset, & \text{otherwise.} \end{cases}$$

Claim 2.6.2. *$\langle (C^\bullet_\alpha, S^\bullet_\alpha) \mid \alpha < \lambda^+ \rangle$ is a $\boxed{\Diamond}_\lambda$-sequence.*

Proof. It is clear that for all limit $\alpha < \lambda^+$, $C^\bullet_\alpha$ is a club subset of α of order-type $\leq \lambda$. Next, suppose that $\alpha < \lambda^+$ and $\bar\alpha \in \mathrm{acc}(C_\alpha)$.

▶ If $\sup(E \cap C_\alpha) = \alpha$, then $C^\bullet_\alpha = C_\alpha \cap E$ and hence $\bar\alpha \in \mathrm{acc}(C_\alpha)$, and $\sup(E \cap C_{\bar\alpha}) = \sup(E \cap C_\alpha \cap \bar\alpha) = \bar\alpha$. Consequently, $C^\bullet_{\bar\alpha} = C_{\bar\alpha} \cap E = C^\bullet_\alpha \cap \bar\alpha$, $f_{\bar\alpha}(\pi_{\bar\alpha}(i)) = f_\alpha(\pi_\alpha(i)) \cap \bar\alpha$, and

$$S^\bullet_\alpha \cap \bar\alpha = S^i_\alpha \cap \bar\alpha = \{\tau < \bar\alpha \mid \psi(i,\tau) \in f_\alpha(\pi_\alpha(i))\}.$$

As $\sup(E \cap C_{\bar\alpha})$, we get in particular that $\bar\alpha \in E$, and hence the right hand side of the preceding is equal to

$$\{\tau < \bar\alpha \mid \psi(i,\tau) \in f_\alpha(\pi_\alpha(i)) \cap \bar\alpha\} = S^i_{\bar\alpha} = S^\bullet_{\bar\alpha}.$$

▶ If $\sup(E \cap \alpha) < \alpha$, then $C^\bullet_\alpha = C_\alpha \setminus \sup(E \cap \alpha)$, and $\bar\alpha \in \mathrm{acc}(C_\alpha)$. Consequently,

$$C^\bullet_{\bar\alpha} = C_{\bar\alpha} \setminus \sup(E \cap \bar\alpha) = C_\alpha \cap \bar\alpha \setminus \sup(E \cap \bar\alpha) = C_\alpha \cap \bar\alpha \setminus \sup(E \cap \alpha) = C^\bullet_\alpha \cap \bar\alpha,$$

and $S^\bullet_\alpha \cap \bar\alpha = \emptyset = S^\bullet_{\bar\alpha}$.

Finally, let $X \subseteq \lambda^+$ and a club $D \subseteq \lambda^+$ be arbitrary. By the choice of i, let us pick a limit ordinal $\alpha < \lambda^+$ with $\mathrm{otp}(\mathrm{acc}(C_\alpha)) = \lambda$ such that $S^i_\alpha = X \cap \alpha$ and $C_\alpha \subseteq D \cap E$. Then $S^\bullet_\alpha = S^i_\alpha = X \cap \alpha$, as sought. □

□

3. The strongest principle holds in L

Theorem 3.1. *$\boxed{\Diamond}^\dagger_\lambda$ holds in L for all infinite cardinals λ.*

Proof. We follow the proof of [18, Theorem 11.64], making use of arguments from [1], [11], and [14]. We assume $V = L$. Let λ denote an arbitrary infinite cardinal.

Consider the set $C = \{\alpha < \lambda^+ \mid J_\alpha \prec_{\Sigma_\omega} J_{\lambda^+}\}$, which is a club subset of λ^+ consisting of limit ordinals above λ. Note that $C \cap E^{\lambda^+}_{>\omega} \subseteq \mathrm{acc}(C)$. Let $\varphi : \lambda^+ \to C$ be the monotone enumeration of C.

Let $\alpha \in C$. Obviously, λ is the largest cardinal of J_α. Note that as $J_\alpha \models \mathsf{ZFC}^-$, we have $\rho_\omega(J_\alpha) = \alpha$. We may therefore define $\nu(\alpha)$ to be the largest $\nu > \alpha$ such that α is a cardinal in the viewpoint of J_ν.[6]

Note also that if $\beta < \alpha$, then by $J_\alpha \models |J_\beta| = \lambda$, we have that $\mathcal{P}(\lambda) \cap J_\beta \subsetneq \mathcal{P}(\lambda) \cap J_\alpha$. So, $\nu(\beta) < \alpha < \nu(\alpha)$ for all $\beta \in C \cap \alpha$.

As $\rho_\omega(J_{\nu(\alpha)}) = \lambda$, let $n(\alpha)$ be the unique $n < \omega$ such that $\lambda = \rho_{n+1}(J_{\nu(\alpha)}) < \alpha \leq \rho_n(J_{\nu(\alpha)})$.

Let us write R for the set of all $\alpha < \lambda^+$ such that $\nu(\varphi(\alpha)) = \bar\nu + \omega$ for some $\bar\nu$ such that α is the only cardinal of $J_{\bar\nu}$ strictly above λ.

For all $\alpha < \lambda^+$, set

$$N_\alpha = \begin{cases} J_{\nu(\varphi(\alpha))+\omega}, & \text{if } \alpha \in R; \\ J_{\nu(\varphi(\alpha))}, & \text{otherwise.} \end{cases}$$

As $\nu \circ \varphi$ is an increasing function from λ^+ to λ^+, we have just established Clause (7) of Definition 2.4.

[6] In particular, $J_{\nu+\omega}$ can see that α has cardinality λ.

Let us now define a $\square_\lambda$-sequence by what became the standard construction, cf. e.g. [18, pp. 270ff.], modulo various crucial adjustments. As we'll have to refer to some details of this construction later on in this proof, let us repeat this construction here for the convenience of the reader.

First, for $\alpha \in C$, we define D_α as follows. We let D_α consist of all $\bar{\alpha} \in C \cap \alpha$ such that $n(\bar{\alpha}) = n(\alpha)$ and there is a weakly $r\Sigma_{n(\alpha)+1}$-elementary embedding

$$\sigma\colon J_{\nu(\bar{\alpha})} \longrightarrow J_{\nu(\alpha)}$$

such that $\sigma \restriction \bar{\alpha} = \mathrm{id}$, $\sigma(p_{n(\bar{\alpha})+1}(J_{\nu(\bar{\alpha})})) = p_{n(\alpha)+1}(J_{\nu(\alpha)})$, and if $\bar{\alpha} \in J_{\nu(\bar{\alpha})}$, then $\alpha \in J_{\nu(\alpha)}$ and $\sigma(\bar{\alpha}) = \alpha$. It is easy to see that if $\bar{\alpha} \in D_\alpha$, then there is exactly one map σ witnessing this, namely the one which is given by

$$h^{n(\bar{\alpha})+1, p_{n(\bar{\alpha})+1}(J_{\nu(\bar{\alpha})})}_{J_{\nu(\bar{\alpha})}}(i, \vec{x}) \mapsto h^{n(\alpha)+1, p_{n(\alpha)+1}(J_{\nu(\alpha)})}_{J_{\nu(\alpha)}}(i, \vec{x}), \tag{1}$$

where $i < \omega$ and $\vec{x} \in [\lambda]^{<\omega}$. We here make use of the notation for fine structural iterated Σ_1 Skolem functions as presented e.g. in [18, Equation (11.29) on p. 252]. We shall denote the unique map as given by (1) by $\sigma_{\bar{\alpha},\alpha}$.

Notice that if $\alpha \in C$, then

$$J_{\nu(\alpha)} = h^{n(\alpha)+1, p_{n(\alpha)+1}(J_{\nu(\alpha)})}_{J_{\nu(\alpha)}} “(\omega \times [\lambda]^{<\omega}),$$

so that if $\bar{\alpha} \in D_\alpha$, then

$$\mathrm{Im}(\sigma_{\bar{\alpha},\alpha}) \nsubseteq h^{n(\alpha)+1, p_{n(\alpha)+1}(J_{\nu(\alpha)})}_{J_{\nu(\alpha)}} “(\omega \times [\lambda]^{<\omega}),$$

which means that there must be $i < \omega$ and $\vec{x} \in [\lambda]^{<\omega}$ such that the left hand side of (1) is undefined, whereas the right hand side of (1) is defined.

Also notice that the maps $\sigma_{\bar{\alpha},\alpha}$ commute, i.e., if $\bar{\alpha} \in D_\alpha$ and $\alpha \in D_{\alpha'}$, then $\bar{\alpha} \in D_{\alpha'}$ and

$$\sigma_{\bar{\alpha},\alpha'} = \sigma_{\alpha,\alpha'} \circ \sigma_{\bar{\alpha},\alpha}.$$

Having constructed $\langle D_\alpha \mid \alpha \in C\rangle$, we claim:

Claim 3.1.1. *Let $\alpha \in C$. All of the following hold true:*

(a) *D_α is closed;*
(b) *If $\mathrm{cf}(\alpha) > \omega$, then D_α is unbounded in α;*
(c) *If $\bar{\alpha} \in D_\alpha$ then $D_\alpha \cap \bar{\alpha} = D_{\bar{\alpha}}$.*

Proof. This is Claim 11.65 of [18]. □

Notation. *For $\alpha \in C, i < \omega$ and $\vec{x} \in [\lambda]^{<\omega}$, we shall denote:*

$$h_\alpha(i, \vec{x}) = h^{n(\alpha)+1, p_{n(\alpha)+1}(J_{\nu(\alpha)})}_{J_{\nu(\alpha)}}(i, \vec{x}).$$

Let $\alpha \in C$. If $\sup(D_\alpha) < \alpha$, let $\theta(\alpha) = 0$. Now, suppose $\sup(D_\alpha) = \alpha$. We shall obtain some limit ordinal $\theta(\alpha)$, and sequences $\langle \mu^\alpha_i \mid i \leq \theta(\alpha)\rangle$ and $\langle \xi^\alpha_i \mid i < \theta(\alpha)\rangle$, by recursion, as follows.

► Set $\mu^\alpha_0 = \min(D_\alpha)$.

► Given μ^α_i with $\mu^\alpha_i < \alpha$, we let ξ^α_i be the least $\xi < \lambda$ such that

$$h_\alpha(k, \vec{x}) \notin \mathrm{Im}(\sigma_{\mu^\alpha_i,\alpha})$$

for some $k < \omega$ and some $\vec{x} \in [\xi]^{<\omega}$. Given ξ^α_i, we let μ^α_{i+1} be the least $\bar{\alpha} \in D_\alpha$ such that

$$h_\alpha(k, \vec{x}) \in \mathrm{Im}(\sigma_{\bar{\alpha},\alpha})$$

for all $k < \omega$ and $\vec{x} \in [\xi^\alpha_i]^{<\omega}$ such that $h_\alpha(k, \vec{x})$ exists.

► Given $\langle \mu^\alpha_j \mid j < i\rangle$, where i is a limit ordinal, we set $\mu^\alpha_i = \sup_{j<i} \mu^\alpha_j$.

Naturally, $\theta(\alpha)$ will be the least i such that $\mu_i^\alpha = \alpha$.

For any $\alpha \in C$, denote $E_\alpha = \{\mu_i^\alpha \mid i < \theta(\alpha)\}$.

Claim 3.1.2. *Let $\alpha \in C$. All of the following hold true:*
(a) *$\langle \xi_i^\alpha \mid i < \theta(\alpha)\rangle$ is a strictly increasing sequence of ordinals below λ;*
(b) *$\mathrm{otp}(E_\alpha) = \theta(\alpha) \leq \lambda$;*
(c) *$\theta(\alpha) > 0$ iff E_α is a club in α;*
(d) *If $\bar\alpha \in \mathrm{acc}(E_\alpha)$, then $E_\alpha \cap \bar\alpha = E_{\bar\alpha}$.*

Proof. (a) is immediate, and it implies (b).

(c) is trivial.

(d) Let $\bar\alpha \in \mathrm{acc}(D_\alpha)$. We have $E_\alpha \subseteq D_\alpha$, and $D_{\bar\alpha} = D_\alpha \cap \bar\alpha$ by Claim 3.1.1(c). We now show that $\langle \mu_i^{\bar\alpha} \mid i < \theta(\bar\alpha)\rangle = \langle \mu_i^\alpha \mid i < \theta(\bar\alpha)\rangle$ and $\langle \xi_i^{\bar\alpha} \mid i < \theta(\bar\alpha)\rangle = \langle \xi_i^\alpha \mid i < \theta(\bar\alpha)\rangle$ by induction:

Say $\mu_i^{\bar\alpha} = \mu_i^\alpha$, where $i+1 \in \theta(\bar\alpha) \cap \theta(\alpha)$. Write $\mu = \mu_i^{\bar\alpha} = \mu_i^\alpha$. As $\sigma_{\mu,\alpha} = \sigma_{\bar\alpha,\alpha} \circ \sigma_{\mu,\bar\alpha}$, for all $k < \omega$ and $\vec{x} \in [\lambda]^{<\omega}$,

$$h_{\bar\alpha}(k, \vec{x}) \in \mathrm{Im}(\sigma_{\mu,\bar\alpha}) \Longrightarrow h_\alpha(k, \vec{x}) \in \mathrm{Im}(\sigma_{\mu,\alpha}) \neq \emptyset.$$

This gives $\mu_{i+1}^\alpha \leq \mu_{i+1}^{\bar\alpha}$ On the other hand, $\mathrm{Im}(\sigma_{\mu_{i+1}^\alpha,\alpha})$ contains the relevant witness so as to guarantee conversely that $\mu_{i+1}^{\bar\alpha} \leq \mu_{i+1}^\alpha$. □

Recall that $\varphi : \lambda^+ \to C$ denotes the monotone enumeration of C. For all $\alpha \in \mathrm{acc}(\lambda^+)$, we have $\varphi(\alpha) \in \mathrm{acc}(C)$, so we set

$$C'_\alpha = \varphi^{-1}``E_{\varphi(\alpha)}.$$

Since $E_{\varphi(\alpha)} \subseteq D_{\varphi(\alpha)} \subseteq C = \mathrm{Im}(\varphi)$, we have $\mathrm{otp}(C'_\alpha) = \theta(\varphi(\alpha)) \leq \lambda$, and C'_α is a club in α iff $\theta(\varphi(\alpha)) > 0$.

Because $E_\mu \subseteq D_\mu \subseteq C \cap \mu$ for every $\mu \in C$, we have that if $\bar\alpha \in \mathrm{acc}(C'_\alpha)$, then $C'_{\bar\alpha} = C'_\alpha \cap \bar\alpha$.

Claim 3.1.3. *Let $\alpha \in E^{\lambda^+}_\omega$ be such that $\varphi(\alpha) = \alpha$.*

Then there exists a cofinal subset d of α of order-type ω satisfying the following:
(a) *If $c \in J_{\nu(\alpha)}$ is a club in α, then $d \subseteq^* c$;*
(b) *$d \in N_{\alpha'}$ whenever $\alpha < \alpha' < \lambda^+$.*

Proof. The argument is a simplified version of the proof of [11, Theorem 4.15].

Fix $\alpha \in E^{\lambda^+}_\omega$ such that $\varphi(\alpha) = \alpha$, so that $\alpha \in C$. Recall that $\lambda = \rho_{n(\alpha)+1}(J_{\nu(\alpha)}) < \alpha \leq \rho_{n(\alpha)}(J_{\nu(\alpha)})$. Let us write $\nu = \nu(\alpha)$, $n = n(\alpha)$, and

$$\tau = \sup(\{\xi < \nu \mid J_\nu \models |\xi| \leq \alpha\}).$$

I.e., either α is the largest cardinal of J_ν in which case $\tau = \nu$, or else $\tau = \alpha^{+J_\nu}$.

Case 1. $\mathrm{cf}(\tau) = \omega$.

Let $(\tau^m \mid m < \omega)$ be a sequence witnessing $\mathrm{cf}(\tau) = \omega$. For each $m < \omega$, we may inside J_ν pick some club $c^m \subseteq \alpha$ such that $c^m \subseteq^* c$ for every club $c \subseteq \alpha$, $c \in J_{\tau^m}$. E.g. we may let c^m be the diagonal intersection of all clubs $c \subseteq \alpha$, $c \in J_{\tau^m}$, as being given by some enumeration in J_ν in order type α. Let then $(\rho^m \mid m < \omega)$ be a strictly increasing sequence which is cofinal in α and such that for every $m < \omega$, $\rho^m \in \bigcap_{k \leq m} c^m$. Set $d = \{\rho^m \mid m < \omega\}$. Then for every $c \in J_\nu$ which is a club in α, we have $d \subseteq^* c$.

Case 2. $\mathrm{cf}(\tau) > \omega$.

By [14, Lemma 1.2], we must then have that $\rho_n(J_\nu) = \alpha$, and if $\ell < n$ is largest such that $\rho_\ell(J_\nu) > \alpha$, then $\mathrm{cf}(\rho_\ell(J_\nu)) = \mathrm{cf}(\tau) > \omega$. We then have $\ell + 1 \leq n$ and

$$\rho_\ell(J_\nu) > \rho_{\ell+1}(J_\nu) = \rho_n(J_\nu) = \alpha > \rho_{n+1}(J_\nu) = \lambda.$$

Let $(\alpha^m \mid m < \omega)$ be a sequence witnessing $\mathrm{cf}(\alpha) = \omega$. By $\mathrm{cf}(\tau) \neq \omega$, there must be some $m < \omega$ such that

$$h_{J_\nu^\ell}``(\alpha^m \cup \{p(J_\nu^\ell)\}) \cap \tau$$

is cofinal in τ. Note that as $\omega = \mathrm{cf}(\alpha) < \mathrm{cf}(\rho_\ell(J_\nu))$ and α is a regular cardinal in J_ν,

$$h_{J_\nu^\ell}``(\alpha^m \cup \{p(J_\nu^\ell)\}) \cap \alpha$$

cannot be cofinal in α. Set

$$H = h_{J_\nu^\ell}``(\alpha^m \cup \{p(J_\nu^\ell)\}) \text{ and } \tilde{\alpha} = \sup(H \cap \alpha).$$

Let us write $\theta = \mathrm{cf}(\rho_\ell(J_\nu)) = \mathrm{cf}(\tau)$, and let $(\tau^i \colon i < \theta)$ be a strictly increasing sequence witnessing $\mathrm{cf}(\tau) = \theta$, where $\tau^i \in H$ for every $i < \theta$. For each $i < \theta$, in much the same way as in Case 1 we may inside H pick some club $c^i \subseteq \alpha$ such that $c^i \subseteq^* c$ for every $c \in J_{\tau^i}$ which is a club in α. Notice that we may arrange $c^i \in H$ by $\tau^i \in H$. We may assume without loss of generality that $c^i \in J_{\tau^{i+1}}$ for every $i < \theta$. We then get

$$H \models c_i \subseteq^* c_j$$

for all $j < i < \theta$, which by the choice of $\tilde{\alpha}$ readily implies that

$$c_i \setminus \tilde{\alpha} \subseteq c_j \setminus \tilde{\alpha}$$

for all $j < i < \theta$. But as $\mathrm{cf}(\alpha) \neq \theta$, this gives that

$$\tilde{c} = \bigcap_{i<\theta} (c_i \setminus \tilde{\alpha})$$

is club in α.

We may then let d be a cofinal subset of $\tilde{c}$ of order type ω. Then for every club $c \in J_\nu$ which is a club in α, we have $d \subseteq^* c$.

We thus in both cases found a set d which satisfies (a) from Claim 3.1.3. The existence of some such d is a Σ_1–statement in the parameter $J_{\nu(\alpha)}$. Therefore, if we pick d $<_L$–least with (a), then by $N_\alpha \in J_{\varphi(\alpha')} \prec J_{\lambda^+}$ for $\alpha' > \alpha$ we will also get (b) from Claim 3.1.3. $\square$

Let us now for each $\alpha \in E^{\lambda^+}_\omega$ define $c_\alpha \subseteq \alpha$ as follows. If $\varphi(\alpha) = \alpha$, then we let c_α be the $<_L$–least d which satisfies the conclusion of Claim 3.1.3. If $\varphi(\alpha) \neq \alpha$, then just let c_α be the $<_L$-least cofinal subset of α of order-type ω.

Finally, for all $\alpha < \lambda^+$, let

$$C_\alpha = \begin{cases} \emptyset, & \text{if } \alpha = 0; \\ \{\beta\}, & \text{if } \alpha = \beta + 1; \\ C'_\alpha, & \text{if } \alpha \in \mathrm{acc}(\lambda^+) \ \&\ \theta(\varphi(\alpha)) > 0; \\ c_\alpha, & \text{otherwise.} \end{cases}$$

Clearly, C_α is a club in α of order-type $\leq \lambda$. By Claim 3.1.1(c) and Claim 3.1.2(d), if $\bar{\alpha} \in \mathrm{acc}(C_\alpha)$, then $C_{\bar{\alpha}} = C_\alpha \cap \bar{\alpha}$. Altogether, $\langle C_\alpha \mid \alpha < \lambda^+ \rangle$ satisfies Clauses (1) and (4a) of Definition 2.3.

Let $\Gamma \colon \mathrm{OR} \to \omega \times [\mathrm{OR}]^{<\omega}$ be some simply-definable enumeration of $\omega \times [\mathrm{OR}]^{<\omega}$. Using Kleene's "$\simeq$" notation, for $\alpha < \lambda^+$, let g_α be the partial function with $\mathrm{dom}(g_\alpha) \subseteq \lambda$ such that

$$g_\alpha(\chi) \simeq h_{\varphi(\alpha)}(\Gamma(\chi)).$$

Claim 3.1.4. *Let $\alpha < \lambda^+$. All of the the following hold:*

(a) $J_{\nu(\varphi(\alpha))} = \{g_\alpha(i) \mid i < \lambda, g_\alpha(i)$ *is defined*$\}$*;*
(b) *If* $\sup(\mathrm{acc}(C_\alpha)) = \alpha$*,* $\chi < \mathrm{otp}(C_\alpha)$*, and* $g_\alpha(\chi)$ *is defined, then there exists some* $\bar\alpha \in \mathrm{acc}(C_\alpha)$ *such that* $g_{\bar\alpha}(\chi)$ *is defined;*
(c) *If* $\bar\alpha \in \mathrm{acc}(C_\alpha), \chi < \mathrm{otp}(C_{\bar\alpha})$ *and* $g_{\bar\alpha}(\chi) \in {}^{\varphi(\bar\alpha)}2$*, then* $g_\alpha(\chi) \in {}^{\varphi(\alpha)}2$ *and* $g_{\bar\alpha}(\chi) = g_\alpha(\chi) \restriction \varphi(\bar\alpha)$*.*

Proof. (a) Since $J_{\nu(\varphi(\alpha))} = h_{\varphi(\alpha)}``(\omega \times [\lambda]^{<\omega})$, as pointed out earlier.

(b) is clear from the construction, as $J_{\nu(\varphi(\alpha))}$ results from the direct limit of the system $(J_{\nu(\varphi(\beta))}, \sigma_{\varphi(\beta),\varphi(\beta')} \mid \beta \leq \beta' \in C_\alpha)$.

(c) Let α, $\bar\alpha$, and χ be as in the hypothesis of (c). Then $g_\alpha(\chi) \in {}^{\varphi(\alpha)}2$ follows immediately from the fact that $g_{\bar\alpha}(\chi) \in {}^{\varphi(\bar\alpha)}2$, as $\sigma_{\varphi(\bar\alpha),\varphi(\alpha)}$ is Σ_0–elementary and sends its critical point $\varphi(\bar\alpha)$ to $\varphi(\alpha)$. Moreover, if $\xi < \varphi(\bar\alpha)$, then $g_\alpha(\xi) = \sigma_{\varphi(\bar\alpha),\varphi(\alpha)}(g_{\bar\alpha})(\xi) = \sigma_{\varphi(\bar\alpha),\varphi(\alpha)}(g_{\bar\alpha}(\xi)) = g_{\bar\alpha}(\xi)$. □

We have already defined $\langle (C_\alpha, N_\alpha) \mid \alpha < \lambda^+\rangle$, and our next goal is to define $\langle f_\alpha \mid \alpha < \lambda^+\rangle$.

▶ If $\lambda = \aleph_0$, then for every $\alpha \in \mathrm{acc}(\lambda^+)$, we have $\mathrm{acc}(C_\alpha) = \emptyset$, and hence we simply let $f_\alpha : C_\alpha \to \mathcal{P}(\alpha) \cap N_\alpha$ be the $<_L$-least surjection. For $\alpha \in \lambda^+ \setminus \mathrm{acc}(\lambda^+)$, we let $f_\alpha : C_\alpha \to \{0\}$ be constant.

▶ If $\lambda > \aleph_0$, we do the following. Let $\psi : \lambda \to \lambda$ be the $<_L$-least function such that $\psi^{-1}\{i\}$ is cofinal in λ for all $i < \lambda$. Given $\alpha \in \mathrm{acc}(\lambda^+)$, let $\phi_\alpha : C_\alpha \to \mathrm{otp}(C_\alpha)$ denote the order-preserving isomorphism. Then, for all $\beta \in C_\alpha$, set

$$Z^\beta_\alpha = \{\delta \in \mathrm{acc}(C_\alpha) \cap \beta \mid \psi(\phi_\alpha(\beta)) < \phi_\alpha(\delta) \ \&\ g_\delta(\psi(\phi_\alpha(\beta))) \in {}^{\varphi(\delta)}2\},$$

and

$$f_\alpha(\beta) = \begin{cases} \{\varepsilon < \alpha \mid g_\alpha(\psi(\phi_\alpha(\beta)))(\varepsilon) = 1\}, & \text{if } Z^\beta_\alpha \neq \emptyset; \\ \emptyset, & \text{otherwise.} \end{cases}$$

For $\alpha \in \lambda^+ \setminus \mathrm{acc}(\lambda^+)$, let $f_\alpha : C_\alpha \to \{0\}$ be constant.

Having constructed $\langle f_\alpha \mid \alpha < \lambda^+\rangle$, we claim:

Claim 3.1.5. *Let $\alpha < \lambda^+$ be arbitrary. All of the following hold:*

(a) f_α *is a (well-defined) function from* C_α *to* $\mathcal{P}(\alpha) \cap N_\alpha$*. Moreover,* $\mathrm{Im}(f_\alpha) \subseteq J_{\nu(\varphi(\alpha))}$*;*
(b) *If* $\bar\alpha \in \mathrm{acc}(C_\alpha)$ *and* $\beta \in C_{\bar\alpha}$*, then* $f_{\bar\alpha}(\beta) = f_\alpha(\beta) \cap \bar\alpha$*;*
(c) *If* $\mathrm{otp}(C_\alpha) = \lambda$*, then* $\mathrm{Im}(f_\alpha) = \mathcal{P}(\alpha) \cap J_{\nu(\varphi(\alpha))}$*;*
(d) f_α *is surjective iff* $\alpha \in \lambda^+ \setminus R_<$*, where*

$$R_< = R \cup \{\alpha < \lambda^+ \mid \mathrm{otp}(C_\alpha) < \lambda\}.$$

Proof. To avoid trivialities, assume $\lambda > \aleph_0$ and $\alpha \in \mathrm{acc}(\lambda^+)$.

(a) Let $\beta \in C_\alpha$ be arbitrary. If $Z^\beta_\alpha = \emptyset$, then $f_\alpha(\beta) = \emptyset$ which is indeed an element of $\mathcal{P}(\alpha) \cap J_{\nu(\varphi(\alpha))}$. Suppose that $Z^\beta_\alpha \neq \emptyset$. Write $\chi = \psi(\phi_\alpha(\beta))$. Fix $\delta \in Z^\beta_\alpha$. Then $\delta \in \mathrm{acc}(C_\alpha)$ and $\chi < \phi_\alpha(\delta)$. Consequently, $\chi < \mathrm{otp}(C_\delta)$ and $g_\delta(\chi) \in {}^{\varphi(\delta)}2$, and then by Clause (c) of Claim 3.1.4, $g_\alpha(\chi) \in {}^{\varphi(\alpha)}2$. By Clause (a) of Claim 3.1.4, $g_\alpha(\chi) \in J_{\nu(\varphi(\alpha))}$. Since the latter is rud-closed, it follows that $f_\alpha(\beta) \in \mathcal{P}(\alpha) \cap J_{\nu(\varphi(\alpha))}$. By definition of N_α (cf. page 6), we have $J_{\nu(\varphi(\alpha))} \subseteq N_\alpha$.

(b) Suppose $\bar\alpha \in \mathrm{acc}(C_\alpha)$ and $\beta \in C_{\bar\alpha}$. Then $C_{\bar\alpha} = C_\alpha \cap \bar\alpha$, $\phi_{\bar\alpha} = \phi_\alpha \restriction \bar\alpha$, and $\psi(\phi_{\bar\alpha}(\beta)) = \psi(\phi_\alpha(\beta))$, say, it is χ. As $Z^\beta_\alpha \subseteq C_\alpha \cap \beta$ and $Z^\beta_{\bar\alpha} \subseteq C_{\bar\alpha} \cap \beta$, we altogether infer that $Z^\beta_\alpha = Z^\beta_{\bar\alpha}$. In particular, if the latter is empty, then $f_{\bar\alpha}(\beta) = \emptyset = f_\alpha(\beta) \cap \bar\alpha$.

Next, suppose that $Z^\beta_{\bar\alpha}$ is nonempty, and fix a witnessing element δ. By $\delta \in Z^\beta_{\bar\alpha}$, we know that $\chi < \mathrm{otp}(C_\delta)$ and $g_\delta(\chi) \in {}^{\varphi(\delta)}2$, and then by Clause (c) of Claim 3.1.4, we know that $g_{\bar\alpha}(\chi) \in {}^{\varphi(\bar\alpha)}2$.

By $\bar{\alpha} \in \operatorname{acc}(C_\alpha)$ and $\chi < \operatorname{otp}(C_\delta) < \operatorname{otp}(C_{\bar{\alpha}})$ and Clause (c) of Claim 3.1.4, we then know that $g_\alpha(\chi) \in {}^{\varphi(\alpha)}2$ and $g_{\bar{\alpha}}(\chi) \restriction \bar{\alpha} = g_\alpha(\chi) \restriction \bar{\alpha}$. Consequently, $f_{\bar{\alpha}}(\beta) = f_\alpha(\beta) \cap \bar{\alpha}$.

(c) Suppose that $\operatorname{otp}(C_\alpha) = \lambda$, and let x be an arbitrary element of $\mathcal{P}(\alpha) \cap J_{\nu(\varphi(\alpha))}$. By $x \subseteq \alpha \subseteq \varphi(\alpha)$, let $\varrho_x : \varphi(\alpha) \to 2$ denote the characteristic function of x. Since $\varphi(\alpha) \in J_{\nu(\varphi(\alpha))}$ and since the latter is rud-closed, we have $\varrho_x \in J_{\nu(\varphi(\alpha))}$. By Clause (a) of Claim 3.1.4, we may fix some ordinal $i < \lambda$ such that $g_\alpha(i) = \varrho_x$. Since $\operatorname{otp}(C_\alpha) = \lambda$ is an uncountable cardinal, we have $\sup(\operatorname{acc}(C_\alpha)) = \sup(C_\alpha) = \alpha$, and then by Clauses (b) and (c) of Claim 3.1.4, let us fix a large enough $\bar{\alpha} \in \operatorname{acc}(C_\alpha)$ such that $i < \operatorname{otp}(C_{\bar{\alpha}})$ and $g_{\bar{\alpha}}(i) = g_\alpha(i) \restriction \varphi(\bar{\alpha})$. Now, by the choice of ψ, there exists a large enough j with $\phi_\alpha(\bar{\alpha}) < j < \lambda$ such that $\psi(j) = i$. As $\operatorname{otp}(C_\alpha) = \lambda > j$, let $\beta \in C_\alpha$ be the unique element to satisfy $\phi_\alpha(\beta) = j$. Then:

$$Z_\alpha^\beta = \{\delta \in \operatorname{acc}(C_\alpha) \cap \beta \mid i < \phi_\alpha(\delta) \ \&\ g_\delta(i) \in {}^{\varphi(\delta)}2\}.$$

By $i < \operatorname{otp}(C_{\bar{\alpha}}) = \phi_\alpha(\bar{\alpha})$ and $g_{\bar{\alpha}}(i) \in {}^{\varphi(\bar{\alpha})}2$, we have $\bar{\alpha} \in Z_\alpha^\beta$. So

$$\begin{aligned} f_\alpha(\beta) &= \{\varepsilon < \alpha \mid g_\alpha(\psi(\phi_\alpha(\beta)))(\varepsilon) = 1\} \\ &= \{\varepsilon < \alpha \mid g_\alpha(\psi(j))(\varepsilon) = 1\} \\ &= \{\varepsilon < \alpha \mid g_\alpha(i)(\varepsilon) = 1\} \\ &= \{\varepsilon < \alpha \mid \varrho_x(\varepsilon) = 1\} \\ &= x, \end{aligned}$$

as sought.

(d) If $\alpha \in R$, then by Clause (a), $\operatorname{Im}(f_\alpha) \subseteq J_{\nu(\varphi(\alpha))} \subsetneq J_{\nu(\varphi(\alpha))+\omega} = N_\alpha$. If $\operatorname{otp}(C_\alpha) < \lambda$, then $|\operatorname{Im}(f_\alpha)| < \lambda = |N_\alpha|$, so α is not onto. Finally, if $\alpha \notin R$ and $\operatorname{otp}(C_\alpha) = \lambda$, then by Clause (c) and the definition of N_α in this case, $\operatorname{Im}(f_\alpha) = \mathcal{P}(\alpha) \cap J_{\nu(\varphi(\alpha))} = \mathcal{P}(\alpha) \cap N_\alpha$. □

Thus, we have established Clauses (3) and (4b) of Definition 2.4.

Claim 3.1.6. *Let $\alpha < \lambda^+$. All of the following hold:*

(a) $\{C_\beta \mid \beta < \alpha\} \subseteq N_\alpha$. *Moreover:*
(b) $\{f_\beta \mid \beta < \alpha\} \subseteq N_\alpha$.

Proof. Denote $\gamma = \varphi(\alpha)$ and $\nu = \nu(\gamma)$. Clearly, $\alpha \leq \gamma < \nu$ and $N_\alpha = J_\nu$ or $N_\alpha = J_{\nu+\omega}$. Let $\beta < \alpha$ be arbitrary.

(a) If $C_\beta = c_\beta$, then by $\gamma \in C \setminus (\beta + 1)$, we get that $J_\gamma \prec_{\Sigma_\omega} J_{\lambda^+}$ and $J_\gamma \models \operatorname{cf}(\beta) = \omega$. In particular, if c_β is the $<_L$-least cofinal subset of β of order-type ω, then $c_\beta \in J_\gamma \subseteq J_\nu$. If c_β is not of this form, then it was obtained by Claim 3.1.3 which also ensures that $c_\beta \in N_\alpha$.

Next, suppose that $C_\beta = C'_\beta$. Notice first that

$$\varphi(\beta) < \nu(\varphi(\beta)) < \nu(\varphi(\beta)) + \omega < \varphi(\alpha) < \nu(\varphi(\alpha)) = N_\alpha \cap \mathrm{OR}. \tag{2}$$

We have that $C_\beta = \phi^{-1}``E_{\varphi(\beta)}$, where $\phi = \varphi \restriction \beta$. By definition of C and since $\varphi(\beta) \in C$, we know that $C \cap \varphi(\beta)$ is Σ_1-definable over $J_{\varphi(\beta)+\omega}$. By Equation (2), $J_{\varphi(\beta)+\omega} \in N_\alpha$ and $\phi \in N_\alpha$, and it suffices to show that $E_{\varphi(\beta)} \in N_\alpha$. But an inspection of the construction of $E_{\varphi(\beta)}$ yields that $E_{\varphi(\beta)}$ is Σ_1-definable over $J_{\nu(\varphi(\beta))+\omega}$. Hence by Equation (2) again, $E_{\varphi(\beta)} \in N_\alpha$.

(b) We have already shown that $C_\beta \in N_\alpha$. This immediately gives $\phi_\beta \in N_\alpha$. We have that g_β is definable over $J_{\nu(\varphi(\beta))}$, so that $g_\beta \in N_\alpha$ by $N_\beta \in N_\alpha$. Certainly, $\psi \in N_\alpha$. Taken together, $f_\beta \in N_\alpha$. □

Having Clause (c) of Claim 3.1.5 in mind, we now turn to verify Clause (5) of Definition 2.4.

Claim 3.1.7. *Suppose that $A \subseteq \lambda^+$ is some set, and $B \subseteq \lambda^+$ is a club.*

Then there exists a club $D \subseteq \{\alpha \in C \mid \varphi(\alpha) = \alpha\}$ such that for all $\alpha \in D$:

(a) $A\cap\alpha, B\cap\alpha\in J_{\nu(\alpha)}$;
(b) $D\cap\alpha\in N_\alpha$;
(c) $C_\alpha\subseteq^* B$.

Proof. (a) Let $h^\omega_{J_{\lambda^{++}}}$ denote the closure under Σ_n Skolem functions for $J_{\lambda^{++}}$, for all $n<\omega$, which are uniformly definable over all J_γ, $\gamma\geq\omega\cdot\omega$, in a canonical fashion.

Let us recursively define $\langle X_i\mid i<\lambda^+\rangle$ as follows.

$$\begin{aligned} X_0 &= h^\omega_{J_{\lambda^{++}}}(\omega\times[\lambda\cup\{A,B\}]^{<\omega}),\\ X_{i+1} &= h^\omega_{J_{\lambda^{++}}}(\omega\times[\lambda\cup\{A,B,\langle X_j\mid j\leq i\rangle\}]^{<\omega}),\text{ and}\\ X_i &= \bigcup\{X_j\mid j<i\}\ \text{ for limit } i>0.\end{aligned}$$

For $i<\lambda^+$, let $\gamma(i)$ be such that

$$\pi_i\colon J_{\gamma(i)}\cong X_i\prec J_{\lambda^{++}},$$

and write $\alpha(i)=X_i\cap\lambda^+=\lambda^{+J_{\gamma(i)}}$. Notice that $\{C,\varphi\}\subseteq X_0$, and hence $\alpha(i)$ is a closure point under φ, and $\varphi^{-1}(\alpha(i))=\alpha(i)$. Consider the set $D=\{\alpha(i)\mid i<\lambda^+\}$ which is a club subset of $\{\alpha\in C\mid\varphi(\alpha)=\alpha\}$.

For $i<\lambda^+$,

$$J_{\gamma(i)+\omega}\models\alpha(i)\text{ is a cardinal},$$

while the definition of $\nu(\alpha(i))$ entails that $\nu(\alpha(i))\geq\gamma(i)+\omega$, and hence

$$\nu(\alpha(i))>\gamma(i). \tag{3}$$

Of course, $A\cap\alpha(i)=\pi_i^{-1}(A)\in J_{\gamma(i)}$. Thus, $A\cap\alpha\in J_{\nu(\alpha)}$ for every $\alpha\in D$, and likewise, $B\cap\alpha\in J_{\nu(\alpha)}$ for every $\alpha\in D$.

(b) First, we point out that for all $i<\lambda^+$, Equation (3) implies:

$$\langle\pi_i^{-1}``X_j\mid j<i\rangle\in N_{\alpha(i)}. \tag{4}$$

To see this, notice that the elementary embedding π_i will respect the uniformly defined Σ_n Skolem functions for $J_{\lambda^{++}}$ and $J_{\gamma(i)}$, respectively, and hence

$$\begin{aligned} \pi_i^{-1}``X_0 &= h^\omega_{J_{\gamma(i)}}(\omega\times[\lambda\cup\{A\cap\alpha(i),B\cap\alpha(i)\}]^{<\omega}),\\ \pi_i^{-1}``X_{i+1} &= h^\omega_{J_{\gamma(i)}}(\omega\times[\lambda\cup\{A\cap\alpha(i),B\cap\alpha(i),\langle\pi_i^{-1}``X_j\mid j\leq i\rangle\}]^{<\omega}),\text{ and}\\ X_i &= \bigcup\{X_j\mid j<i\}\ \text{ for limit } i>0.\end{aligned}$$

This gives that if $\alpha=\alpha(i)\in D$, then the sequence from (4) is Δ_1–definable over $J_{\gamma(i)+\omega}$. However, we obviously have that in this situation α is the only cardinal of $J_{\gamma(i)}$, so that if $\gamma(i)+\omega=\nu(\alpha)$, then $\alpha\in R$ and $N_\alpha\cap\mathrm{OR}=\gamma(i)+\omega\cdot 2$, hence Equation (4) holds true. If $\nu(\alpha)>\gamma(i)+\omega$, then the sequence from Equation (4) is in $J_{\nu(\alpha)}\subseteq N_\alpha$.

Thus, we have verified that $D\cap\alpha\in N_\alpha$ for every $\alpha\in E$.

(c) Let $\alpha\in D$ be arbitrary, and we shall show that $C_\alpha\subseteq^* B$. By $B\cap\alpha\in J_{\nu(\alpha)}$, $\varphi(\alpha)=\alpha$ and Claim 3.1.3, we may assume that $C_\alpha\neq c_\alpha$. That is, $C_\alpha=C'_\alpha=\varphi^{-1}``E_\alpha$, and we must show that $E_\alpha\subseteq^*\varphi``B$.

Let $i<\lambda^+$ be such that $\alpha=\alpha(i)$, and let $\mu\in E_\alpha$ be large enough such that

$$\{\gamma(i),(\varphi``B)\cap\alpha\}\subseteq\mathrm{Im}(\sigma_{\varphi(\mu),\alpha}). \tag{5}$$

Write $\bar{\gamma} = \sigma^{-1}_{\varphi(\mu),\alpha}(\gamma(i))$. We have that $\sigma_{\varphi(\mu),\alpha} \restriction J_{\bar{\gamma}} \colon J_{\bar{\gamma}} \to J_{\gamma(i)}$ is fully elementary, so that in fact $\varphi(\mu)$ is a limit point of $(\varphi``B) \cap \alpha$, and hence $\varphi(\mu) \in (\varphi``B) \cap \alpha$, i.e., $\mu \in B$. As Equation (5) holds true for a tail end of $\mu \in E_\alpha$, we have $E_\alpha \subseteq^* \varphi``B$. □

We now verify Clause (6) of Definition 2.4, using an argument from [1, §2]. Assume (6) were to fail. Recalling Clause (d) of Claim 3.1.5, there are n, $m < \omega$, some Π^n_m-sentence η valid in a structure $(\lambda^+, \in, \langle A_i \mid i < \omega\rangle)$, and some club $D \subseteq \lambda^+$ such that for every $\alpha \in D$, $\alpha \in R_<$ or else

$$N_\alpha \models \text{ “}\eta \text{ is not valid in } (\alpha, \in, \langle A_i \restriction \alpha \mid i < \omega\rangle).\text{”}$$

Let $(D, \langle A_i \mid i < \omega\rangle)$ be the $<_L$–least such pair. Notice that C, φ, and $R_<$ are all definable over $J_{\lambda^+ + \omega}$ by some formulas with no parameters. Also, D and $\langle A_i \mid i < \omega\rangle$ are both definable over $J_{\lambda^{+n}}$ by some formulas with no parameters. But as λ^+ and λ^{+n}, and hence $J_{\lambda^+ + \omega}$ and $J_{\lambda^{+n}}$, are both Σ_1–definable over $J_{\lambda^{+n+1}}$ from the parameter λ^{+n}, we get that D, $\langle A_i \mid i < \omega\rangle$, C, and φ, and also λ, λ^+, λ^{+2}, …, λ^{+n} are all Σ_1–definable over $J_{\lambda^{+n+1}}$ from the parameter λ^{+n}.

Let us here and in what follows use the notation from [18, p. 194] which for $X \subseteq J_\gamma$ writes $h_{J_\gamma}(X)$ for $h_{J_\gamma}``(\omega \times [X]^{<\omega})$, where h_{J_γ} is the canonical Σ_1 Skolem function for J_γ.

We now have, setting $D^* = \varphi``D$,

$$\{D, \langle A_i \mid i < \omega\rangle, C, \varphi, D^*, \lambda, \lambda^+, \lambda^{+2}, \dots, \lambda^{+n}\} \subseteq h_{J_{\lambda^{+n+1}}}(\{\lambda^{+n}\}).$$

Let ν be such that

$$\sigma \colon J_\nu \cong h_{J_{\lambda^{+n+1}}}(\lambda \cup \{\lambda^{+n}\}) \prec_{\Sigma_1} J_{\lambda^{+n+1}}, \tag{6}$$

and write $\alpha = \sigma^{-1}(\lambda^+) = \mathrm{crit}(\sigma)$ and $\beta = \sigma^{-1}(\lambda^{+n})$.

Of course, $J_\nu = h_{J_\nu}(\lambda \cup \{\beta\})$, so that $\rho_1(J_\nu) = \lambda$ and $p_1(J_\nu) \leq^* \{\beta\}$.[7] However, if we had $p_1(J_\nu) <^* \{\beta\}$, then $\beta \in h_{J_\nu}(\beta)$, so that $\lambda^{+n} \in h_{J_{\lambda^{+n+1}}}(\lambda^{+n})$; but it easily follows from the Condensation Lemma that $h_{J_{\lambda^{+n+1}}}(\lambda^{+n}) \subseteq J_{\lambda^{+n}}$. Therefore, $p_1(J_\nu) = \{\beta\}$.

Obviously, $\nu(\alpha) = \nu$. By $\{D, \varphi, D^*\} \subseteq h_{J_{\lambda^{+n+1}}}(\{\lambda^{+n}\})$, we have that

$$D \cap \alpha, \varphi \cap (\alpha \times \alpha), D^* \cap \alpha = \sigma^{-1}(D, \varphi, D^*) \in J_\nu,$$

so that $\alpha \in D \cap D^*$, and we also get that α is a closure point of φ, and $\alpha = \varphi(\alpha)$.

By $\alpha = \varphi(\alpha)$, we have $\nu(\varphi(\alpha)) = \nu$. As $\langle A_i \mid i < \omega\rangle \in h_{J_{\lambda^{+n+1}}}(\{\lambda^{+n}\})$, $A_i \restriction \alpha = \sigma^{-1}(A_i)$ for every $i < \omega$. By elementarity then,

$$N_\alpha = J_\nu \models \text{ “}\eta \text{ is valid in } (\alpha, \in, \langle A_i \restriction \alpha \mid i < \omega\rangle)\text{”}. \tag{7}$$

Claim 3.1.8. (a) *There is no $\xi < \lambda$ such that $h_{J_\nu}(\xi \cup \{\beta\}) \cap \alpha$ is cofinal in α, so that in particular $\{\xi^\alpha_i \mid i < \theta(\alpha)\}$ is cofinal in λ;*
(b) *For every $i < \theta(\alpha)$, $|\xi^\alpha_{i+1}| = |\xi^\alpha_i|$;*
(c) *$\theta(\alpha) = \lambda$;*
(d) *$\alpha \notin R_<$.*

Proof. Write $d = \{(n, \vec{x}) \in \omega \times [\lambda]^{<\omega} \mid h_{J_\nu}(n, \vec{x} \cup \{\beta\}) \text{ is defined}\}$. We now introduce the following notation. For $\xi < \lambda$, let us write $\zeta(\xi) \leq \lambda^{+n+1}$ for the least ζ such that if $(n, \vec{x}) \in d \cap (\omega \times [\xi]^{<\omega})$, then $h_{J_\zeta}(n, (\vec{x}, \lambda^{+n}))$ is defined. As λ^{+n+1} is regular, $\zeta(\xi) < \lambda^{+n+1}$.

(a) Assume that $\xi < \lambda$ were such that $h_{J_\nu}(\xi \cup \{\beta\}) \cap \alpha$ is cofinal in α. Using the map σ of Equation (6), $h_{J_{\lambda^{+n+1}}}(\xi \cup \{\lambda^{+n}\}) \cap \alpha$ is then cofinal in α. Let $\zeta = \zeta(\xi) < \lambda^{+n+1}$. Trivially, $d \cap (\omega \times [\xi]^{<\omega}) \in h_{J_{\lambda^{+n+1}}}(\lambda)$, so that

$$\zeta \in h_{J_{\lambda^{+n+1}}}(\{d \cap (\omega \times [\xi]^{<\omega}), \lambda^{+n}\}) \subseteq h_{J_{\lambda^{+n+1}}}(\lambda \cup \{\lambda^{+n}\}) = \mathrm{Im}(\sigma),$$

[7] Here, $\leq^*$ is the canonical well–ordering of finite sets of ordinals, cf. [18, Problem 5.19 and p. 254].

so that by using σ again,
$$\alpha=\sup(h_{J_{\sigma^{-1}(\zeta)}}(\xi\cup\{\beta\})\cap\alpha).$$
However, $h_{J_{\sigma^{-1}(\zeta)}}(\xi\cup\{\beta\})\in J_\nu$ and α is regular in J_ν. This is a contradiction.

(b) This follows from the proof of (a). We have that
$$\zeta(\xi_i^\alpha)\in h_{J_{\lambda^{+n+1}}}(\{d\cap(\omega\times[\xi_i^\alpha]^{<\omega}),\lambda^{+n}\}),$$
and then
$$\begin{aligned}\tilde\mu:=\sup(h_{J_\nu}(\xi_i^\alpha\cup\{\beta\})\cap\alpha) &= \sup(h_{J_{\sigma^{-1}(\zeta(\xi_i^\alpha))}}(\xi_i^\alpha\cup\{\beta\})\cap\alpha)\\ &\in h_{J_\nu}(\{d\cap(\omega\times[\xi_i^\alpha]^{<\omega}),\beta\}).\end{aligned}$$
However, by the Condensation Lemma, $d\cap(\omega\times[\xi_i^\alpha]^{<\omega})\in h_{J_{\lambda^{+n+1}}}((\xi_i^\alpha)^+)$, so that
$$\tilde\mu\in h_{J_\nu}((\xi_i^\alpha)^+\cup\{\beta\}).$$
But this means that there is $(n,\vec{x})\in\omega\times[(\xi_i^\alpha)^+]^{<\omega}$ such that
$$\tilde\mu=h_{J_\nu}(n,(\vec{x},\beta)).$$
But $\tilde\mu\notin h_{J_\nu}(\xi_i^\alpha\cup\{\beta\})$, which now readily gives $\xi_{i+1}^\alpha<(\xi_i^\alpha)^+$.

(c) By Clause (b), for every $i<\theta(\alpha)$,
$$\mathrm{otp}(\{j<\theta(\alpha)\mid\xi_j^\alpha<|\xi_i^\alpha|^+\})=|\xi_i^\alpha|^+,$$
which together with Clause (a) gives that $\theta(\alpha)=\lambda$.

(d) As ν is certainly a limit of limit ordinals, $\alpha\notin R$. But then Clause (c) gives $\alpha\notin R_<$. □

Altogether $\alpha\in D\setminus R_<$, contradicting Equation (7). □

4. Applications

4.1. **Preliminaries.** For a family of functions T and a set of ordinals D, write $T\restriction D=\{f\in T\mid \mathrm{dom}(f)\in D\}$, and $\mathrm{succ}_\omega(D)=\{\delta\in D\mid 0<\mathrm{otp}(D\cap\delta)<\omega\}$.

Definition 4.1. We say that T is a *κ-tree*, whenever there exists a set Ω of size $\leq\kappa$, for which

- $T\subseteq {}^{<\kappa}\Omega$;
- $\{\mathrm{dom}(f)\mid f\in T\}=\kappa$;
- for every $f\in T$, we have $\{f\restriction\alpha\mid\alpha<\kappa\}\subseteq T$;
- $T_\alpha:=\{f\in T\mid\mathrm{dom}(f)=\alpha\}$ has size $<\kappa$ for all $\alpha<\kappa$.

A *κ-Aronszajn tree* is a κ-tree with no cofinal branches. A *κ-Kurepa tree* is a κ-tree admitting at least κ^+ many cofinal branches. A λ^+-tree is *special* if it may be covered by λ many antichains. Following [2], which generalizes the case $\lambda=\aleph_0$ from [7], we say that a λ^+-tree T is *almost Souslin* if for every antichain $A\subseteq T$, the set $\{\mathrm{dom}(z)\mid z\in A\}\cap E^{\lambda^+}_{\mathrm{cf}(\lambda)}$ is nonstationary.

Of course, *almost Souslin* and *special* are contradictory concepts.

In [3],[4], the parameterized principle $\mathrm{P}^-(\kappa,\mu,\mathcal{R},\theta,\mathcal{S},\nu,\sigma,\mathcal{E})$ was introduced and studied in relation with κ-Souslin tree constructions. Here, we shall only give the definition of the special case
$$(\kappa,\mu,\mathcal{R},\theta,\mathcal{S},\nu,\sigma,\mathcal{E})=(\kappa,2,\sqsubseteq,\kappa,\{S\},2,\omega,(\mathcal{P}(\kappa))^2),$$
which, for simplicity, is denoted by $\boxtimes(S)$.

Definition 4.2. For any regular uncountable cardinal κ, and stationary $S\subseteq\kappa$, $\boxtimes(S)$ asserts the existence of a sequence $\langle C_\alpha\mid\alpha<\kappa\rangle$ such that:

- C_α is a club subset of α for every limit ordinal $\alpha<\kappa$;

- $C_{\bar\alpha} = C_\alpha \cap \bar\alpha$ for every ordinal $\alpha < \kappa$ and every $\bar\alpha \in \operatorname{acc}(C_\alpha)$;
- for every sequence $\langle A_i \mid i < \kappa\rangle$ of cofinal subsets of κ, there exist stationarily many $\alpha \in S$ such that $\sup\{\beta < \alpha \mid \operatorname{succ}_\omega(C_\alpha \setminus \beta) \subseteq A_i\} = \alpha$ for all $i < \alpha$.

Note that for $S \subseteq S' \subseteq \kappa$, every $\boxtimes(S)$-sequence is also a $\boxtimes(S')$-sequence. Clearly, every $\boxtimes(\kappa)$-sequence is in particular a $\square(\kappa)$-sequence.

Definition 4.3. The principle $\boxtimes_\lambda(S)$ asserts the existence of a $\boxtimes(S)$-sequence $\langle C_\alpha \mid \alpha < \sup(S)\rangle$ with the additional property that $\operatorname{otp}(C_\alpha) \leq \lambda$ for all α.

Note that every $\boxtimes_\lambda(\lambda^+)$-sequence is in particular a $\square_\lambda$-sequence.

Definition 4.4 ([2])**.** Suppose that $T \subseteq {}^{<\kappa}\Omega$ is a κ-tree, and S is stationary in κ.

We say that T is $\boxtimes(S)$*-respecting* if there exist a subset $\S \subseteq S$ and a sequence

$$\langle \mathbf{b}^\alpha : T \restriction C_\alpha \to {}^\alpha\Omega \cup \{\emptyset\} \mid \alpha < \kappa\rangle$$

such that:

(1) $\langle C_\alpha \mid \alpha < \kappa\rangle$ is a $\boxtimes(\S)$-sequence;
(2) $T_\alpha \subseteq \operatorname{Im}(\mathbf{b}^\alpha)$ for every $\alpha \in \S$;
(3) if $\bar\alpha \in \operatorname{acc}(C_\alpha)$ and $x \in T \restriction C_{\bar\alpha}$, then $\mathbf{b}^{\bar\alpha}(x) = \mathbf{b}^\alpha(x) \restriction \bar\alpha$.

The notion of $\boxtimes_\lambda(S)$-respecting is defined in a similar fashion.

4.2. **Walks on ordinals.** In this subsection, we address the trees obtained from *walks on ordinals*, as introduced in [21] (see also [23]). Suppose that $\vec{C} = \langle C_\alpha \mid \alpha < \kappa\rangle$ is a C-sequence over some fixed regular uncountable cardinal κ. That is, C_α is a club in α for all limit $\alpha < \kappa$, and $C_{\alpha+1} = \{\alpha\}$ for all $\alpha < \kappa$. Recall few of the characteristic functions of walks on ordinals:

Definition 4.5 ([21],[23])**.** Define $\operatorname{Tr} : [\kappa]^2 \to {}^\omega\kappa$, $\rho_2 : [\kappa]^2 \to \omega$, $\rho_1 : [\kappa]^2 \to \kappa$ and $\rho_0 : [\kappa]^2 \to {}^{<\omega}\kappa$ as follows.

For all $\alpha < \delta < \kappa$, let

- $\operatorname{Tr}(\alpha,\delta)(n) := \begin{cases} \delta, & n = 0; \\ \min(C_{\operatorname{Tr}(\alpha,\delta)(n-1)} \setminus \alpha), & n > 0,\ \&\ \operatorname{Tr}(\alpha,\delta)(n-1) > \alpha; \\ \alpha, & \text{otherwise}; \end{cases}$
- $\rho_2(\alpha,\delta) := \min\{n < \omega \mid \operatorname{Tr}(\alpha,\delta)(n) = \alpha\}$;
- $\rho_1(\alpha,\delta) := \max(\rho_0(\alpha,\delta))$, where
- $\rho_0(\alpha,\delta) := \langle \operatorname{otp}(C_{\operatorname{Tr}(\alpha,\delta)(i)} \cap \alpha) \mid i < \rho_2(\alpha,\delta)\rangle$.

Definition 4.6 ([15])**.** Define $\varphi_2 : [\kappa]^2 \to 2$ by stipulating $\varphi_2(\alpha,\delta) = 1$ iff $\alpha \in \operatorname{acc}(C_{\operatorname{Tr}(\alpha,\delta)(\rho_2(\alpha,\delta)-1)})$.

Definition 4.7 ([21],[23])**.** For all $\delta < \kappa$, let $\rho_{0\delta} : \delta \to {}^{<\omega}\delta$, $\rho_{1\delta} : \delta \to \delta$ and $\rho_{2\delta} : \delta \to \omega$ denote the fiber maps $\alpha \mapsto \rho_0(\alpha,\delta)$, $\alpha \mapsto \rho_1(\alpha,\delta)$ and $\alpha \mapsto \rho_2(\alpha,\delta)$, respectively. Then, put

- $\mathcal{T}(\rho_0) := \{\rho_{0\delta} \restriction \beta \mid \beta \leq \delta < \kappa\}$;
- $\mathcal{T}(\rho_1) := \{\rho_{1\delta} \restriction \beta \mid \beta \leq \delta < \kappa\}$;
- $\mathcal{T}(\rho_2) := \{\rho_{2\delta} \restriction \beta \mid \beta \leq \delta < \kappa\}$.

It is easy to see that if $|\{C_\alpha \cap \beta \mid \alpha < \kappa\}| < \kappa$ for all $\beta < \kappa$, then $\mathcal{T}(\rho_0), \mathcal{T}(\rho_1)$ and $\mathcal{T}(\rho_2)$ are κ-trees.

Fact 4.8 ([21],[23])**.** *Suppose that $\mathcal{T}(\rho_0)$ is derived from walks along a $\square_\lambda$-sequence,*[8] *then $\mathcal{T}(\rho_0)$ is a special λ^+-Aronszajn tree.*

[8]This means that the C-sequence that was used to define ρ_0 in Definition 4.5 is a $\square_\lambda$-sequence, $\langle C_\alpha \mid \alpha < \lambda^+\rangle$.

Fact 4.9 ([21],[23]). *Suppose that λ is a regular cardinal, and $\mathcal{T}(\rho_1)$ is derived from walks along a C-sequence $\langle C_\alpha \mid \alpha < \lambda^+\rangle$ for which $\mathrm{otp}(C_\alpha) \le \lambda$ for all $\alpha < \lambda^+$. Then:*

(1) $\mathcal{T}(\rho_1) \subseteq {}^{<\lambda^+}\lambda$;
(2) *for every $z \in \mathcal{T}(\rho_1)$ and $i < \lambda$, the set $z^{-1}\{i\}$ has size $< \lambda$;*
(3) *for every $\gamma < \delta < \lambda^+$, the set $\{\xi \le \gamma \mid \rho_{1\gamma}(\xi) \neq \rho_{1\delta}(\xi)\}$ has size $< \lambda$.*

Theorem 4.10. *If $\vec{C}$ is a $\boxtimes(S)$-sequence, then the corresponding trees $\mathcal{T}(\rho_0), \mathcal{T}(\rho_1), \mathcal{T}(\rho_2)$ are $\boxtimes(S)$-respecting, as witnessed by the very same $\vec{C}$.*

Proof. Suppose that $\vec{C} = \langle C_\alpha \mid \alpha < \kappa\rangle$ is a $\boxtimes(S)$-sequence. Fix bijections $\pi : \kappa \leftrightarrow 2 \times \kappa$ and $\psi : \kappa \leftrightarrow {}^{<\omega}\{C_\alpha \cap \beta \mid \alpha, \beta < \kappa\}$. By the coherence of $\vec{C}$, we have $|\{C_\alpha \cap \beta \mid \alpha < \kappa\}| < \kappa$ for all $\beta < \kappa$, and hence, the set

$$E := \{\beta < \kappa \mid \pi[\beta] = 2 \times \beta \ \&\ {}^{<\omega}\{C_\alpha \cap \beta \mid \alpha < \kappa, \sup(C_\alpha \cap \beta) < \beta\} = \psi[\beta]\}$$

is a club in κ. Fix a surjection $\varphi : \kappa \to \kappa$ such that the preimage of any singleton is cofinal in κ. Put

$$\S := \{\alpha \in S \cap E \mid \alpha \subseteq \varphi[C_\alpha]\}.$$

Claim 4.10.1. *$\vec{C}$ is a $\boxtimes(\S)$-sequence.*

Proof. Given a sequence $\langle A_i \mid i < \kappa\rangle$ of cofinal subsets of κ, write:

$$A'_i := \begin{cases} A_j, & \text{if } \pi(i) = (0, j); \\ \varphi^{-1}\{j\}, & \text{if } \pi(i) = (1, j). \end{cases}$$

As $\vec{C}$ is a $\boxtimes(S)$-sequence, the following set is stationary:

$$R := \{\alpha \in S \cap E \mid \forall i < \alpha\ \sup\{\beta \in C_\alpha \mid \mathrm{succ}_\omega(C_\alpha \setminus \beta) \subseteq A'_i\} = \alpha\}.$$

Let $\alpha \in R$ be arbitrary.

▶ By $\pi[\alpha] \supseteq \{0\} \times \alpha$, we have $\sup\{\beta \in C_\alpha \mid \mathrm{succ}_\omega(C_\alpha \setminus \beta) \subseteq A_i\} = \alpha$ for all $i < \alpha$.

▶ By $\pi[\alpha] \supseteq \{1\} \times \alpha$, we have $\sup\{\beta \in C_\alpha \mid \mathrm{succ}_\omega(C_\alpha \setminus \beta) \subseteq \varphi^{-1}\{i\}\} = \alpha$ for all $i < \alpha$. In particular, $\alpha \in \S$. □

For all $\varepsilon < \kappa$ and nonzero limit $\beta < \kappa$, let ψ^β_ε denote $\psi(\varepsilon)^\frown C_\beta$. Then define $\Sigma^\beta_\varepsilon : \beta \to {}^{<\omega}\kappa$, as follows. Given $\alpha < \beta$, put

- $j' = \min\{j \in \mathrm{dom}(\psi^\beta_\varepsilon) \mid \psi^\beta_\varepsilon(j) \setminus \alpha \neq \emptyset\}$;
- $\alpha^+ = \min(\psi^\beta_\varepsilon(j') \setminus \alpha)$, and
- $\Sigma^\beta_\varepsilon(\alpha) = \langle \mathrm{otp}(\psi^\beta_\varepsilon(j) \cap \alpha) \mid j \le j'\rangle^\frown\langle \mathrm{otp}(C_{\mathrm{Tr}(\alpha,\alpha^+)(i+1)} \cap \alpha) \mid i + 1 < \rho_2(\alpha, \alpha^+)\rangle$.

Let $m : {}^{<\omega}\kappa \to \kappa$ denote a map that satisfies $\sigma \mapsto \max(\sigma)$ for all nonempty sequence σ. Let $\ell : {}^{<\omega}\kappa \to \omega$ denote the map that satisfies $\sigma \mapsto |\sigma|$ for all sequence σ.

Denote $\Omega_0 := {}^{<\omega}\kappa$, $\Omega_1 := \kappa$ and $\Omega_2 := \omega$. For all $i < 3$, define $\vec{\mathbf{b}_i} = \langle \mathbf{b}^\beta_i : \mathcal{T}(\rho_i) \restriction C_\beta \to {}^\beta\Omega_i \mid \beta < \kappa\rangle$ by stipulating:

$$\begin{aligned} \mathbf{b}^\beta_0(x) &= \Sigma^\beta_{\varphi(\mathrm{dom}(x))}, \\ \mathbf{b}^\beta_1(x) &= m \circ \Sigma^\beta_{\varphi(\mathrm{dom}(x))}, \\ \mathbf{b}^\beta_2(x) &= \ell \circ \Sigma^\beta_{\varphi(\mathrm{dom}(x))}. \end{aligned}$$

Claim 4.10.2. *Suppose $i < 3$, $\bar\beta \in \mathrm{acc}(C_\beta)$ and $x \in \mathcal{T}(\rho_i) \restriction C_{\bar\beta}$. Then $\mathbf{b}^{\bar\beta}_i(x) = \mathbf{b}^\beta_i(x) \restriction \bar\beta$.*

Proof. By $\bar\beta \in \operatorname{acc}(C_\beta)$, we have $C_{\bar\beta} = C_\beta \cap \bar\beta$, and it suffices to show that $\Sigma^{\bar\beta}_\varepsilon = \Sigma^\beta_\varepsilon \restriction \bar\beta$ for all $\varepsilon < \kappa$. But the latter is straight-forward to verify. □

Claim 4.10.3. *$\mathcal{T}(\rho_i)_\beta \subseteq \operatorname{Im}(\mathbf{b}^\beta_i)$ for every $i < 3$ and $\beta \in \S$.*

Proof. We concentrate on the case $i = 0$. Let $\beta \in \S$ and $z \in \mathcal{T}(\rho_0)_\beta$ be arbitrary. Pick $\delta \in [\beta, \kappa)$ such that $z = \rho_{0\delta} \restriction \beta$. Let $n = \rho_2(\beta, \delta) - \varphi_2(\beta, \delta)$. Define $\sigma : n \to \mathcal{P}(\beta)$, by stipulating $\sigma(j) := C_{\operatorname{Tr}(\beta,\delta)(j)} \cap \beta$. By $\beta \in \S \subseteq E$ and the definition of n, there exists some $\varepsilon < \beta$ such that $\psi(\varepsilon) = \sigma$. By $\beta \in \S$, there exists some $\gamma \in C_\beta$ such that $\varphi(\gamma) = \varepsilon$. Let $x = z \restriction \gamma$. By $z \in \mathcal{T}(\rho_0)_\beta$, we have $x \in \mathcal{T}(\rho_0)_\gamma$, let alone $x \in \mathcal{T}(\rho_0) \restriction C_\beta$.

We have $\varphi(\operatorname{dom}(x)) = \varepsilon$, and so, to show that $\mathbf{b}^\beta_0(x) = z$, it suffices to prove that $\Sigma^\beta_\varepsilon = z$. First, we make the following observation.

▶ If $\varphi_2(\beta, \delta) = 0$, then

$$\begin{aligned}\psi^\beta_\varepsilon &= \psi(\varepsilon)^\frown C_\beta = \sigma^\frown C_\beta \\ &= \langle C_{\operatorname{Tr}(\beta,\delta)(j)} \cap \beta \mid j < \rho_2(\beta,\delta)\rangle^\frown C_\beta \\ &= \langle C_{\operatorname{Tr}(\beta,\delta)(j)} \cap \beta \mid j \le n\rangle.\end{aligned}$$

▶ If $\varphi_2(\beta, \delta) = 1$, then $\beta \in \operatorname{acc}(C_{\operatorname{Tr}(\beta,\delta)(\rho_2(\beta,\delta)-1)})$, and hence $C_\beta = C_{\operatorname{Tr}(\beta,\delta)(\rho_2(\beta,\delta)-1)} \cap \beta$. Consequently

$$\begin{aligned}\psi^\beta_\varepsilon &= \psi(\varepsilon)^\frown C_\beta = \sigma^\frown C_\beta \\ &= \langle C_{\operatorname{Tr}(\beta,\delta)(j)} \cap \beta \mid j < \rho_2(\beta,\delta) - 1\rangle^\frown C_\beta \\ &= \langle C_{\operatorname{Tr}(\beta,\delta)(j)} \cap \beta \mid j < \rho_2(\beta,\delta) - 1\rangle^\frown (C_{\operatorname{Tr}(\beta,\delta)(\rho_2(\beta,\delta)-1)} \cap \beta) \\ &= \langle C_{\operatorname{Tr}(\beta,\delta)(j)} \cap \beta \mid j \le n\rangle.\end{aligned}$$

Now, let $\alpha < \beta$ be arbitrary. By $z = \rho_{0\delta} \restriction \beta$ and definition of $\rho_{0\delta}$, we have:

$$z(\alpha) = \langle \operatorname{otp}(C_{\operatorname{Tr}(\alpha,\delta)(j)} \cap \alpha) \mid j < \rho_2(\alpha,\delta)\rangle.$$

Let $j' = \min\{j \in \operatorname{dom}(\psi^\beta_\varepsilon) \mid \psi^\beta_\varepsilon(j) \setminus \alpha \neq \emptyset\}$. Then, for all $j < j'$, we have $(C_{\operatorname{Tr}(\beta,\delta)(j)} \cap \beta) \setminus \alpha = \emptyset$, and hence $\min(C_{\operatorname{Tr}(\beta,\delta)(j)} \setminus \alpha) = \min(C_{\operatorname{Tr}(\beta,\delta)(j)} \setminus \beta)$. As $\operatorname{Tr}(\alpha,\delta)(0) = \delta = \operatorname{Tr}(\beta,\delta)(0)$, it follows that

$$\operatorname{Tr}(\alpha,\delta) \restriction j' + 1 = \operatorname{Tr}(\beta,\delta) \restriction j' + 1,$$

and hence

$$z(\alpha) \restriction j' + 1 = \Sigma^\beta_\varepsilon \restriction j' + 1.$$

Let $\alpha^+ := \operatorname{Tr}(\alpha,\delta)(j')$. By definition

$$z(\alpha) = (z(\alpha) \restriction j' + 1)^\frown \langle \operatorname{otp}(C_{\operatorname{Tr}(\alpha,\alpha^+)(i+1)} \cap \alpha) \mid i + 1 < \rho_2(\alpha,\alpha^+)\rangle.$$

By $\operatorname{Tr}(\beta,\delta)(j') = \operatorname{Tr}(\alpha,\delta)(j') = \alpha^+$, we have $\min(\psi^\beta_\varepsilon(j') \setminus \alpha) = \alpha^+$. Altogether:

$$z(\alpha) = \Sigma^\beta_\varepsilon(\alpha).$$

□

So for each $i < 3$, $\S$ and $\overrightarrow{\mathbf{b}_i}$ witness that $\mathcal{T}(\rho_i)$ is $\boxtimes(S)$-respecting. □

Theorem 4.11. *Suppose that $\diamondplus^*_\lambda$ holds for a given regular uncountable cardinal λ. Then there exists a $\boxtimes_\lambda(E^{\lambda^+}_\lambda)$-sequence $\langle D_\alpha \mid \alpha < \lambda^+\rangle$ satisfying the following. For every stationary $S \subseteq E^{\lambda^+}_\lambda$, there are $\delta \in S$ and $\gamma \in \operatorname{nacc}(D_\delta) \cap S$ such that $D_\delta \cap \gamma \sqsubseteq D_\gamma$.*

Proof. Let $\langle (C_\alpha, X_\alpha, f_\alpha) \mid \alpha < \lambda^+\rangle$ witness $\diamondsuit^*_\lambda$. Fix a surjection $\psi : \lambda \to \lambda$ with the property that for all $j < \lambda$, the set $\{i < \lambda \mid \psi``(i, i+\omega) = \{j\}\}$ has size λ. We may assume that $C_{\alpha+1} = \emptyset$ for all $\alpha < \kappa$. Denote $\Lambda = \{\alpha < \lambda^+ \mid \operatorname{otp}(C_\alpha) = \lambda\}$. Clearly, $\Lambda = E^{\lambda^+}_\lambda$. Write $\kappa = \lambda^+$.

For all $\alpha < \kappa$, let $\pi_\alpha : \operatorname{otp}(C_\alpha) \to C_\alpha$ denote the monotone enumeration of C_α. We now define a sequence of functions $\overrightarrow{\sigma} = \langle \sigma_\alpha : \operatorname{otp}(C_\alpha) \to \alpha \mid \alpha < \kappa\rangle$ by recursion over $\alpha < \kappa$. For this, suppose $\alpha < \kappa$, and $\overrightarrow{\sigma} \restriction \alpha$ has already been defined. The definition of $\sigma_\alpha : \operatorname{otp}(C_\alpha) \to \alpha$ is obtained by recursion over $i < \operatorname{otp}(C_\alpha)$. For this, suppose $i < \operatorname{otp}(C_\alpha)$ and $\sigma_\alpha \restriction i$ has already been defined. Let

- $X^i_\alpha = \{\xi \in f_\alpha(\pi_\alpha(\psi(i))) \mid \pi_\alpha(i) < \xi \leq \pi_\alpha(i+1), \psi(i) \leq i\}$, and
- $Y^i_\alpha = \{\xi \in X^i_\alpha \mid \sigma_\alpha \restriction (i+1) = \sigma_\xi \restriction (i+1)\}$.

Then, let

$$\sigma_\alpha(i) = \begin{cases} \min(Y^{i'}_\alpha), & \text{if } i = i'+1, Y^{i'}_\alpha \neq \emptyset;\\ \min(X^{i'}_\alpha), & \text{if } i = i'+1, Y^{i'}_\alpha = \emptyset, X^{i'}_\alpha \neq \emptyset;\\ \pi_\alpha(i), & \text{otherwise.}\end{cases}$$

Put $D_\alpha = \operatorname{Im}(\sigma_\alpha)$.

Claim 4.11.1. *D_α is a club in α, $\operatorname{otp}(D_\alpha) = \operatorname{otp}(C_\alpha)$, $\operatorname{acc}(D_\alpha) = \operatorname{acc}(C_\alpha)$ and if $\bar\alpha \in \operatorname{acc}(D_\alpha)$, then $D_{\bar\alpha} = D_\alpha \cap \bar\alpha$.*

Proof. For all $i < \operatorname{otp}(C_\alpha)$, we have

(a) $\pi_\alpha(i) < \sigma_\alpha(i+1) \leq \pi_\alpha(i+1)$;
(b) $\sigma_\alpha(i) = \pi_\alpha(i)$ for all limit $i < \operatorname{otp}(C_\alpha)$, including $i = 0$.

So $\operatorname{otp}(D_\alpha \cap \gamma) \leq \operatorname{otp}(C_\alpha \cap \gamma)+1$ for all $\gamma < \alpha$, and $\operatorname{acc}(D_\alpha) = \operatorname{acc}(C_\alpha)$. Towards a contradiction, suppose that $\bar\alpha \in \operatorname{acc}(D_\alpha)$, while $D_\alpha \cap \bar\alpha \neq D_{\bar\alpha}$. Let $i < \operatorname{otp}(C_{\bar\alpha})$ be the least such that $\sigma_\alpha(i) \neq \sigma_{\bar\alpha}(i)$. By $\bar\alpha \in \operatorname{acc}(D_\alpha) = \operatorname{acc}(C_\alpha)$, we have $C_{\bar\alpha} = C_\alpha \cap \bar\alpha$, $\pi_{\bar\alpha} \subseteq \pi_\alpha$. So by Clause (b), i must be a successor ordinals, say $i = i'+1$. We have:

$$\begin{aligned} X^{i'}_{\bar\alpha} &= \{\xi \in f_{\bar\alpha}(\pi_{\bar\alpha}(\psi(i'))) \mid \pi_{\bar\alpha}(i') < \xi \leq \pi_{\bar\alpha}(i+1), \psi(i') \leq i'\}\\ &= \{\xi \in f_\alpha(\pi_\alpha(\psi(i'))) \cap \bar\alpha \mid \pi_\alpha(i') < \xi \leq \pi_\alpha(i+1), \psi(i') \leq i\} = X^{i'}_\alpha. \end{aligned}$$

By minimality of i, we then also have $Y^{i'}_{\bar\alpha} = Y^{i'}_\alpha$. But then $\sigma_\alpha(i) = \sigma_{\bar\alpha}(i)$. This is a contradiction. □

So $\vec D := \langle D_\alpha \mid \alpha < \kappa\rangle$ is a $\square_\lambda$-sequence. Next, we prove that $\vec D$ is a $\boxtimes_\lambda(\S)$-sequence for every stationary $\S \subseteq \Lambda$.

Claim 4.11.2. *For every stationary $\S \subseteq \Lambda$, and every sequence $\langle A_\delta \mid \delta < \kappa\rangle$ of cofinal subsets of κ, there exist stationarily many $\alpha \in \S$ such that for every $\delta < \alpha$, we have*

$$\sup\{\beta \in D_\alpha \mid \operatorname{succ}_\omega(D_\alpha \setminus \beta) \subseteq A_\delta\} = \alpha.$$

Proof. Let $\S$ and $\langle A_\delta \mid \delta < \kappa\rangle$ be as in the hypothesis. By Clause (5) of Definition 2.2, for every $\delta < \kappa$, fix a club $E_\delta \subseteq \operatorname{acc}^+(A_\delta)$ such that $A_\delta \cap \alpha \in N_\alpha$ for all $\alpha \in E_\delta$. Let $C := \Delta_{\delta<\kappa} E_\delta$. By Clause (5) of Definition 2.2, let us fix $\alpha \in \S$ such that $C_\alpha \subseteq^* C$.

Let $\epsilon, \delta < \alpha$ be arbitrary. We shall find $\beta \in D_\alpha \setminus \epsilon$ such that

$$\operatorname{succ}_\omega(D_\alpha \setminus \beta) \subseteq A_\delta.$$

Without loss of generality, ϵ is also large enough so that $C_\alpha \setminus \epsilon \subseteq C$.

By $\delta \in \alpha \in C$, we have $\alpha \in E_\delta$, and hence $A_\delta \cap \alpha \in X_\alpha$. Since $\alpha \in \S \subseteq \Lambda$, we appeal to Clause (3) of Definition 2.2 to obtain some $j < \operatorname{otp}(C_\alpha)$ such that $f_\alpha(\pi_\alpha(j)) = A_\delta \cap \alpha$. Fix a large enough $i < \operatorname{otp}(C_\alpha)$ such that $\psi``(i, i+\omega) = \{j\}$, and $\pi_\alpha(i) > \max\{\epsilon, \delta, \pi_\alpha(j)\}$. Write $\beta := \min(D_\alpha \setminus \pi_\alpha(i)+1)$. Then $\beta > \epsilon$ and $\operatorname{succ}_\omega(D_\alpha \setminus \beta) = \sigma_\alpha``(i, i+\omega)$.

Let $i' \in [i, i+\omega)$ be arbitrary. We shall show that $\sigma_\alpha(i'+1) \in A_\delta$. We have

$$\begin{aligned} X_\alpha^{i'} &= \{\xi \in f_\alpha(\pi_\alpha(\psi(i'))) \mid \pi_\alpha(i') < \xi \leq \pi_\alpha(i'+1), \psi(i') \leq i'\} = \\ &= \{\xi \in f_\alpha(\pi_\alpha(j)) \mid \pi_\alpha(i') < \xi \leq \pi_\alpha(i'+1)\} = \\ &= \{\xi \in A_\delta \cap \alpha \mid \pi_\alpha(i') < \xi \leq \pi_\alpha(i'+1)\}. \end{aligned}$$

By $\pi_\alpha(i'+1) > \max\{\epsilon, \delta\}$, we have $\pi_\alpha(i'+1) \in E_\delta \subseteq \mathrm{acc}^+(A_\delta)$, and hence $X_\alpha^{i'} \neq \emptyset$. Consequently, $\sigma_\alpha(i'+1) \in X_\alpha^{i'} \subseteq A_\delta$, as sought. $\square$

Claim 4.11.3. *For every stationary $S \subseteq \Lambda$, there are $\delta \in S$ and $\gamma \in \mathrm{nacc}(D_\delta) \cap S$ such that $D_\delta \cap \gamma \sqsubseteq D_\gamma$.*

Proof. Fix a surjection $\varphi : \kappa \to \{D_\alpha \cap \beta \mid \alpha, \beta < \kappa\}$. Note that by Claim 4.11.1, the following set is a club

$$D := \{\delta < \kappa \mid \{D_\alpha \cap \beta \mid \alpha, \beta < \kappa, \sup(D_\alpha \cap \beta) < \delta\} = \varphi[\delta]\}.$$

Let S be an arbitrary stationary subset of Λ. Define a function $g : \kappa \times \kappa \to \kappa$ by letting for every $\alpha < \kappa$:

$$g(\alpha, \zeta) := \begin{cases} \sup\{\xi \in S \mid \varphi(\alpha) \sqsubseteq D_\xi\} + 1, & \text{if } \sup\{\xi \in S \mid \varphi(\alpha) \sqsubseteq D_\xi\} < \kappa; \\ \min\{\xi \in S \mid \xi > \zeta, \varphi(\alpha) \sqsubseteq D_\xi\}, & \text{otherwise.} \end{cases}$$

Let $E := \{\delta < \kappa \mid g[\delta \times \delta] \subseteq \delta\}$. Now, fix $\delta \in S \cap E$ and $\epsilon < \delta$ satisfying the following:

- $C_\delta \setminus \epsilon \subseteq D \cap E$;
- $S \cap \delta \in X_\delta$.

Since $\delta \in \Lambda$ and $S \cap \delta \in X_\delta$, let us fix some $j < \mathrm{otp}(C_\delta)$ such that $f_\delta(\pi_\delta(j)) = S \cap \delta$. Fix a large enough $i < \mathrm{otp}(C_\delta)$ such that $\psi(i) = j$, and $\pi_\delta(i) > \max\{\epsilon, \pi_\alpha(j)\}$.

By $\pi_\delta(i+1) \in D$, let us fix some $\alpha < \pi_\delta(i+1)$ such that $\varphi(\alpha) = D_\delta \cap \pi_\delta(i+1)$. By $\alpha \in \delta \in E$, we have $g[\{\alpha\} \times \delta] \subseteq \delta$. As $\varphi(\alpha) \sqsubseteq D_\delta$ and $\delta \in S$, we thus infer that $g(\alpha, \zeta) > \zeta$ for all $\zeta < \kappa$. In particular, by $\pi_\delta(i+1) \in E$, we have $\pi_\delta(i) < g(\alpha, \pi_\delta(i)) < \pi_\delta(i+1)$. Recalling that

$$X_\delta^i = \{\xi \in S \cap \delta \mid \pi_\delta(i) < \xi \leq \pi_\delta(i+1)\},$$

and

$$Y_\delta^i = \{\xi \in X_\delta^i \mid \sigma_\delta \restriction (i+1) = \sigma_\xi \restriction (i+1)\} = \{\xi \in X_\delta^i \mid \varphi(\alpha) \sqsubseteq D_\xi\},$$

we see that $g(\alpha, \pi_\delta(i))$ witnesses that Y_δ^i is nonempty. Write $\gamma := \sigma_\delta(i+1)$. Then $\gamma \in Y_\delta^i \cap \mathrm{nacc}(D_\delta) \cap S$, and hence $D_\delta \cap \gamma \sqsubseteq D_\gamma$. $\square$

$\square$

We now address the trees $\mathcal{T}(\rho_0)$ and $\mathcal{T}(\rho_1)$. Note that the latter is a projection of the former.[9]

Corollary 4.12. *Suppose that $\diamondplus^*_\lambda$ holds for a given uncountable cardinal $\lambda = \lambda^{<\lambda}$. Then there exists a $\boxtimes_\lambda(E^{\lambda^+}_\lambda)$-sequence which is respected by the corresponding trees $\mathcal{T}(\rho_0)$ and $\mathcal{T}(\rho_1)$. Moreover:*

1. *$\mathcal{T}(\rho_0)$ is a special λ^+-Aronszajn tree;*
2. *$\mathcal{T}(\rho_1)$ is an almost Souslin, λ^+-Aronszajn tree;*
3. *$\mathcal{T}(\rho_1)$ can be made special by means of a cofinality-preserving forcing.*

[9]Indeed, let m be some map satisfying $\sigma \mapsto \max(\sigma)$ for every nonempty sequence of ordinals, σ. Then $\mathcal{T}(\rho_1)$ is the image of $\mathcal{T}(\rho_0)$ under the map $t \mapsto m \circ t$.

Proof. Let $\vec{D} = \langle D_\alpha \mid \alpha < \lambda^+\rangle$ be given by Theorem 4.11. In particular, $\vec{D}$ is a $\boxtimes_\lambda(E^{\lambda^+}_\lambda)$-sequence. Let $\mathcal{T}(\rho_0)$ and $\mathcal{T}(\rho_1)$ denote the trees derived from walks along $\vec{D}$. By Theorem 4.10, $\mathcal{T}(\rho_0)$ and $\mathcal{T}(\rho_1)$ are $\boxtimes_\lambda(E^{\lambda^+}_\lambda)$-respecting as witnessed by our $\vec{D}$.

(1) Since $\vec{D}$ is in particular a $\square_\lambda$-sequence, we get from Fact 4.8 that $\mathcal{T}(\rho_0)$ is special.

(2) It is easy to see that $\mathcal{T}(\rho_1)$ is a λ^+-tree. By Clause (2) of Fact 4.9, $\mathcal{T}(\rho_1)$ is moreover Aronszajn. To see that $\mathcal{T}(\rho_1)$ is almost Souslin, suppose that $A \subseteq \mathcal{T}(\rho_1)$, and that $S = \{\mathrm{dom}(z) \mid z \in A\} \cap E^{\lambda^+}_\lambda$ is stationary. For all $\delta \in S$, fix $z_\delta \in A$ such that $\mathrm{dom}(z_\delta) = \delta$. We now run the arguments from [10]. For all $\delta \in S$, put

$$y_\delta = z_\delta \restriction \{\xi < \delta \mid z_\delta(\xi) \neq \rho_{1\delta}(\xi)\}.$$

By $\lambda^{<\lambda} = \lambda$ and Fact 4.9, we may find a stationary subset $S' \subseteq S$ such that $\{y_\delta \mid \delta \in S'\}$ is a singleton. It follows that $\{z_\delta \mid \delta \in S'\}$ is an antichain iff $\{\rho_{1\delta} \mid \delta \in S'\}$ is an antichain. So, let us show that the latter is not an antichain. By the choice of $\vec{D}$, let us fix $\delta \in S'$ and $\gamma \in \mathrm{nacc}(D_\delta) \cap S'$ such that $D_\delta \cap \gamma \sqsubseteq D_\gamma$. Let $\alpha < \gamma$ be arbitrary, and we shall show that $\rho_{1\gamma}(\alpha) = \rho_{1\delta}(\alpha)$. Write $\beta = \sup(D_\delta \cap \gamma)$. Clearly, $\beta \in D_\gamma \cap D_\delta$ and $D_\gamma \cap \beta = D_\delta \cap \beta$.

▶ If $\alpha < \beta$, then there exists some $\beta' \in (D_\gamma \cap D_\delta \cap (\beta + 1))$ for which $\rho_{0\delta}(\alpha) = \max\{\mathrm{otp}(D_\delta \cap \alpha), \rho_{0\delta}(\beta')\}$ and $\rho_{0\gamma}(\alpha) = \max\{\mathrm{otp}(D_\gamma \cap \alpha), \rho_{0\gamma}(\beta')\}$. Since $\mathrm{otp}(D_\delta \cap \alpha) = \mathrm{otp}(D_\delta \cap \beta \cap \alpha) = \mathrm{otp}(D_\gamma \cap \beta \cap \alpha) = \mathrm{otp}(D_\gamma \cap \alpha)$, we infer that $\rho_{1\gamma}(\alpha) = \rho_{1\delta}(\alpha)$.

▶ If $\beta \leq \alpha < \gamma$, then $\min(C_\delta \setminus \alpha) = \gamma$. Consequently $\rho_{1\delta}(\alpha) = \max\{\mathrm{otp}(C_\delta \cap \alpha), \rho_{1\gamma}(\alpha)\}$. By definition, we have $\rho_{1\gamma}(\alpha) \geq \mathrm{otp}(C_\gamma \cap \alpha)$. As $C_\gamma \sqsupseteq C_\delta \cap \gamma$, we have $\mathrm{otp}(C_\gamma \cap \alpha) \geq \mathrm{otp}(C_\delta \cap \gamma \cap \alpha) = \mathrm{otp}(C_\delta \cap \alpha)$, and hence $\rho_{1\delta}(\alpha) = \rho_{1\gamma}(\alpha)$.

(3) Let $\mathbb{P}$ denote the collection of all partial specializing functions of size $< \lambda$. That is, $p \in \mathbb{P}$ iff it is a function with $\mathrm{dom}(p) \in [\mathcal{T}(\rho_1)]^{<\lambda}$, $\mathrm{Im}(p) \subseteq \lambda$, such that $p(y) \neq p(z)$ for all $y \subsetneq z$ in $\mathrm{dom}(p)$. Clearly, $\mathbb{P}$ is $(< \lambda)$-closed. It remains to verify that $\mathbb{P}$ has the λ^+-cc.

Towards a contradiction, suppose that $\mathbb{P}$ admits an antichain of size λ^+. Then by $\lambda^{<\lambda} = \lambda$ and a standard Δ-system argument, one could find some cardinal $\mu < \lambda$ and a family $\mathcal{F} \subseteq [\mathcal{T}(\rho_1)]^\mu$ consisting of λ^+ many pairwise disjoint sets with the property that for every two distinct $a, b \in \mathcal{F}$, there exist $x \in a$ and $y \in b$ such that x and y are comparable. For all $a \in \mathcal{F}$, let $\{a(i) \mid i < \mu\}$ be some enumeration of a. For every $\delta \in E^{\lambda^+}_\lambda$, pick $a_\delta \in \mathcal{F}$ such that $\min\{\mathrm{dom}(x) \mid x \in a\} > \delta$, and define $f_\delta : \mu \to {}^\delta\lambda$ by stipulating $f_\delta(i) = a(i) \restriction \delta$. Then, for all $\gamma, \delta \in E^{\lambda^+}_\lambda$ there exist $i, j < \mu$ such that $f_\gamma(i)$ and $f_\delta(j)$ are compatible. For all $\delta \in E^{\lambda^+}_\lambda$, let $D_\delta := \{\xi < \delta \mid \exists i < \mu[f_\delta(i)(\xi) \neq \rho_{1\delta}(\xi)]\}$. By Clause (3) of Fact 4.9, $|D_\delta| < \lambda$. By $\lambda^{<\lambda} = \lambda$, we may find a stationary set $S \subseteq E^{\lambda^+}_\lambda$ for which

$$\{(\langle f_\delta(i) \restriction D_\delta) \mid i < \mu\rangle, (\rho_{1\delta} \restriction D_\delta)) \mid \delta \in S\}$$

is a singleton. Consequently, $\{\rho_{1\delta} \mid \delta \in S\}$ forms a chain in $\mathcal{T}(\rho_1)$, contradicting the fact that $\mathcal{T}(\rho_1)$ is Aronszajn. □

If one is willing to give away the *respecting* feature of the preceding, then it is possible to relax $\boxed{\diamondsuit}^*_\lambda$ to just $\diamondsuit^*(\lambda^+)$:

Corollary 4.13. *Suppose that $\diamondsuit^*(\lambda^+)$ holds for a given infinite cardinal $\lambda = \lambda^{<\lambda}$. Then there exists a C-sequence for which the corresponding trees $\mathcal{T}(\rho_0)$ and $\mathcal{T}(\rho_1)$ satisfy Clauses (1)–(3) of Corollary 4.12.*

Notice that the same ideas of this section provides a proof to the following, which unlike Corollary 4.12, also apply to the case of λ singular:

Corollary 4.14. *If $\boxed{\diamondsuit}_\lambda$ holds for a given uncountable cardinal λ, then there exists a $\boxtimes_\lambda(E^{\lambda^+}_{\mathrm{cf}(\lambda)})$-sequence which is respected by the corresponding trees $\mathcal{T}(\rho_0)$ and $\mathcal{T}(\rho_1)$. Moreover:*

(1) $\mathcal{T}(\rho_0)$ *is a special* λ^+*-Aronszajn tree;*
(2) $\mathcal{T}(\rho_1)$ *is a nonspecial* λ^+*-Aronszajn tree.*

4.3. Kurepa.

Theorem 4.15. *Suppose that* λ *is an uncountable cardinal, and* $\langle (C_\alpha, N_\alpha, f_\alpha) \mid \alpha < \lambda^+\rangle$ *is a* $\boxtimes\!\!\!\!\diamondsuit^+_\lambda$*-sequence. Denote* $\Lambda = \{\alpha < \lambda^+ \mid \mathrm{otp}(C_\alpha) = \lambda\}$*.*

Then, there exist a $\boxtimes_\lambda(\Lambda)$*-sequence* $\vec{D} = \langle D_\alpha \mid \alpha < \lambda^+\rangle$ *and* $T \subseteq {}^{<\lambda^+}2$ *such that:*

- *for every stationary* $S \subseteq \Lambda$*, there are* $\delta \in S$ *and* $\gamma \in \mathrm{nacc}(D_\delta) \cap S$ *such that* $D_\delta \cap \gamma \sqsubseteq D_\gamma$*;*
- T *is a* $\boxtimes_\lambda(\Lambda)$*-respecting* λ^+*-Kurepa tree, as witnessed by* $\vec{D}$*.*

Proof. Let $\langle (C_\alpha, N_\alpha, f_\alpha) \mid \alpha < \lambda^+\rangle$ witness $\boxtimes\!\!\!\!\diamondsuit^+_\lambda$. Use $\langle (C_\alpha, f_\alpha) \mid \alpha < \lambda^+\rangle$ to construct the sequence $\vec{D} = \langle D_\alpha \mid \alpha < \lambda^+\rangle$ exactly as in the proof of Theorem 4.11. Then:

(1) $\mathrm{otp}(D_\alpha) = \mathrm{otp}(C_\alpha)$ and $\mathrm{acc}(D_\alpha) = \mathrm{acc}(C_\alpha)$ for all $\alpha < \lambda^+$;
(2) $\vec{D}$ is a $\boxtimes_\lambda(\S)$-sequence for every stationary $\S \subseteq \Lambda$;
(3) for every stationary $S \subseteq \Lambda$, there are $\delta \in S$ and $\gamma \in \mathrm{nacc}(D_\delta) \cap S$ such that $D_\delta \cap \gamma \sqsubseteq D_\gamma$.

Let $\S = \{\alpha < \lambda^+ \mid f_\alpha \text{ is surjective}\}$. Then, $\S$ is a stationary subset of Λ. Denote $\kappa = \lambda^+$, and

- $\mathcal{B} = \{b \in {}^\kappa 2 \mid \forall \alpha < \kappa[(b \restriction \alpha) \in N_\alpha]\}$,
- $T = \{b \restriction \alpha \mid b \in \mathcal{B}, \alpha < \kappa\}$.

Then $T \subseteq {}^{<\kappa}2$ is downward closed, and $|T_\alpha| \leq |N_\alpha| \leq \lambda < \kappa$ for all $\alpha < \kappa$. For all $\alpha < \kappa$, since N_α is rud-closed and $\alpha \in N_\alpha$, we know that the constant function from α to $\{0\}$ belongs to T, and so $\{\mathrm{dom}(f) \mid f \in T\} = \kappa$.

We have shown that T is a κ-tree. Let us show it is Kurepa.

Claim 4.15.1. *T has at least κ^+ many cofinal branches.*

Proof. For all $\alpha < \kappa$, since $\alpha \in N_\alpha$ and the latter is rud-closed, a subset of α is N_α iff its characteristic function is in N_α. Thus, it suffices to show that the following set has size $> \kappa$:

$$\mathcal{A} := \{A \subseteq \kappa \mid \forall \alpha < \kappa[(A \cap \alpha) \in N_\alpha]\}.$$

Suppose not. Then we can find an enumeration (possibly with repetitions) $\{A_\beta \mid \beta < \kappa\}$ of $\mathcal{A}$. Consider the club $Z = \Delta_{\beta<\kappa} \mathrm{acc}^+(A_\beta)$. That is,

$$Z = \{\zeta < \kappa \mid \forall \beta < \zeta(\sup(A_\beta \cap \zeta) = \zeta)\}.$$

Pick a club $D \subseteq \kappa$ such that for every $\alpha \in D$, we have $\{D \cap \alpha, Z \cap \alpha\} \subseteq \mathcal{P}(\alpha) \cap N_\alpha$. Then $E := Z \cap D$ is a club. Let $\{\zeta_\beta \mid \beta < \kappa\}$ be the increasing enumeration of E. For all $\beta < \kappa$, we have $\beta \leq \zeta_\beta < \zeta_{\beta+1}$ and $\zeta_{\beta+1} \in Z$, and hence $\sup(A_\beta \cap \zeta_{\beta+1}) = \zeta_{\beta+1} > \zeta_\beta = \sup(E \cap \zeta_{\beta+1})$. Consequently, $A_\beta \neq E$ for all $\beta < \kappa$, and there must exist some $\alpha < \kappa$ such that $E \cap \alpha \notin N_\alpha$. Fix such an α, and let $\alpha' := \sup(E \cap \alpha)$. As $(E \cap \alpha) \setminus \alpha'$ is a finite subset of α, it is an element of N_α. Since N_α is closed under unions, the set $E \cap \alpha'$ must be outside of N_α. In particular, it is nonempty, and $\alpha' \in \mathrm{acc}(E) \subseteq D$. But then, $\{D \cap \alpha', Z \cap \alpha'\} \subseteq N_{\alpha'} \subseteq N_\alpha$ and since N_α is closed under intersections, we get that $E \cap \alpha' = (D \cap \alpha') \cap (Z \cap \alpha')$ is in N_α. This is a contradiction. □

Next, we show that T is respecting $\vec{D}$, by defining $\langle \mathbf{b}^\alpha : T \restriction D_\alpha \to {}^\alpha 2 \mid \alpha < \kappa\rangle$, as follows. Given $\alpha < \kappa$, let $\psi_\alpha : D_\alpha \leftrightarrow C_\alpha$ denote the order-preserving bijection, and let $g_\alpha : D_\alpha \to {}^\alpha 2$ be such that for all $\beta \in D_\alpha$, $g_\alpha(\beta)$ is the characteristic function of $f_\alpha(\psi_\alpha(\beta))$.

Now, for all $x \in T \restriction D_\alpha$, let

$$\mathbf{b}^\alpha(x) = g_\alpha(\mathrm{dom}(x)).$$

Suppose that $\bar\alpha \in \operatorname{acc}(D_\alpha)$. Then $\bar\alpha \in \operatorname{acc}(C_\alpha)$ and $\psi_{\bar\alpha} \subseteq \psi_\alpha$, so that for all $\beta \in D_{\bar\alpha}$, $f_{\bar\alpha}(\psi_{\bar\alpha}(\beta)) = f_\alpha(\psi_\alpha(\beta)) \cap \bar\alpha$ and $g_{\bar\alpha}(\beta) = g_\alpha(\beta) \restriction \bar\alpha$. Consequently, $\mathbf{b}^{\bar\alpha}(x) = \mathbf{b}^\alpha(x) \restriction \bar\alpha$ for all $x \in T \restriction D_{\bar\alpha}$.

Finally, let $\alpha \in \S$, and we shall show that $T_\alpha \subseteq \operatorname{Im}(\mathbf{b}^\alpha)$. Let $y \in T_\alpha$ be arbitrary. By definition of T, we have $y \in N_\alpha$. Since N_α is rud-closed, the set $\{\gamma < \alpha \mid y(\gamma) = 1\}$ is in N_α, and so by $\alpha \in \S$, there exists some $\beta \in C_\alpha$ such that $f_\alpha(\beta) = \{\gamma < \alpha \mid y(\gamma) = 1\}$. Let $\beta' = \psi_\alpha^{-1}(\beta)$. Then $g_\alpha(\beta') = y$. Now, put $x := y \restriction \beta'$. Then $x \in T \restriction D_\alpha$, and by definition of $\mathbf{b}^\alpha$, we have $\mathbf{b}^\alpha(x) = y$, as sought. $\square$

Corollary 4.16. *Suppose that $\boxed{\diamondsuit}^\dagger_\lambda$ holds for a given uncountable cardinal λ. Then there exists a $\boxtimes_\lambda(E^{\lambda^+}_{\operatorname{cf}(\lambda)})$-respecting λ^+-Kurepa tree that has no λ^+-Aronszajn subtrees.*

Proof. The construction of all involved objects is identical to that of the proof of Theorem 4.15, but this time we consult with a $\boxed{\diamondsuit}^\dagger_\lambda$-sequence rather than $\boxed{\diamondsuit}^+_\lambda$. Consequently, the reflection argument of [5, Theorem 2] shows that the λ^+-Kurepa tree will contain no λ^+-Aronszajn subtrees. $\square$

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References

[1] U. Abraham, S. Shelah, and R. M. Solovay. Squares with diamonds and Souslin trees with special squares. *Fund. Math.*, 127(2):133–162, 1987.

[2] Ari M. Brodsky and Assaf Rinot. Reduced powers of Souslin trees. *Forum Math. Sigma*, 5(e2): 1–82, 2017.

[3] Ari M. Brodsky and Assaf Rinot. A microscopic approach to Souslin-tree constructions. Part I. *arXiv:1601.01821*, 2016. `http://www.assafrinot.com/paper/22`

[4] Ari M. Brodsky and Assaf Rinot. A microscopic approach to Souslin-tree constructions. Part II. In preparation, 2017, `http://www.assafrinot.com/paper/23`

[5] Keith J. Devlin. The combinatorial principle $\diamondsuit^\sharp$. *J. Symbolic Logic*, 47(4):888–899 (1983), 1982.

[6] Keith J. Devlin. *Constructibility*. Perspectives in Mathematical Logic. Springer-Verlag, Berlin, 1984.

[7] Keith J. Devlin and Saharon Shelah. Souslin properties and tree topologies. *Proc. London Math. Soc. (3)*, 39(2):237–252, 1979.

[8] Matthew Foreman and Peter Komjath. The club guessing ideal: commentary on a theorem of Gitik and Shelah. *J. Math. Log.*, 5(1):99–147, 2005.

[9] Charles William Gray. *Iterated forcing from the strategic point of view*. ProQuest LLC, Ann Arbor, MI, 1980. Thesis (Ph.D.)–University of California, Berkeley.

[10] Michael Hrušák and Carlos Martínez Ranero. Some remarks on non-special coherent Aronszajn trees. *Acta Univ. Carolin. Math. Phys.*, 46(2):33–40, 2005.

[11] Tetsuya Ishiu. Club guessing sequences and filters. *J. Symbolic Logic*, 70(4):1037–1071, 2005.

[12] Tetsuya Ishiu and Paul B. Larson. Some results about (+) proved by iterated forcing. *J. Symbolic Logic*, 77(2):515–531, 2012.

[13] R. Björn Jensen. The fine structure of the constructible hierarchy. *Ann. Math. Logic*, 4:229–308; erratum, ibid. 4 (1972), 443, 1972. With a section by Jack Silver.

[14] Ronald Jensen, Ernest Schimmerling, Ralf Schindler, and John Steel. Stacking mice. *J. Symbolic Logic*, 74(1):315–335, 2009.

[15] Assaf Rinot. Chain conditions of products, and weakly compact cardinals. *Bull. Symb. Log.*, 20(3):293–314, 2014.
[16] Assaf Rinot. Putting a diamond inside the square. *Bull. Lond. Math. Soc.*, 47(3):436–442, 2015.
[17] Assaf Rinot. Hedetniemi's conjecture for uncountable graphs. *J. Eur. Math. Soc.*, 19(1): 285–298, 2017.
[18] Ralf Schindler. *Set theory. Exploring independence and truth.* Universitext. Springer, Cham, 2014.
[19] Charles I. Steinhorn and James H. King. The uniformization property for $\aleph_2$. *Israel J. Math.*, 36(3-4):248–256, 1980.
[20] Stevo Todorčević. Trees, subtrees and order types. *Ann. Math. Logic.*, 20(3):233–268, 1981.
[21] Stevo Todorčević. Partitioning pairs of countable ordinals. *Acta Math.*, 159(3-4):261–294, 1987.
[22] Stevo Todorčević. Representing trees as relatively compact subsets of the first Baire class. *Bull. Cl. Sci. Math. Nat. Sci. Math.*, 30:29–45, 2005.
[23] Stevo Todorcevic. *Walks on ordinals and their characteristics*, volume 263 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 2007.

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