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A ZFC example of a forcing which is semi-proper but not proper

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The forcing which Shelah had used to show SPFA $\Rightarrow$ MM gives an example of a forcing which is semi-proper but not proper.

We write $X \sqsubset Y$ for $X \subset Y \wedge Y \cap \omega_1 = X \cap \omega_1$.

Let us fix $\omega_1 \leq \kappa < \lambda < \theta$, $\overline{\overline{H_\kappa}} < \lambda$.

Let $T \subset [H_\kappa]^\omega$ be stationary, and let $S = \{ X \subset H_\lambda : \overline{\overline{X}} = \aleph_0 \wedge X \cap H_\kappa \in T \}$. Let us also write

$$S^\perp = \{ X \subset H_\lambda : \overline{\overline{X}} = \aleph_0 \wedge \text{ there is no } Y, \; X \sqsubset Y \subset H_\lambda, \; Y \in S \}.$$

Let $f : \omega \times [H_\theta]^{<\omega} \longrightarrow H_\theta$ be a Skolem function for H_θ by which we mean that

$$f'' \omega \times [x]^{<\omega} \prec (H_\theta ; \in)$$

for <u>every</u> $x \subset H_\theta$. For $s \in [H_\theta]^{<\omega}$, let us write $f^s : \omega \times [H_\kappa]^{<\omega} \longrightarrow H_\kappa$ for the (partial) function which is defined as follows.

$$f^s(n,t) = \begin{cases} f(n, t \cup s) & \text{if } f(n, t\cup s) \in H_\kappa \\ \uparrow & \text{otherwise} \end{cases}$$

<u>Claim 1.</u> Let $X \prec (H_\theta; \in, (f^s : s \in [H_\theta]^{<\omega}), \kappa, \lambda)$, $\overline{\overline{X}} = \aleph_0$. Then there is some $Y \sqsupset X$, $\overline{\overline{Y}} = \aleph_0$, $Y \cap H_\lambda \in S \cup S^\perp$, and $Y \prec (H_\theta; \in)$.

Proof. We may assume that $X \cap H_\lambda \notin S^\perp$, so that there is some $\overline{Y}$, $\overline{Y} \sqsupset X \cap H_\lambda$, $\overline{Y} \in S$.

Let
$$Y = f'' \omega \times [(\overline{Y} \cap H_\kappa) \cup X]^{<\omega}.$$

We claim that $Y \cap H_\kappa = \overline{Y} \cap H_\kappa$: Let $x = f(n, t \cup s)$, where $x \in H_\kappa$, $n < \omega$, $t \in [\overline{Y} \cap H_\kappa]^{<\omega}$, $s \in [X]^{<\omega}$. We then have $f^s \in X \cap H_\lambda \subset \overline{Y}$, so that $x = f(n, t\cup s) = f^s(n,t) \in \overline{Y}$.

But now $Y \sqsupset X$, $\overline{\overline{Y}} = \aleph_0$, $Y \cap H_\lambda \in S$ (as $(Y \cap H_\lambda) \cap H_\kappa = \overline{Y} \cap H_\kappa \in T$), and $Y \prec (H_\theta; \in)$. $\dashv$

Now let $\mathbb{P}$ be the forcing which shoots a club thru $S \cup S^\perp$, i.e.,

$p \in \mathbb{P}$ iff $p = (X_i : i \leq \alpha)$, where $\alpha < \omega_1$ and $(X_i : i \leq \alpha)$ is a continuous increasing chain of elements of $S \cup S^\perp$. We order $\mathbb{P}$ by end-extension.

<u>Claim 2.</u> $\mathbb{P}$ is semi-proper.

Proof. Let $p \in \mathbb{P}$, and let $X \prec (H_\theta; \in, (f^s : s \in [H_\theta]^{<\omega}), \triangleleft, \lambda)$, $\bar{\bar{X}} = \aleph_0$, with $p \in X$. Let Y be as in Claim 1. Set $\alpha = X \cap \omega_1 = Y \cap \omega_1$. We aim to construct a sequence $(X_i : i \leq \alpha)$, cont. + increasing, of elements of $S \cup S^\perp$ such that $X_\alpha = Y \cap H_\lambda$ and for some $\alpha_n \nearrow \alpha$, $n < \omega$, $p = (X_i : i \leq \alpha_0)$, every $p_n = (X_i : i \leq \alpha_n)$ is in Y, and for every name $\tau \in Y$ for a countable ordinal, there is some $n < \omega$ s.t. $p_n \Vdash \tau \in \check{\alpha}$.

There is no problem with this construction once we proved the following.

(*) Let $x \subset H_\lambda$ be countable. Then is then some $X \in S \cup S^\perp$ with $x \subset X$.

(*) however trivially follows from Claim 1. $\dashv$

We don't need Claim 1 to verify ~~that~~ (*), though. It is easy to see that with T, S is also stationary.

We now follow [Cox] to argue that $\mathbb{P}$ need not be proper.

Let us assume that κ is regular, and let $A, B \subset \kappa$ be such that $cf(\xi) = \omega$ for all $\xi \in A \cup B$, $A \cap B = \emptyset$, and both A and B are stationary.

Let $T = \{ X \in [H_\kappa]^\omega : \sup(X \cap \kappa) \in B \}$.

It is easy to see that T is stationary.

We claim that if S, $S^\perp$ are defined on p. 1, then $S \cup S^\perp$ is costationary.

Let $(H_\lambda ; \ldots)$ be any model of countable type.

Let $Z_0 \prec (H_\lambda ; \ldots)$ be of size $< \kappa$ s.t. $\kappa \in Z_0$, $Z_0 \cap \kappa \in \kappa$, and in fact $Z_0 \cap \kappa \in A$. Let $Z_1 \prec (H_\lambda ; \ldots)$ be of size $< \kappa$ s.t. $Z_1 \cap \kappa \in B$ and $Z_0 \in Z_1$. Let $Y \prec Z_1$ be such that Y is countable, $Z_0 \in Y$, and $\sup(Y \cap \kappa) = Z_1 \cap \kappa$. Let $X = Y \cap Z_0$. Then

$Y \sqsupset X$ and $Y \in S$, as $\sup(Y \cap \kappa) \in \mathcal{B}$, i.e., $Y \cap H_\kappa \in T$.

As $Z_0 \in Y$, $Z_0 \cap \kappa \in Y$, and hence $\sup(Y \cap (Z_0 \cap \kappa)) = Z_0 \cap \kappa$, i.e. $\sup(X \cap \kappa) \in A$, so that $X \notin S$.

We thus have that $X \notin S$, but there is $Y \sqsupset X$ with $Y \in S$, i.e., $X \notin S \cup S^\perp$.

Notice that if φ is a formula, and if $\vec{x} \in X$, then $(H_\lambda; \ldots) \models \cancel{\varphi}\ \exists v\, \varphi(v, \vec{x})$ implies $(H_\lambda; \ldots) \models \exists v \in Z_0\ \varphi(v, \vec{x})$ by $Z_0 \prec (H_\lambda; \ldots)$, which in turn gives $Y \models \exists v \in Z_0\ \varphi(v, \vec{x})$ by $Z_0 \in Y$ and $Y \prec Z_1 \prec (H_\lambda; \ldots)$, so that there is some $y \in Z_0 \cap Y = X$ with $(H_\lambda; \ldots) \models \varphi(y, \vec{x})$.
Hence $X \prec (H_\lambda; \ldots)$.

We have shown that $S \cup S^\perp$ is costationary.
This implies, for T and S as being chosen on p. 4 :

<u>Claim 3.</u> $\mathbb{P}$ is semiproper but not proper.

Proof: Let $X \prec H_\theta$ be such that $X \cap H_\lambda \notin S \cup S^\perp$. Suppose that $p \in X$ and $q \leq p$ is X-generic. Let α be the sup of all $\bar{\alpha}$ s.t. $r > q$ for some $r \in X$ and $r = (X_i : i \leq \bar{\alpha})$, some X_i, $i \leq \bar{\alpha}$. It is easy to see that $\alpha = X \cap \omega_1$ and $X_\alpha \sqsupseteq X$, $X_\alpha \in S \cup S^\perp$ (as q is a condition). However, as q is X-generic, we must have $X_\alpha = X \cap H_\lambda$ (otherwise $X[(X_i : i < \alpha)] \cap H_\theta$ could not be equal to X); but $X \cap H_\lambda \notin S \cup S^\perp$. Contradiction!

<u>Questions.</u>

(1) Without any further hypothesis, does BSPFA prove a statement which does not follow from BPFA?

(2) Is BPFA + $\neg$BSPFA stronger than BPFA (in consistency strength)?