Notes on V as a derived model of HOD

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We sketch a proof of a thm. of Woodin:

<u>Theorem</u> (Woodin) Assume $AD_{\mathbb{R}}$ and either $cf(\Theta) = \omega$, or Θ is regular. Then there is a G V-generic ($Col(\omega, <\Theta)$) such that if $\mathbb{R}^*_G$, Hom^*_G are the associated derived model of HOD, then there is an elementary

$$i : L(\mathbb{R}, P(\mathbb{R}))^V \longrightarrow L(\mathbb{R}^*_G, Hom^*_G).$$

<u>Proof</u> ~~Fix G~~. We may as well assume "$V = L(P(\mathbb{R}))$" holds in V.

Fix G s.t. $x \in \mathbb{R}^V \Longrightarrow x \in HOD[G \cap Col(\omega, \lambda)]$, some $\lambda < \Theta$.

~~[illegible]~~ (Use Vopenka to get $\mathbb{G}$.) ①

Let $x \in V[g]$, g on $\mathrm{Coll}(\omega, \lambda)$ V-generic, $\lambda < \Theta$

g coded in $\mathbb{R}^*_{\alpha}$

For $A \subseteq \mathbb{R}$ in V:

$$A \in \mathcal{U}_x \iff \exists T \; (T \text{ a tree on some } \omega \times \gamma \text{, for } \gamma < \Theta, \text{ and}$$

$$\bullet \; p[T] \subseteq A$$

$$\bullet \; x \in p[T] \;)$$

<u>Claim 1.</u>

<u>$\mathcal{U}_x$ is an ultrafilter:</u>

Let $x = \tau_g$, τ a term. So τ is essentially a subset of λ. We have

$$x \in \mathrm{HOD}_{\tau, \bar{\mu}}[g],$$

~~Force over HOD_τ~~ where $\bar{\mu}$ is a homog. system for A. (I.e. $S_{\bar{\mu}} = A$.)

In $\mathrm{HOD}_{\tau, \bar{\mu}}$ we have (by the OD tower-replicating fcn, cf. derived model paper proof of Scale(Σ^2_1)) for each $\gamma < \Theta$ a homog. sys. for A with γ-complete measures. So we have

$$(\langle T_\gamma, T^*_\gamma \rangle \mid \gamma < \Theta) \text{ in } \mathrm{HOD}_{\tau, \bar{\mu}} \text{ where}$$

$\mathrm{HOD}_{\tau, \bar{\mu}} \models$ ~~these~~ T_γ, T^*_γ are γ-abs. compl. and $p[T_\gamma] = A$, $p[T^*_\gamma] = -A$ ← (every real is generic over HOD)

So

$$x \in p[T_\gamma], \text{ for all suff. lg. } \gamma$$

or

$$x \in p[T^*_\gamma], \text{ for all suff. lg } \gamma$$

(one γ is enough). $\boxtimes$

Rmk See pages 2a – 2d.

Rmk What happens if we force over V with the modified Gandy – Harrington conditions are trees T on some $\omega \times \lambda$, and order ...?

Claim 2
Łoś' thm for ~~ult~~ $V^{\mathbb{R}}/\mathcal{U}_x$: ← fcns in V.

I can only see how to do this if $cf(\Theta) = \omega$, or Θ is regular.

Say ~~ult~~ $A \in \mathcal{U}_x$, and

$$\forall z \in A \quad V \models \exists v\, \varphi[v, f(z)]$$

where $f \in V$, $f: \mathbb{R} \to V$.

2a

<u>Remark</u> Let $A \subseteq \mathbb{R}$ be in V. We have a homog. system $\bar{\mu}$ s.t. $A = S_{\bar{\mu}}$ in V. ~~In HOD, there is a~~ Let $\gamma < \Theta$ be s.t. all measures μ_s are definable in V from ords $< \gamma$. We have od definable, continuous tower-replicating and tower-flipping functions F^i_ξ, $\xi < \Theta$, ~~which~~ associated s.t. $\forall f \in \gamma^\omega$

(1) f codes a homog. sys. $\bar{\eta}$ $\Rightarrow$

$\bigcup_n F^0_\xi(f \restriction n)$ codes a homog sys.

$\bar{\nu}$ s.t. the ν_s are ξ-complete, and $S_{\bar{\nu}} = S_{\bar{\eta}}$

(2) Same as (1), but with $F^1_\xi(f)$ coding $\bar{\nu}$ s.t. $S_{\bar{\nu}} = -S_{\bar{\mu}}$.

2b

Note HOD can reconstruct the embs of a homog. sys. (restricted to itself) from the ordinals coding its measures.

Thus picking Θ_α s.t. $\gamma < \Theta_\alpha < \Theta$ and using Vopenka to get $f \in \gamma^\omega$ s.t. ~~$\mu_f \wedge A$~~ f codes $\bar{\mu}$ generic at Θ_α over HOD:

$\mathrm{HOD}[f] \models$ "$S_{\bar{\mu}_f}$ is $<\Theta$ homog.".

So we get $\langle (T_\xi, T^*_\xi) \mid \xi < \Theta \rangle$ in $\mathrm{HOD}[f]$ s.t.

$\mathrm{HOD}[f] \models (T_\xi, T^*_\xi)$ are ξ-absolute complements, and $p[T_\xi] = S_{\bar{\mu}}$

moreover, $p[T_\xi] = A$ holds in V. (By the construction.) In this circumstance, we call $\langle f, \langle (T_\xi, T^*_\xi) \mid \xi < \Theta \rangle \rangle$ a <u>code of A over HOD</u>.

Note that the codes of A over HOD are in V.

<u>Lemma</u> Let $\langle f, \langle (T_\xi, T_\xi)^* \mid \xi < \Theta \rangle\rangle$ and $\langle g, \langle (S_\xi, S_\xi^*) \mid \xi < \Theta \rangle\rangle$ be codes of A over HOD. Then

$$\bigcup_{\xi<\Theta} p[T_\xi] \cap \mathbb{R}^*_G = \bigcup_{\xi<\Theta} p[S_\xi] \cap \mathbb{R}^*_G$$

<u>Proof</u> $\mathrm{HOD}[f,g]$ is a small extension of both $\mathrm{HOD}[f]$ and $\mathrm{HOD}[g]$. Moreover, $\mathrm{HOD}[f,g] \subseteq V$, where $S_{\bar{\mu}_g} = S_{\bar{\mu}_g} = A$ holds.

$\boxtimes$

So for $A \subseteq \mathbb{R}$ in V, and $\langle f, \langle T_\xi, T_\xi^* \mid \xi < \Theta \rangle\rangle$ a code of A over HOD, put

$$A^* = \bigcup_{\xi<\Theta} p[T_\xi] \cap \mathbb{R}^*_G .$$

This is the <u>canonical blowup</u> of A to ~~HOD~~ $\mathbb{R}^*_G$.
It is indep. of the code.

The proof of claim 1 showed

<u>Lemma</u> For $x \in \mathbb{R}^+_G$, and $A \subseteq \mathbb{R}$ in V

$$x \in A^+ \text{ iff } A \in \mathcal{U}_x$$

<u>Pf</u> $\Rightarrow$ is clear. But if $x \notin A^*$, then $x \in (-A)^*$, so $-A \in \mathcal{U}_x$.

But $\mathcal{U}_x$ is a filter. (~~By~~ Since $p[T_0] \cap p[T_1] = \phi$ in $V \Rightarrow$ same true in $V[G]$.) Thus $A \notin \mathcal{U}_x$.

$\boxtimes$

In V, every v is OD from some set of reals B. Run

$\underline{\Theta \text{ reg}}$: Get $\eta < \Theta$ s.t. $\forall z \in A$

$$V \models \exists B \in P_\eta(\mathbb{R})\ \exists v \in OD(B)\ \varphi[v, f(z)].$$

Fix $|B_0|_w = \eta$

$$V \models \forall z \in A\ \exists \sigma\ \exists v \in OD(\sigma^{-1}{}''B_0)\ \varphi[v, f(z)]$$

Now uniformize to get $g(z) =$ some such σ, etc.

$\underline{cf(\Theta) = \omega}$ Let ~~t~~ η_n's be cof in Θ.

$$Z_n = \{z \mid \exists B \in P_{\eta_n}(\mathbb{R})\ \exists v \in OD(B)\ \varphi[v, f(z)]\}$$

$\langle Z_n \mid n \in \omega \rangle$ is Suslin. This gives a seq. ~~ttt~~ $\bar{\mu}_n$ of homog. ~~[illegible]~~ systems, with $S_{\bar{\mu}_n} = Z_n$. Work over

$$HOD_{\langle \bar{\mu}_n \mid n < \omega \rangle, T, \bar{\nu}} \quad \text{where } S_{\bar{\nu}} = A$$

and do the argument of part 1.

Now $x \in \left(S_{\bar\nu}\right)^{HOD_{\langle \bar\mu_n | n<\omega\rangle, \Upsilon, \bar\nu}[g]}$

so for some n, $x \in \left(S_{\bar\mu_n}\right)^{HOD_{\langle \bar\mu_n | n<\omega\rangle, \Upsilon \ldots}}$

because the relationship $S_{\bar\nu} = \bigcup_n S_{\bar\mu_n}$ holds in V, and is absolute.

This gives an n s.t.

$Z_n \in \mathcal{U}_x$. Now use uniformization, as in Θ reg. case.

$\boxtimes$

Claim 3.

$V^{\mathbb{R}}/\mathcal{U}_x$ is wellfounded

If not, $L(P(\mathbb{R})) \models$ "$p \Vdash Ult(V^{\mathbb{R}}, \mathcal{U}_{x_g})$ is illfdd" for some $p \in Col(\omega, \gamma)$

By AD^+ reflection

$$L_\alpha(P_\beta(\mathbb{R})) \models p \Vdash Ult(V, \mathcal{U}_{x_g}) \text{ is illfdd.}$$

In V, we have full DC_ω for relations on $L_\alpha(P_\beta(\mathbb{R}))$. Using this we can find ~~[illegible]~~ in V (by a Gandy-style fusion argument) sequences $\langle p_n \rangle$, $\langle T^n \rangle$, $\langle f_n \rangle$, $\langle (s^n_k, u_k) \rangle$ (for $k > n$) s.t.:

(1) $p_n \in \mathrm{Col}(\omega, \gamma)$, $p_{n+1} \leq p_n$

(2) $p_n \Vdash p[T^n] \in \mathcal{U}_{\dot{g}}$

(3) $f_n : p[T^n] \to \mathrm{ORD}$

(4) for $k > n$:

(i) ~~[illegible]~~ $(u_k, s^n_k) \in T^n$

(ii) $j < k \to u_j \subsetneq u_k$ and $s^n_j \subsetneq s^n_k$

(iii) $p[T^k] \subseteq p[T^n_{(u_k, s^n_k)}]$

(5) $f_{n+1}(z) < f_n(z)$ for all $z \in p[T^{n+1}]$

⑥

But now let

$$z = \bigcup_k u_k$$

$$f^n = \bigcup_k s^n_k$$

then $(z, f^n) \in [T^n]$ for all n. Thus $f_{n+1}(z) < f_n(z)$ for all n, contra.

$\boxtimes$

Note there is a natural embedding

$$\pi_{\vec{x}, \vec{y}} : \mathrm{Ult}(V, \mathcal{U}_{\vec{x}}) \to \mathrm{Ult}(V, \mathcal{U}_{\vec{y}})$$

whenever $\vec{x}$ is (a real coding) a subsequence of a (real coding) $\vec{y}$. Set

$$V^\infty = \operatorname{dir\,lim}_{\vec{x} \in \mathbb{R}^\#_\alpha} \mathrm{Ult}(V, \mathcal{U}_{\vec{x}}).$$

<u>Claim 4</u> V^∞ is wellfounded.

<u>Proof</u> like that for claim 3.

$\boxtimes$

Claim 5 $\mathbb{R} \cap V^\infty = \mathbb{R}^*_G$

Proof If $x \in \mathbb{R}^*_G$, then $x \in \mathrm{Ult}(V, \mathcal{U}_x)$, so $x \in V^\infty$ as $i_{x,\infty}(x) = x$.

If $x \in V^\infty$, ~~say $x =$~~ then we have $y \in \mathbb{R}^*_g$ s.t. $x \in \mathrm{Ult}(V, \mathcal{U}_y)$. But then $x \in V[g]_\theta$. So we have $\gamma < \theta$ s.t.

$$x \in V[g]$$

where $g = G \cap \mathrm{Col}(\omega, <\gamma)$. Say

$$x = \tau^g$$

where $\tau \in V$, $\tau \subseteq \mathrm{Col}(\omega, <\gamma) \times \omega$.

Note that if $\gamma < \eta < \theta$, then τ is generic over $\mathrm{HOD}[G \cap \mathrm{Col}(\omega, <\eta)]$

for the ~~$\mathrm{HOD}[G \upharpoonright \eta]$~~ ~~extender algebra~~ ~~Vopenka algebra~~

extender algebra of HOD at the least $\Theta_\alpha > \eta$.

<u>Rmk</u> Could use the Vopenka algebra, and T generic over HOD, and then switch ~~order~~ $G \cap Col(\omega, <\eta)$ generic over $HOD[T]$.

~~Since this~~ But then T can be absorbed into a collapse of Θ_α over $HOD[G \cap Col(\omega, <\eta)]$. Since G is generic over V, this does happen at some such η, with the further collapse being G itself.

I.e.

$$T \in V[G \cap Col(\omega, <\Theta_\alpha)]$$

some α. That gives

$$x = T_g \in V[G \cap Col(\omega, <\Theta_\alpha)]$$

as desired.

$\boxtimes$

Claim 6 Let $A \subseteq \mathbb{R}$, $A \in V$. Then

$$i_{0,\infty}(A) = A^*.$$

Proof Let $x \in \mathbb{R}^*_G$. Then

$$x \in A^* \text{ iff } A \in \mathcal{U}_x \quad (\text{earlier lemma})$$

$$\text{iff } x \in i_{0,x}(A) \quad \left(\begin{array}{l}\text{as} \\ x = [\lambda z . z]_{\mathcal{U}_x}\end{array}\right)$$

$$\text{iff } x \in i_{0,\infty}(A).$$

$\boxtimes$

Claim 7 $P(\mathbb{R}) \cap V^\infty \subseteq \mathrm{Hom}^*_G$

Prf ~~[illegible]~~ $\subseteq$! Since Θ is regular ~~of~~ of cof ω, $\sup i_{0,\infty}{}''\,\Theta = \Theta^{V^\infty}$. So it is enough to see $i_{0,\infty}(A) \in \mathrm{Hom}^*_G$ whenever $A \subseteq \mathbb{R}$ and $A \in V$. But $i_{0,\infty}(A) = A^*$, and clearly $A^* \in \mathrm{Hom}^*_G$.

$\supseteq$ Strangely, we seem to need an indirect argument.

Let $\kappa = (\Theta^+)^V$

= least γ not surj. image of Θ in V

= least γ not surj. image of $P(\mathbb{R})$ in V.

We can assume $G \in V[H]$, where H is V-gen for $Coll(\omega, P(\mathbb{R}))$.

So

$V[H] \models ZFC + P(\mathbb{R})^V$ is countable $+ \kappa = \omega_1 +$ GCH (in fact, V is $L[z]$, for some real z).

In $V[H]$ $i_{0\infty}(\alpha)$ is ctbl. for all $\alpha < \kappa$, and $i_{0\infty}$ is cont. at κ, so

$i_{0\infty}(\kappa) = \kappa$.

Thus

$$V^\infty \models \kappa = \Theta^+.$$

But ~~$V^\infty \subseteq \mathrm{Hom}^*_G \ldots$~~ if $P(\mathbb{R})^{V^\infty}$ is a proper Wadge initial segment of Hom^*_G, then $(\Theta^+)^{V^\infty}$ is no longer a cardinal in $L(\mathbb{R}^*_G, \mathrm{Hom}^*_G)$, hence κ is not a cardinal in $V[H]$. Contradiction.

$\boxtimes$

This completes the proof of the theorem. For no important reason, we now relate V^∞ to the generic ultrapower direct limit we would get via Kechris-Woodin's generic codes construction.

~~We don't know whether $\mathrm{Hom}_G^{*} \subseteq V^{\omega}$.~~

~~But there is a natural larger generic ultrapower of V given by the Kechris-Woodin generic codes construction:~~

For $\lambda < \Theta$ and g: $\mathrm{Col}(\omega, \lambda)$-generic over V, one gets an ultrafilter on $({}^{\omega}\lambda)^V$ by setting for A in V, $A \subseteq ({}^{\omega}\lambda)^V$:

$$A \in U_g \text{ iff } \exists n \,(A \cap N_{g \restriction n} \text{ is comeager in } N_{g \restriction n}).$$

Here $N_s = \{h \in {}^{\omega}\lambda \mid s \subseteq h\}$, ~~and comeager~~

<u>Claim 8</u> Los' theorem holds for $\mathrm{Ult}(V, U_g)$.

<u>Proof</u> This is essentially the same as the proof of claim 2: we use $\mathrm{cf}(\Theta) = \omega$ or Θ regular.

We also use that any $h : {}^\omega\lambda \to {}^\omega\omega$ can be uniformized on a comeager set. (The whole argument here is just Kechris-Woodin, though they only address directly $V = L(\mathbb{R})$.) 11

☒

<u>Claim 9</u> (KW) Let $\mathcal{U}_\lambda$ be the supercompactness measure on $P_{\omega_1}(\lambda)$; then true in V:

for every ~~$F : {}^\omega\lambda \to ORD$ in V, there is an $F^* : P_{\omega_1}(\lambda) \to ORD$ and a comeager~~

nonmeager $A \subseteq {}^\omega\lambda$ and $F : A \to ORD$, there is a $B \subseteq A$ nonmeager s.t. $\forall f, g \in B$

$$\operatorname{ran}(f) = \operatorname{ran}(g) \Rightarrow F(f) = F(g)$$

<u>Pf</u> Kechris-Woodin: for the typical $\sigma \in P_{\omega_1}(\lambda)$, find $s : n \to \sigma$ s.t. F is constant on ~~[illegible]~~ a comeager subset

of $N^\sigma_s = \{ f \in {}^\omega\sigma \mid s \subseteq f \}$.
(Comeager in ${}^\omega\sigma$.) Now fix $s = s_\sigma$ for a.e. σ.

$\boxtimes$

By claim 9, $\mathrm{Ult}(\overset{HWO}{\forall}, \mathcal{U}_g)$ and $\mathrm{Ult}(HWO, \mathcal{U}_\lambda)$ are isomorphic, where HWO = class of hereditarily wo'd sets, and $\mathcal{U}_\lambda$ = supercptness measure on $P_{\omega_1}(\lambda)$. So

<u>Claim 10</u> $\mathrm{Ult}(V, \mathcal{U}_g)$ is wellfdd.

Similarly, given $\lambda_1 \dots \lambda_n < \Theta$, and $g_1 \dots g_n$ generic for $\mathrm{Col}(\omega, \lambda_1) \times \dots \times \mathrm{Col}(\omega, \lambda_n)$, we can define $\mathcal{U}_{\langle g_1 \dots g_n \rangle}$ on $({}^\omega\lambda_1 \times \dots \times {}^\omega\lambda_n)^V$ using comeager sets in the product top.

Again, Łoś' thm holds for $\mathrm{Ult}(V, \mathcal{U}_{\langle g_1 \ldots g_n \rangle})$, which is w.f.,

~~Note that~~ and so far as ordinals go, isom to $\mathrm{Ult}(V, \cancel{\ldots} \mathcal{U}_{\langle \lambda_1 \times \ldots \times \lambda_n \rangle})$

(where $\mathcal{U}_{\langle \lambda_1 \ldots \lambda_n \rangle}$ = supercptness measure on $P_{\omega_1}(\lambda_1 \times \ldots \times \lambda_n)$).

Also, if $\vec{h}$ is a subseq. of $\vec{g}$, there is a natural elem. emb.

$$\pi_{\vec{h},\vec{g}} : \mathrm{Ult}(V, \mathcal{U}_{\vec{h}}) \to \mathrm{Ult}(V, \mathcal{U}_{\vec{g}}).$$

(Have to record which subseq of $\vec{g}$ is $\vec{h}$, if more than one.) Set

$$\tilde{V}^{\infty} = \operatorname*{dir\,lim}_{\vec{g} \text{ coded in } \mathbb{R}^*_a} \mathrm{Ult}(V, \mathcal{U}_{\vec{g}})$$

<u>Claim 11</u> $\tilde{V}^{\infty}$ is wellfounded.

<u>Proof</u> Similar to that for V^{∞}. Omitted for now. $\boxtimes$

<u>Claim</u> 12 $\mathbb{R} \cap \check{V}^{\infty} = \mathbb{R}^*_G$.

<u>Prf</u> If $x \in \mathbb{R}^*_g$, then $x = \tau^g$ for some g V-gen / $\mathrm{Coll}(\omega, \lambda)$, with $\lambda < \Theta$. Kechris–Woodin show that $x \in \mathrm{Ult}(V, \mathcal{U}_g)$.

($x = [F]_{\mathcal{U}_g}$, where F is a cont. fcn on ω^{λ} ~~which looks for where~~ with $F(f)(n) = k$ iff for some $p \subseteq f$, $p \Vdash \tau(n) = k$.)

If $x \in \check{V}^{\infty}$, then for some g coded in $\mathbb{R}^*_G$, $x \in \mathrm{Ult}(V, \mathcal{U}_g)$. But then $x \in V[g]$, so $x \in \mathbb{R}^*_G$ as in Claim 5. $\boxtimes$

~~Now for $x \in \mathbb{R}^*_G$, say $x = \tau^g$ where g on $\mathrm{Coll}(\omega, \lambda)$.~~

~~then~~

We can now embed V^∞ into $\tilde{V}^\infty$: we have

$$\begin{tikzcd} V \arrow[r, "i"] \arrow[dr, "j"'] & \tilde{V}^\infty \\ & V^\infty \arrow[u, "k"'] \end{tikzcd}$$

where k is defined by:

$$k(x) = x \quad \text{for} \quad x \in \mathbb{R}^*_G$$

and

$$k(j(t)) = i(t) \quad \text{for} \quad t \in V.$$

(These two clauses are mutually consistent.)

Since V^∞ is the Skolem closure of $\mathbb{R}^*_G$ union $\operatorname{ran}(j)$, k is well-defined and elementary, provided we show

<u>Claim 13</u> For $\bar{t} \in V^{<\omega}$, $\bar{x} \in \mathbb{R}^{*\,<\omega}_{G}$, φ wff

$$\tilde{V}^\infty \models \varphi[i(\bar{t}), \bar{x}] \text{ iff } V^\infty \models \varphi[j(\bar{t}), \bar{x}]$$

<u>Proof</u>

Fix $t \in V$ and $x \in \mathbb{R}^*_G$. In V, let

$$A = \{z \in \mathbb{R} \mid V \models \varphi[t,z]\},$$

~~Let $x = \tau_g$~~ Let $x = \tau_g$, g on $\mathrm{Col}(\omega,\lambda)$ coded in $\mathbb{R}^*_G$, $\lambda < \theta$. Let $\langle f, \langle\langle T_\xi, T^*_\xi\rangle \mid \xi < \theta\rangle\rangle$ be a code for A over HOD. We have

$V^\infty \models \varphi[i(t), x]$ iff

$\cancel{\omega_1(V, U_g) \models \varphi[i_g(t), x]}$

iff $x \in i(A)$

iff $x \in p[i(T_\xi)]$, for all ξ

(*) iff $x \in p[T_\xi]$, for all suff. lg. ξ (all $\xi > \lambda$)

iff $x \in A^* = j(A)$

iff $\tilde{V}^\infty \models \varphi[j(t), x]$.

187

To see the equivalence in (*), note that if $x \notin p[T_\xi]$, where $\xi > \lambda$, then $x \in p[T^*_\xi]$, so $x \in p[i(T_\xi^*)] \cap p[i(T_\xi)]$, contrary to absoluteness of wffness. ☒

<u>Claim 14</u> $\tilde{V}^\infty = V^\infty$ and $k = \mathrm{id}$.
(Note $\mathrm{cf}(\theta) = \omega$ or θ regular is part of our hypothesis in getting k.)

<u>Prf</u> If $k \neq \mathrm{id}$, then $\mathrm{crit}\, k = \theta^{V^\infty}$. But both i and j are continuous at θ^V, as either $\mathrm{cf}(\theta) = \omega$ or θ reg. ~~Since $k(j(\theta)) = i(\theta)$~~. As $i = k \circ j$, this gives $k(\theta^{V^\infty}) = \theta^{V^\infty}$. ☒

From $\check{V}^\infty = V^\infty$ we get a second proof that V^∞ is wellfounded. We also get that $i_{0\infty} \restriction ORD$ is determined by supercompactness ultrapowers on $P_{\omega_1}(\lambda)$, $\lambda < \Theta$, so $i_{0\infty} \restriction \Theta$ is in V. We also get $i_{0\infty} \restriction \lambda \in V^\infty$, for all $\lambda < \Theta$. I don't see how to prove these things without using $V^\infty = \check{V}^\infty$.

References

Kechris-Woodin, Generic codes for unctbl. ordinals, re-issued Cabal Seminar volume I.