Burstin bases and well-ordering the reals

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Reflections on Set Theoretic Reflection
Sant Bernat, Montseny, Nov 19, 2018

"Paradoxical" sets of reals

Definition

Let $A \subseteq \mathbb{R}$ uncountable. We say that A is

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Let $A \subseteq \mathbb{R} \times \mathbb{R}$. We say that A is

- a Mazurkiewicz set iff $|A \cap \ell| = 2$ for every straight line $\ell \subset \mathbb{R} \times \mathbb{R}$.

Folklore and classical results

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All these classical constructions may be obtained by assuming ZF plus the existence of a well-ordering of $\mathbb{R}$ (or, ZF plus there is a well-ordering of $\mathbb{R}$ of order type ω_1 in the case of Luzin and Sierpiński sets).

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Recall the Cohen-Halpern-Lévy model: Let g be $\mathbb{C}(\omega)$-generic over L ($\mathbb{C}(\omega)$ being the finite support product of ω Cohen forcings), and let $A = \{c_n : n < \omega\}$ be the set of Cohen reals added by g.

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Theorem (D. Pinkus and K. Prikry, S. Feferman, 1975)

In the Cohen-Halpern-Lévy model H, in which A is an infinite set of reals with no (infinite) countable subset (i.e., $AC_\omega(\mathbb{R})$ fails), there is a Luzin set as well as a Vitali set.

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Question (D. Pincus and K. Prikry, 1975)

"We would be interested in knowing whether a Hamel basis for $\mathbb{R}$ over $\mathbb{Q}$ (the rationals) exists in H or in any other model in which $\mathbb{R}$ cannot be well ordered."

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Question (variant 1 of Pinkus-Prikry)

Is the existence of a Hamel basis (or, the simultaneous existence of all of those "paradoxical" sets of reals) compatible with ZF *plus the negation of $AC_\omega(\mathbb{R})$?*

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Theorem (A. Blass, 1984)

In ZF, *if* every *vector space has a basis, then the Axiom of Choice holds true.*

Burstin bases and non-$AC_\omega(\mathbb{R})$

Theorem (Beriashvili, Sch., Wu and Yu, 2018)

In the Cohen-Halpern-Lévy model H there is a Hamel basis and a Bernstein set (but there are no Sierpiński sets).

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Theorem (Beriashvili, Sch., Wu and Yu, 2018)

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In H, there is also a Hamel basis which contains a perfect set.

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Let H^* be the following variant of the Cohen-Halpern-Lévy model: Let h be $\mathbb{S}(\omega)$-generic over L ($\mathbb{S}(\omega)$ being the finite support product of ω Sacks forcings). Let $B = \{d_n : n < \omega\}$ be the set of Sacks reals added by h.

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By replacing Sacks forcing $\mathbb{S}$ above by a refinement of Sacks forcing which is due to Jensen, one obtains a model H^{**} of ZF plus non-$AC_\omega(\mathbb{R})$ plus there is Δ^1_3 Sierpiński set, a Δ^1_3 Luzin set, a Δ^1_3 Hamel basis which contains a perfect set, as well as a Δ^1_3 Burstin basis.

Burstin bases in ZF plus DC plus "no w.o. of $\mathbb{R}$"

Theorem (Brendle, Castiblanco, Sch., Wu, Yu)

There is a model W of $\mathsf{ZF} + \mathsf{DC}$ such that in W the reals cannot be well-ordered and W contains Luzin as well as Sierpiński sets and also a Burstin basis.

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Lemma (Folklore)

Let $\mathbb{P}$ be a forcing notion satisfying the Sacks property and let G be a $\mathbb{P}$-generic filter over V. Then:

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(2) *Similarly, for every meager set $M \subseteq {}^{\omega}\omega$ in $V[G]$, there is a meager set $\bar{M} \subseteq {}^{\omega}\omega$ coded in V such that $M \subseteq \bar{M}$.*

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Corollary

If $\mathbb{P}$ has the Sacks property, then $\mathbb{P}$ preserves Luzin and Sierpiński sets.

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(c) there is no Vitali set (and hence no Hamel basis) in $L(\mathbb{R}^*)$.

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Problem: $L(\mathbb{R}^*)[b] \models$ ZFC plus CH.

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Extendability: If $p \in \mathbb{P}_B$ is such that $L[x] \models$ "p is a Burstin basis" and if $y \in \mathbb{R}^{L[x,y]} \setminus L[x]$, then there is some $q \leq_{\mathbb{P}_B} p$ such that q is a Burstin basis in $\mathbb{R}^{L[x,y]}$.

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A set $X \subseteq \mathbb{R}$ is in s^0 if and only if for every perfect set P there is a perfect subset $Q \subseteq P$ with $Q \cap X = \varnothing$.

s_0 is an σ-ideal which does not contain any perfect set.

Theorem (M. Groszek, T. Slaman, 1998)

Let $W \subseteq V$ be an inner model such that $W \models \mathsf{CH}$. If $\mathbb{R} \cap V \setminus W \neq \varnothing$, then

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$$V \models \mathbb{R} \cap W \in s^0$$

Corollary

Let x, y be reals such that $y \notin L[x]$, and let $\{z_0, z_1, \dots\} \in L[x, y] \cap [\mathbb{R}]^\omega$. Then

$$\textit{span}((\mathbb{R} \cap L[x]) \cup \{z_0, z_1, \dots\}) \in {s_0}^{L[x,y]}$$

Extendability of $\mathbb{P}_B$

Corollary

Let $b \in L[x]$ be linearly independent, $x \in \mathbb{R}$. Let $y \in \mathbb{R} \setminus L[x]$. There is then some $p \supset b$, $p \in L[x, y]$ such that

$$L[x, y] \models \text{“}p \textit{ is a Burstin basis.”}$$

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Lemma

$L(\mathbb{R}^*)$ thinks that:

(a) *(Extendability) If $p \in \mathbb{P}_B$ is such that $L[x] \models$ “p is a Burstin basis” and if $y \in \mathbb{R}^{L[x,y]} \setminus L[x]$, then there is some $q \leq_{\mathbb{P}_B} p$ such that q is a Burstin basis in $\mathbb{R}^{L[x,y]}$.*

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By these arguments, if in the definition of $\mathbb{P}_B$ be replace “Burstin” by “Hamel,” then the generic added over $L(\mathbb{R}^*)$ will still automatically be a Burstin basis. But there is a variant of $\mathbb{P}_B$ which does add a Hamel basis over $L(\mathbb{R}^*)$ which is not a Burstin basis.

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Lemma

Let b be $\mathbb{P}_B$-generic over $L(\mathbb{R}^*)$. Then

$$L(\mathbb{R}^*)[b] \models \text{“There is no well-ordering of } \mathbb{R}.\text{”}.$$

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Let s be $\mathbb{S}(\omega_1)$-generic over L, and let $\mathbb{R}^* = \mathbb{R} \cap L[s]$. Let (b, m) be $\mathbb{P}_B \times \mathbb{P}_M$ generic over $L(\mathbb{R}^*)$. Then $\mathbb{R}^* = \mathbb{R} \cap L(\mathbb{R})$ and

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Per molts anys, Joan!