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Taylor's proof that $MA_{\omega_1} \Rightarrow NS_{\omega_1}$ is not ω_1-dense.

(A.D. Taylor, "Regularity properties of ideals and ultrafilters", AML 16 (1979), pp. 33-55.)

Theorem (A. Taylor) Assume MA_{ω_1}. NS_{ω_1} is not ω_1-dense.

Proof. Let $(S_\alpha : \alpha < \omega_1)$ be given such that each $S_\alpha \subset \omega_1$ is stationary.

Let $(\alpha_n^\xi : \xi < \omega_1, n < \omega)$ be a ladder system, i.e., for each limit $\xi < \omega_1$, $(\alpha_n^\xi : n < \omega)$ is strictly monotone and cofinal in ξ.

Let us recursively define $(T_\alpha : \alpha < \omega_1)$, $(\rho_\alpha : \alpha < \omega_1)$ as follows, where we maintain that each T_α is a stationary subset of S_α and $(\rho_\alpha : \alpha < \omega_1)$ is a strictly increasing sequence of countable ordinals.

Let $(T_\beta : \beta < \alpha)$, $(\rho_\beta : \beta < \alpha)$ be given. Write $\bar\rho_\alpha = \sup\{\rho_\beta : \beta < \alpha\}$. Look at

$$f_\alpha = f : S_\alpha \setminus \bar\rho_\alpha \longrightarrow \omega_1$$

$$\xi \longmapsto \alpha_n^\xi, \text{ where } n<\omega \text{ is least such that } \alpha_n^\xi \geq \bar\rho_\alpha.$$

Let $T_\alpha \subset S_\alpha \setminus \bar\rho_\alpha$ be stationary and $\rho_\alpha < \omega_1$ such that $f''T_\alpha = \{\rho_\alpha\}$. This finishes the construction.

<u>Claim 1.</u> If $X \subset \omega_1$, $\bigcap\{T_\alpha : \alpha \in X\} \neq \emptyset$, then $\mathrm{otp}(X) \leq \omega$.

Proof: Let $\xi \in T_\alpha \cap T_{\alpha'}$, where $\alpha < \alpha'$. Then

$$f_\alpha(\xi) = \alpha_n^\xi = \rho_\alpha < \bar{\rho}_{\alpha'} \leq \rho_{\alpha'} = \alpha_m^\xi = f_{\alpha'}(\xi),$$

some $n < m$. $\dashv$ (Claim)

Let us now recursively define $(U_\alpha : \alpha < \omega_1)$ as follows, where we maintain that each U_α is a stationary subset of T_α. Let $(U_\beta : \beta < \alpha)$ be given. If $T_\alpha \cap U_\beta$ is nonstationary for all $\beta < \alpha$, then set $U_\alpha = T_\alpha \setminus \bigcup\{U_\beta : \beta < \alpha\}$.

Otherwise pick $\beta < \alpha$ s.t. $T_\alpha \cap U_\beta$ is stationary. Look at

$$g : T_\alpha \cap U_\beta \longrightarrow \alpha$$
$$\xi \longmapsto \text{the largest } \gamma < \alpha \text{ s.t. } \xi \in U_\gamma$$

By Claim 1, g is well-defined.

Pick $U_\alpha \subset T_\alpha \cap U_\beta$ s.t. g is constant on U_α. This finishes the construction.

It is easy to see that the following holds true.

<u>Claim 2.</u> If $\alpha < \alpha'$, then either $U_\alpha \cap U_{\alpha'} = \emptyset$ or else $U_{\alpha'} \subset U_\alpha$.

Let us now suppose that $(S_\alpha : \alpha < \omega_1)$ was dense in $NS^+_{\omega_1}$ to begin with; then $(U_\alpha : \alpha < \omega_1)$ is also dense, and the conclusions of Claims 1 + 2 hold true for $(U_\alpha : \alpha < \omega_1)$.

Let us "reindex" $(U_\alpha : \alpha < \omega_1)$, as follows. Let w.l.o.g. $U_0 = \omega_1$. Consider

$$h : \omega_1 \setminus \{0\} \longrightarrow \omega_1$$
$$\xi \longmapsto \text{the largest } \eta \ (\leq \xi) \text{ s.t. } A_\eta \supset A_\xi .$$

This is well-defined by Claims 1+2.

Let us define a tree $\overline{T}$ of height ω as follows. The bottom node is $U_0 = \omega_1$. If U is a node, $U = U_\alpha$, some α, then the immediate successors of U are all U_γ with $h(\gamma) = \alpha$, $\gamma > \alpha$.

By Claim 2, any two different immediate successors of a given node U are disjoint. Also, every U_α occurs on the tree: otherwise let α be least such that U_α doesn't, so $\alpha > 0$; then $U_{h(\alpha)}$ is on the tree and U_α is an immediate successor of $U_{h(\alpha)}$, contradiction!

For each U on $\bar{T}$, the set of all (eventual) $\bar{T}$-successors is dense below U. The set of all $U \in \bar{T}$ with $\aleph_1$ immediate $\bar{T}$-successors must then be dense in $\bar{T}$. We may then thin out $\bar{T}$ to get T as follows.

$U_0 = \omega_1$ is still the bottom node. Say U was already decided to be a node of T. Let U' be an (eventual) $\bar{T}$-successor of U with $\aleph_1$ immediate $\bar{T}$-successors, $(U'_i : i < \omega_1)$. Then let $\{U'_i : i < \omega_1\}$ be ~~the~~ immediate successors of U in T, as well as all immediate $\bar{T}$-successors of U which are disjoint from U'.
We may write T as $(U_s : s \in {}^{<\omega}\omega_1)$, where $U_\emptyset = \omega_1$ is the bottom node and $\{U_{s^\frown\xi} : \xi \in \omega_1\}$ is the set of immediate T-successors of U_s.

Let us summarize the properties of $(U_s : s \in {}^{<\omega}\omega_1)$.

$\{U_s : s \in {}^{<\omega}\omega_1\}$ is dense,

$U_{s^\frown\xi} \subset U_s$ for all $s \in {}^{<\omega}\omega_1$, $\xi < \omega_1$, and

$U_{s^\frown\xi} \cap U_{s^\frown\xi'} = \emptyset$ for all $s \in {}^{<\omega}\omega_1$, $\xi, \xi' < \omega_1$, $\xi \neq \xi'$.

Let us now consider the following poset.

$\mathbb{P} \ni (p, A)$ iff $p : \omega_1 \rightarrowtail \omega_1$ is a partial regressive function with $\mathrm{dom}(p)$ being finite, $A \subset \omega_1 \times {}^{<\omega}\omega_1$ is finite, and if $(\alpha, s) \in A$, $\xi \in \mathrm{dom}(p) \cap U_s$, then $p(\xi) \neq \alpha$.

<u>Claim 3.</u> For all $\xi < \omega_1$, $\omega \leq \xi$, $D^\xi = \{(p, A) \in \mathbb{P} : \xi \in \mathrm{dom}(p)\}$ is dense.

~~Claim 4. For all $(\alpha, s) \in \omega_1 \times {}^{<\omega}\omega_1$, $D_{\alpha, s} =$ $\{(p, A) \in \mathbb{P}$~~

<u>Claim 4.</u> For all $\alpha < \omega_1$ and $S \subset \omega_1$ stationary,

$$D_{\alpha, S} = \{(p, A) \in \mathbb{P} : \exists s \ (\alpha, s) \in A \wedge U_s \subset S \bmod NS_{\omega_1}\}$$

is dense.

Claim 3 is trivial. To see Claim 4, fix α

and S. Let $(p, A) \in \mathbb{P}$. It is easy to find $s \in {}^{<\omega}\omega_1$ s.t. $U_s \subset S \mod NS_{\omega_1}$ and $U_s \cap \mathrm{dom}(p) = \emptyset$. Then $(p, A \cup \{(\alpha, s)\}) \leq (p, A)$ in $\mathbb{P}$ and $(p, A \cup \{(\alpha, s)\}) \in D_{\alpha, S}$.

Suppose g is $\{D^\xi : \xi < \omega_1\} \cup \{D_{\alpha, S_\beta} : \alpha, \beta < \omega_1\}$- generic. g yields $f : \omega_1 \longrightarrow \omega_1$ regressive, and say $S \subset \omega_1$ is stationary and $\alpha < \omega_1$ s.t. $f''S = \{\alpha\}$. Let $S_\beta \subset S \mod NS_{\omega_1}$. Pick

$$(p, A) \in D_{\alpha, S_\beta} \cap g .$$

Let $s \in {}^{<\omega}\omega_1$ be s.t. $(\alpha, s) \in A$, $U_s \subset S_\beta \subset S \mod NS_{\omega_1}$. If $\xi \in U_s \cap S$, then $f(\xi) = p(\xi) \neq \alpha$. Contradiction!

It thus remains to be shown that $\mathbb{P}$ has the c.c.c.

<u>Claim 5.</u> $\mathbb{P}$ has the c.c.c.

Proof: Say $(p_i, A_i : i < \omega_1)$ is a large antichain. We may suppose w.l.o.g. that there is a root r for the p_i, i.e., if $i \neq j$ then $\mathrm{dom}(p_i) \cap \mathrm{dom}(p_j) = \mathrm{dom}(r)$ and $p_i \restriction \mathrm{dom}(r) = p_j \restriction \mathrm{dom}(r) = r$.

We may then also assume that all $\mathrm{dom}(p_i)$ have the same size, and that there is a root A for the A_i, i.e., if $i \neq j$, then $A_i \cap A_j = A$, and that $\max\{s : \exists \alpha\ (\alpha, s) \in A_i\}$ is the same for all i.

Let us define $g : [\omega_1]^2 \longrightarrow 2$ as follows.

Let $i < j < \omega_1$. Then $g(\{i,j\}) = 0$ iff there is some $(\alpha, s) \in A_i$ and $\xi \in \mathrm{dom}(p_j) \cap U_s$ with $p_j(\xi) = \alpha$; so $g(\{i,j\}) = 1$ ~~iff~~ gives that there is $(\alpha, s) \in A_j$ and $\xi \in \mathrm{dom}(p_i) \cap U_s$ with $p_i(\xi) = \alpha$.

Let us use $\omega_1 \longrightarrow (\omega+1, \omega)^2$. We then have one of the two cases to follow.

<u>1st case.</u> Have $(i_n : n \leq \omega)$ strictly increasing s.t. $g(\{i_n, i_m\}) = 0$ for all $n < m \leq \omega$. Say $(\alpha_n, s_n) \in A_{i_n}$, $\xi_n \in \mathrm{dom}(p_{i_\omega}) \cap U_{s_n}$, $p_{i_\omega}(\xi_n) = \alpha_n$, $n < \omega$.
There is then some $\xi = \xi_n \in \mathrm{dom}(p_{i_\omega})$ and some $X \subset \omega$ infinite s.t. $p_{i_\omega}(\xi) = \alpha_n = \alpha$, all $n \in X$. There is then some infinite $Y \subset X$ s.t. all s_n, $n \in Y$, have the same length, and as $U_{s_n} \cap U_{s_m} \ni \xi$ for all $n, m \in Y$, ~~they must~~ all the s_n, $n \in Y$,

must be equal with each other, call it s.

But then $(\alpha, s) \in A$ = root for the A_i, which clearly gives a contradiction, as $(\alpha, s) \in A_{i_\omega}$.

<u>2nd case.</u> Same argument with i_0 playing the role of i_ω.

<u>Addendum:</u> Proof of $\omega_1 \longrightarrow (\omega+1, \omega)^2$ (and even more):

Let $F : [\omega_1]^2 \longrightarrow 2$. Suppose that there is no $X \subset \omega_1$ with $\mathrm{otp}(X) = \omega+1$ which is 0-homogeneous. Then for all ξ we may pick some finite $X_\xi \subset \xi$ such that

- $X_\xi = \emptyset$ or $X_\xi \cup \{\xi\}$ is 0-homogeneous, and
- no $Y \supsetneq X_\xi \cup \{\xi\}$ is 0-homogeneous.

Let $S \subset \omega_1$ be stationary and $X \in [\omega_1]^{<\omega}$ s.t. $X_\xi = X$ f.a. $\xi \in S$. If $\xi < \xi'$ are both in S, then $F(\{\xi, \xi'\}) = 1$, as o.w. $X \cup \{\xi, \xi'\}$ would be 0-homogeneous, but $X \cup \{\xi, \xi'\} \supsetneq X_{\xi'} \cup \{\xi'\}$. In other words, S is 1-homogeneous, so that in fact

$$\omega_1 \longrightarrow (\omega+1, \text{stationary})^2 .$$