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The Triangular Embedding Theorem for $L(\mathbb{R})$

In "Happy and Mad Families in $L(\mathbb{R})$," Neeman and Norwood formulate a triangular version of the embedding theorem for $L(\mathbb{R})$. The aim of this note is to prove a version of this theorem, starting from a remarkable cardinal. According to my memory, the result reported here arose out of a discussion with William Chan at the 1st Irvine conference on descriptive inner model theory and hod mice in July, 2016.

The following is a version of theorem 22 of the above-mentioned paper by Neeman / Norwood.

<u>Theorem.</u> Let κ be a remarkable cardinal in L. Let g be $Col(\omega, <\kappa)$-generic over L, and let $\mathbb{P} \in L[g]$ be s.t. $L[g] \models$ "$\mathbb{P}$ is a proper poset." Let θ be an L-cardinal $> \kappa$ s.t. $\mathbb{P} \in H_\theta^{L[g]}$.

Inside $L[g]$ there is then a stationary set of countable $X \prec H_\theta^{L[g]}$ such that if $\pi : L_\alpha[\bar{g}] \cong X$ and $h \in L[g]$ is $\pi^{-1}(\mathbb{P})$-generic over $L_\alpha[\bar{g}]$, then there is an elementary embedding $\tilde{\pi} : L(\mathbb{R})^{L_\alpha[\bar{g}, h]} \longrightarrow L(\mathbb{R})^{L_\theta[g]}$ with

$$\tilde{\pi} \restriction [\alpha \cup (\mathbb{R} \cap L_\alpha[\bar{g}])] = \pi \restriction [\alpha \cup (\mathbb{R} \cap L_\alpha[\bar{g}])] .$$

$$\begin{tikzcd}
H_\theta^{L[g]} = L_\theta[g] \arrow[r, phantom, "\subseteq"] & L(\mathbb{R})^{L_\theta[g]} \\
L_\alpha[\bar{g}] \arrow[u, "\pi"] \arrow[r, phantom, "\subset"] \arrow[d, phantom, "\cap"] & L_\alpha[\bar{g}, h] \arrow[d, phantom, "\cap"] \\
L(\mathbb{R})^{L_\alpha[\bar{g}]} \arrow[uur, "\pi\restriction\ldots"] \arrow[r, hook, "\sigma"'] & L(\mathbb{R})^{L_\alpha[\bar{g}, h]} \arrow[uu, "\tilde{\pi}"']
\end{tikzcd}$$

We will make use of the results of my paper "Proper forcing and remarkable cardinals $\underline{\text{II}}$."

By this paper, there is a (unique) elementary embedding $\sigma : L(\mathbb{R})^{L_\alpha[\bar{g}]} \longrightarrow L(\mathbb{R})^{L_\alpha[\bar{g}, h]}$ with

$$\sigma \upharpoonright [\alpha \cup (\mathbb{R} \cap L_\alpha[\bar{g}])] = id.$$

Proof of the Theorem. Let $X \in [L_\theta[g]]^\omega \cap L[g]$ be such that $X \cap \kappa \in \kappa$ and $otp(X \cap \theta)$ is an L-cardinal ($< \kappa$). As κ is remarkable in L, $L[g]$ has stationarily many such X.

Let $\pi : L_\alpha[g \upharpoonright \beta] \cong X \prec L_\theta[g]$, where $\beta = X \cap \kappa$ and $\alpha = otp(X \cap \theta)$.

α is an L-cardinal, β is inaccessible (in fact, remarkable) in L_α, so that β is also inaccessible in L, and $g \upharpoonright \beta$ is $Col(\omega, < \beta)$-generic over L.

Let us assume that $\mathbb{P} \in ran(\pi)$, and write $\bar{\mathbb{P}} = \pi^{-1}(\mathbb{P})$. Let $h \in L[g]$ be $\bar{\mathbb{P}}$-generic over $L_\alpha[g \upharpoonright \beta]$. h is then in fact also $\bar{\mathbb{P}}$-generic over $L[g \upharpoonright \beta]$.

It remains to be shown that there is an embedding $\tilde{\pi} : L(\mathbb{R})^{L_\alpha[g \upharpoonright \beta, h]} \longrightarrow L(\mathbb{R})^{L_\theta[g]}$, elementary, s.t. $\tilde{\pi} \upharpoonright [\alpha \cup (\mathbb{R} \cap L_\alpha[g \upharpoonright \beta])] = \pi \upharpoonright [\alpha \cup (\mathbb{R} \cap L_\alpha[g \upharpoonright \beta])]$.

In what follows, we also write $\bar{g} = g \upharpoonright \beta$.

By one of the key arguments of "Proper forcing and remarkable cardinals II," if $x \in \mathbb{R} \cap L_\alpha[\bar{g}, h]$, then there is some $\mathbb{Q} \in L_\beta$ s.t. x is $\mathbb{Q}$-generic over L_α. As $L_\alpha[\bar{g}, h] \in L[g]$ is countable in $L[g]$, we may then work in $L[g]$ to build some $g^* \in L[g]$, g^* $Col(\omega, <\beta)$-generic over L_α, such that $\mathbb{R} \cap L_\alpha[g^*] = \mathbb{R} \cap L_\alpha[\bar{g}, h]$ (cf. the proof of Lemma 2.2 of "Proper forcing and remarkable cardinals II"). As $\alpha > \beta$ is an L-cardinal, g^* is also $Col(\omega, <\beta)$-generic over L.

By Solovay, then, there is some $g^{**} \in L[g]$, g^{**} $Col(\omega, [\beta, \kappa))$-generic over $L[g^*]$ such that

$$L[g^*, g^{**}] = L[g],$$

and of course $L[g^*, g^{**}] = L[\bar{g}, h, g^{**}]$.

Now look at $\pi \restriction L_\alpha : L_\alpha \longrightarrow L_\theta$.
We may extend $\pi \restriction L_\alpha$ to

$$\tilde{\tilde{\pi}} : L_\alpha[g^*] \longrightarrow L_\theta[g^*, g^{**}]$$

by $\tilde{\tilde{\pi}}(\tau^{g^*}) = \pi(\tau)^{g^* \frown g^{**}}$ for $\tau \in (L_\alpha)^{Col(\omega, <\beta)}$.

Write $\tilde{\pi} = \tilde{\tilde{\pi}} \restriction L(\mathbb{R})^{L_\alpha[g^*]}$.

As $L(\mathbb{R})^{L_\alpha[g^*]} = L(\mathbb{R})^{L_\alpha[\bar{g}, h]}$ and $L(\mathbb{R})^{L[g^*, g^{**}]}$ $= L(\mathbb{R})^{L[g]}$, $\tilde{\pi}$ is an elementary embedding from $L(\mathbb{R})^{L_\alpha[\bar{g}, h]} = L(\mathbb{R})^{L_\alpha[g \restriction \beta, h]}$ into $L(\mathbb{R})^{L[g]}$.

Trivially, $\tilde{\pi} \restriction \mathbb{R} \cap L_\alpha[\bar{g}] = id = \pi \restriction \mathbb{R} \cap L_\alpha[\bar{g}]$, but also $\tilde{\pi}(\xi) = \tilde{\tilde{\pi}}(\xi) = \tilde{\tilde{\pi}}(\check{\xi}^{g^*}) = (\pi(\xi)^\vee)^{g^* \frown g^{**}} =$ $\pi(\xi)$ for $\xi < \alpha$, so that $\tilde{\pi} \restriction \alpha = \pi \restriction \alpha$.

$\dashv$ (Theorem)

The methods from my "Proper forcing and remarkable cardinals II" show that the conclusion of the Theorem implies that ω_1 be remarkable in L.