SET THEORY COURSE WINTER TERM 2020-21, EXERCISE SHEET NO. 5, NOV 24, 2020

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Hand in by Dec 01, 2020.

Problem 1. Show the following statements in ZF.

(a) There is a function f with domain ω such that $f(0) = \omega$ and for every $n \in \omega$, $f(n+1) = f(n) \cup \{f(n)\}$ $(= f(n) + 1)$.

(b) If f is as in (a) and b is the range of f (i.e., $b = \{f(n) : n \in \omega\}$), then $\omega \cup b$ is an ordinal.

(c) If b is as in (b), then $\omega \cup b$ is the smallest limit ordinal $> \omega$.

(d) If α is any ordinal, then there is a limit ordinal $> \alpha$.

Recall that Problem 3 from sheet no. 4 showed (in ZF) that for every set A there is some well-ordered set $(a, \leq)$ such that there is no injection $f \colon a \to A$.

Problem 2. Show in ZF that for every set A there is some ordinal β such that there is no injection $f \colon \beta \to A$.

For any set A, we denote by A^+ the least ordinal β such that there is no injection $f \colon \beta \to A$.

Problem 3. Show in ZF that for every ordinal α there is some ordinal $\gamma > \alpha$ such that for all $\beta < \gamma$, $\beta^+ < \gamma$.

Problem 4. Use the Recursion Theorem to show that for all ordinals α and δ, there are functions f, g, and h with domain δ such that for all $\xi < \delta$,

(a) $f(0) = \alpha$, $f(\xi) = f(\bar{\xi}) + 1$ if $\xi = \bar{\xi} + 1$, and if ξ is a limit ordinal, then $f(\xi) = \bigcup_{\bar{\xi} < \xi} f(\bar{\xi})$. $f(\xi)$ is usually written $\alpha + \xi$.

(b) $g(0) = 0$, $g(\xi) = g(\bar{\xi}) + \alpha$ if $\xi = \bar{\xi} + 1$, and if ξ is a limit ordinal, then $g(\xi) = \bigcup_{\bar{\xi} < \xi} g(\bar{\xi})$. $g(\xi)$ is usually written $\alpha \cdot \xi$.

(c) $h(0) = 1$, $h(\xi) = h(\bar{\xi}) \cdot \alpha$ if $\xi = \bar{\xi} + 1$, and if ξ is a limit ordinal, then $h(\xi) = \bigcup_{\bar{\xi} < \xi} h(\bar{\xi})$. $h(\xi)$ is usually written α^{ξ}.

Show that $1 + \omega = \omega < \omega + 1$ and $\omega = 2 \cdot \omega < \omega \cdot 2$.

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